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ERRATA.

Page 163, Ex. 8, for $=$ read $+$. P. 183, line 2, for 88 read 80 ; and line 4, for 33 read 25. P. 206, Ex. 17, for tan read sin ; and Ex. 19, for tan b read tan B. P. 381 in the diagram, the point e should be on OE. P. 446, line 18, for exactly read very nearly. P. 449, line 22, after denominator put $\div g$. P. 571, last line but one, omit comma after C.

Types fallen out or defective. Page 27, Ex. 10, $\frac{3}{2}$. P. 64, line 6, the exponent $n+p$.

P. 72, line 8, $\frac{p}{q}$. P. 77, line 19, $\frac{25}{2}$. P. 169, line 29, the minus sign.

P. 224, line 1, the exponent n. P. 225, line 16, the exponent $\frac{1}{n}$.

P. 297, Prob. II., $\frac{y'}{2}$. P. 499, Ex. 5, exponents of x, $z-1$.

P. 515, line 8, $\frac{ds}{dx}$. P. 533, Ex. 1, $\frac{x^2+a}{ax+y^2}$.

A COURSE

OF

ELEMENTARY MATHEMATICS,

ETC., ETC.

THE preparation necessary for the profitable study of the following course of Mathematics is—a knowledge of common Arithmetic, and some acquaintance with the principles of Geometry, as taught in *Euclid's Elements*. A student ignorant of these initiatory, but most important departments of elementary science, would scarcely seek his first lessons therein from a book such as this. The *Elements of Euclid* is a work by itself; universally known and esteemed, and everywhere to be easily procured:—to transfer its pages to the present performance, could be of no possible advantage to the learner. And the same may be said of common Arithmetic:—both this and *Euclid* are more conveniently studied from the ordinary manuals in popular use. We shall therefore commence the volume now in the hands of the reader, with a treatise on ALGEBRA—the indispensable foundation of the entire fabric of modern analytical science.

I. ALGEBRA.

1. **Preliminary Notions.**—Algebra may be regarded simply as an extension of the principles of Arithmetic. In the latter science the symbols of quantity, to which its rules and operations are applied, are limited to the nine digits or figures 1, 2, 3, 4, 5, 6, 7, 8, 9, together with the *cypher* or *zero*, 0. And not only is the *notation* of Arithmetic limited to these ten symbols, but each symbol is employed by every computer in the same sense:—the character or symbol 4, for instance, stands for *four, always;* 6 for *six;* 8 for *eight*, and so on: the symbols of Arithmetic are thus fixed in meaning, as well as limited in number.

It is otherwise in Algebra: in this science the symbols of quantity comprehend not only the figures of arithmetic, but also the letters of the alphabet:—the *figures* being, as in arithmetic, of invariable signification, but the *letters* admitting of arbitrary interpretation. It is this latter circumstance—namely, the possession of a set of symbols which we may employ to represent anything we please—that gives to Algebra its peculiarity and its power. In Arithmetic, known quantities only can be denoted by symbols: in Algebra a quantity altogether unknown, in value, at the outset of an inquiry, may be represented—an alphabetical letter serving this purpose,—and then the rules of the science, to be hereafter developed, will enable us ultimately to interpret its meaning, consistently

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with the conditions which connect it, in that inquiry, with the known quantities concerned.

2. **Definitions—Symbols of Quantity—Signs of Operation.**—As noticed above, the symbols by which the quantities operated upon in algebra are represented, are the figures of ordinary arithmetic, and the letters of the alphabet: the marks or signs by which these operations are indicated, are called *signs of operation*: the principal of these are the following:—

$+$, *plus*, the sign of *addition*, implying that the quantity to which it is prefixed is to be *added*.

$-$, *minus*, the sign of *subtraction*, denoting that the quantity to which it is prefixed is to be subtracted.

Thus $5+2$, which is read 5 plus 2, signifies that 2 is to be added to 5; and $5-2$, which is read 5 minus 2, indicates that 2 is to be subtracted from 5. In like manner $a+b$, or a plus b, implies that b is to be added to a, that is, that the quantity represented by b is to be added to that represented by a. And $a-b$, or a minus b, implies that b is to be subtracted from a. Of course we cannot actually *perform* the addition and subtraction operations thus indicated, till we know what numbers or quantities a and b stand for.

It may be remarked here, that although the letters a, b, &c. are but the representatives of quantities or numerical values, yet, for brevity of expression, we refer to them as the quantities themselves.

The crooked mark $\sim$ placed between two quantities denotes the *difference* between those quantities: thus $a \sim b$ means the difference between a and b, whether that difference be the result of subtracting b from a, or a from b.

$\times$, the sign of *multiplication*, when placed between two quantities, implies that those quantities are to be multiplied together: thus 4×6, or 6×4 means that 4 and 6 are to be multiplied together, and $a \times b$, or $b \times a$, implies in like manner the product of a and b.

Instead of the sign $\times$, a dot placed between the factors is often used for the sign of multiplication: thus 4.6, or 6.4, and $a.b$, or $b.a$, each implies the product of the quantities between which the dot is placed. It must be observed, however, that the dot should range with the *lower* part of the figures or letters, and not with the upper part, to avoid confounding it with the decimal point, as, in the case of *figures*, might otherwise happen: thus 6.4 means 24, but $6 \cdot 4$ means 6 and 4 tenths.

In the case of *letters* however, the dot is usually dispensed with altogether, and the factors simply written side by side, without any intervening sign at all: thus, ab, cx, bxy, $abxz$, &c. mean the same as $a \times b$. $c \times x$, $b \times x \times y$, $a \times b \times x \times z$; or as $a.b$, $c.x$, $b.x.y$, $a.b.x.z$, &c. This suppression of the intervening sign of multiplication between the factors is not allowable when those factors are *numbers*, as is obvious: if 6×4, or 6.4 were written 64, sixty-four would be implied, and not 24, as intended. But when a *single* numerical factor enters with the letters, then the multiplying sign may be omitted, since no ambiguity can arise: thus, $6 \times a \times b$ or $6.a.b$, may be more conveniently written $6ab$, which means 6 times the product of a and b, or as it is more briefly read, 6 times a, b. It is proper, as here, always to place the numerical factor first, and the literal factors afterwards; and also to arrange these latter in the order in which they succeed each other in the alphabet. The numerical factor, thus placed first, is called the *coefficient* of the quantity

multiplied by it: thus 6 is the coefficient of ab in $6ab$, and 15 is the coefficient of xyz in $15xyz$.

$\div$, the sign of *division*, when placed before a quantity, supplies the place of the words "divided by," so that $8\div2$ means 8 divided by 2, $12\div3$ means 12 divided by 3, $a\div b$ means a divided by b, and so on; but, as in common arithmetic, division is more frequently indicated by writing the dividend above and the divisor below a horizontal bar of separation, thus:—

$$\frac{a}{b} \text{ is the same as } a\div b, \text{ and } \frac{3xy}{2ab} \text{ is the same as } 3xy\div 2ab.$$

3. The four signs now explained—indicating the four fundamental operations both of arithmetic and algebra—are, of course, those of most frequent occurrence in calculation. Algebraists, however, economize their signs of operation as much as possible, and never introduce them needlessly. This has been already exemplified in the case of multiplication: the absence of sign between letters, placed side by side, as much implies the multiplication together of the numbers those letters represent, as if each were separated from the others by an oblique cross, or a dot. In like manner, when a row of additive and subtractive quantities are connected together by the proper signs, if the *first* of these quantities be additive, or *plus*, the sign $+$ is suppressed as superfluous; thus: $a-b+c+d-4$ is the same as $+a-b+c+d-4$, and implies that a, c, and d are additive; or, as they are more frequently called, *positive* quantities, and that b, and 4, are subtractive, or *negative* quantities. If the letters a, b, c, d stood respectively for 2, 4, 3, 8, the interpretation of the *expression* just written would be 5.

4. The term *coefficient* has been already defined: it is the numerical multiplier of the algebraic quantity to which it is prefixed: when this numerical multiplier is simply 1, it is not inserted: it is superfluous to introduce unit-factors; $a+b$, is as well understood to be once a plus once b, as $1a+1b$; but if the question were asked—What is the coefficient of a or of b in the expression $a+b$? the answer would be, not *nothing*, but 1.

5. $=$, *equal to*, is the sign of *equality*; it implies that what is written on one side of it is equal to what is written on the other, thus:—

$$7+4=11,\ 7-4=3,\ 7-4+1=4,\ 3x+2x-x=4x, \text{ \&c.}$$

6. Any quantity—of how many letters soever it may consist—is called a *simple* quantity, or a quantity of but one *term*, provided it be not separated into distinct parts by the interposition of a plus or minus sign; thus, each of the following is a simple quantity, or a quantity of but one term:—

$$5abx,\ 7abcy,\ \frac{2ax}{3by},\ \frac{14}{9xyz}, \text{ \&c.}$$

7. Each of the following, however, is a *compound* quantity: the first consists of two terms, the second of three terms, and the third of four terms:—

$$2a+3b,\ 6a-2b+c,\ 5ab+2cd-3m+\frac{4}{5}.$$

8. We shall now add a few exercises by which the learner may satisfy himself as to whether he correctly understands what has already been explained or not.

EXERCISES ON THE DEFINITIONS.—In the following exercises we shall suppose $a=4$, $b=3$, $c=5$, and $d=7$.

Find the values in numbers of the following expressions:—

Expression.	*Interpretation.*	*Value.*
$3a+2b-c$	$12+6-5$	13.
$5b-3a+2d$	$15-12+14$	17.
$4d+2c-6a+1$	$28+10-24+1$	15.
$8c-d-3b+4a$	$40-7-9+16$	40.
$2ab+5cd-16$	$24+175-16$	183.

Consequently $3a+2b-c=13$, $5b-3a+2d=17$, $4d+2c-6a+1=15$, $8c-d-3b+4a=40$, and $2ab+5cd-16=183$; and the values of the following expressions are to be obtained in a similar manner:—

(1) $2b+3a-2d$.
(2) $48-4d+3d$.
(3) $4ac+2cd-ab$.
(4) $3abc+4bcd-60$.
(5) $\frac{abc}{2}+bd-4a$.
(6) $\frac{b+c}{a}+\frac{c+d}{2b}$.
(7) $\frac{ab-c}{d}-\frac{2a}{b+c}$.
(8) $\frac{3cd-4ab-1}{2ab+4}+\frac{ab}{c+d}$.

☞ For the proper values, see answers at the end.

9. **Addition of Algebra.**—In Arithmetic, addition means the collection into one sum of a set of quantities, all of which are additive or positive: in Algebra the term is extended to the finding the aggregate or balance of a set of quantities, some of which only may be positive, and the others negative; the result of such addition being plus or minus according as the sum of the positive, or the sum of the negative terms, preponderate. The sum of a set of algebraic quantities—in the absence of all interpretation of the symbols—can be exhibited only when the quantities are all *alike;* that is, when, so far as the *letters* are concerned, they do not differ from one another. That a set of *like* quantities may be added together, without our requiring to know what those quantities are, is plain: thus, it is clear that $2a+3a+5a=10a$, whatever a may stand for, and that $6ab+3ab-4ab=5ab$, whatever ab may stand for. If however the quantities are not all *like* quantities, then to incorporate the entire set into a single term as here, would be impossible. For instance, if we had the set of quantities $4ax+2ax-3ax+2b$, all we could do would be to actually collect the first three into one term, and then to annex to the sum the fourth term $2b$ with its proper sign; we should thus say that $4ax+2ax-3ax+2b=3ax+2b$, an expression which we cannot further reduce or simplify till we know something about the values of the letters.

10. Addition therefore divides itself into two cases: 1. When the quantities to be added are all *like* quantities, and 2. When they are not all like quantities.

CASE I.—*When the quantities are all* LIKE *quantities, that is, when they differ in nothing but in their coefficients.*

RULE 1. Find the sum of the positive coefficients. 2. Find the sum of the negative coefficients. 3. Take the difference of these two sums, prefix to it the sign of the *greater* sum, and then annex the letters common to all the quantities: the correct sum will thus be exhibited.

NOTE.—When there are two or more columns to be added up, we always commence with the first column on the *left*, and not, as in arithmetic, with the first on the right, as it is more convenient to write

the several results, with their signs, from left to right, than from right to left.

The following four examples are worked: the learner should clearly see how the results are obtained, and satisfy himself of their accuracy, before attempting the exercises below.

(1)	(2)	(3)	(4)
$3x$	$2ax+b$	$6bx-4a$	$-8abx+6\frac{1}{2}ac$
$5x$	$3ax+2b$	$-2bx+3a$	$3abx-2\frac{1}{2}ac$
$9x$	$ax-3b$	$-5bx-2a$	$-5abx+4ac$
$-x$	$5ax+4b$	$12bx-6a$	$-2abx+3ac$
$7x$	$\frac{1}{2}ax-5b$	$8bx+9a$	$4abx+ac$
Sum $23x$	$11\frac{1}{2}ax-b$	$19bx$	$-8abx+12ac$

EXERCISES TO BE WORKED.

(1)	(2)	(3)	(4)
$5a$	$3by+4a$	$7ax+2by-3cz$	$2\frac{1}{2}mx-7ny+3\frac{a}{b}$.
$-7a$	$2by-3a$	$2ax-3by+4cz$	$5\frac{1}{4}mx+2ny-6\frac{a}{b}$.
$2a$	$-by+9a$	$3\frac{1}{2}ax-by+\frac{1}{2}cz$	$-9\frac{1}{2}mx-8ny-6\frac{a}{b}$.
$9a$	$-5by+a$	$11ax+5by-2cz$	$13\frac{1}{4}mx-3ny-7\frac{a}{b}$.
$15a$	$8by-2a$	$-ax+8by-7cz$	

(5) Add the following expressions: $7axy-3b$, $-4axy-2b$, $8axy+6b$.

(6) $5ax+1$, $3ax-2$, $6ax+4$, $-ax+3$, $-7ax-5$, $4ax+3$. (7) $8cz+2x-4$, $-3cz-7x$, $5cz-4x+1$, $-9cz+3x-8$, $6x+2$, -7. (8) $3axy+2bz-4c$, $-6axy-3bz+7c$, $\frac{1}{2}axy+5bz-3c$, $2axy-bz-6c$, $-4\frac{1}{2}axy-bz-12c$, $-13axy+\frac{1}{2}bz+5c$, $axy+4bz+13c$.

11. CASE II.—*When the quantities are* NOT *all* LIKE *quantities.*

RULE.—Collect the LIKE quantities from the several expressions, and add them together: to the sum connect, with their proper signs, those of the quantities which have no like.

NOTE.—Athough it is of no consequence which set of like quantities be added first, yet the custom is to commence with the quantity at the *top* of the *first* column on the *left*, and to put down, under that column, the sum of all the quantities like *it;* then to collect the quantities like that at the top of the second column, and so on, as in the following examples:—

(1)	(2)	(3)
$2x-7y+4z$	$7ax-2by+z$	$4xyz-3xy+2yz$
$8z+2x-y$	$3by+9z-ax$	$5xy+8yz-7xyz$
$-2y+5z+a$	$-6z+2ax+by$	$-9ax+xyz+11$
$4x-3z+7y$	$5by-3ax+6$	$-4yz+7xy-8$
$8x-3y+14z+a$	$5ax+7by+4z+6$	$-2xyz+9xy+6yz-9ax+3$

12.—EXERCISES TO BE WORKED.—In the following examples the several expressions may be taken as they are, and placed one under another as above; or the arrangement of the terms may be changed, so that when placed in vertical columns, the *like* terms may stand one under

another, as in Case I.; or lastly, without transcribing, and writing them one under another, we may pick out the like quantities from the expressions as they stand, and write down the sum of each set at once, afterwards connecting, with their proper signs, the unlike quantities.

(4) $4ax-2y+7$, $3y-8+ax$, $42-3ax+y$. (5) $6x-8y+5a$, $4y+2x$, $20-6x+7y$, $3a+b$. (6) $abx+2$, $5ay-7+2abx$, $14-ay$, $6ay-9$. (7) $8mx-3ny$, $5az+2mx$, $7ny-az$, $23-4mx+9ny$, $4az-3$. (8) $2\frac{x}{y}-7\frac{y}{x}+3$, $9\frac{y}{x}-8$, $7\frac{a}{b}+5$, $3ax-\frac{a}{b}+2$, $8\frac{y}{x}+9\frac{x}{y}+1$. (9) $7abx-9aby+az$, $4az-3ay+abx$, $-ax+7abx-9aby$, $2abz-6ax-aby$, $-12ay-4az+x$, $3abz-5ax-abz$.

13. **Subtraction of Algebra.**—The operations of Arithmetic are all performed with *positive* numbers or quantities. In Algebra, quantities both positive and negative equally enter into our computations: we must know therefore how the operation called subtraction is to be performed, whether the quantities operated upon be positive or negative.

To *subtract* a quantity is to take it away from some other quantity: this we can actually do, provided we can split this other quantity into two parts—one of which shall be equal to the quantity to be taken away: the other part is, of course, the *remainder*.

This simple truth suggests the rule for algebraic subtraction, thus: 1. Suppose we have to subtract 5 from 12: instead of 12 we may write its equivalent $7+5$, so that actually taking the 5 away, the remainder is 7.

2. Suppose we have to subtract -5 from 12: then, instead of 12, writing its equivalent $17-5$, and then taking the -5 away, the remainder is 17.

3. Suppose we have to subtract 5 from -12: then, instead of -12, writing its equivalent $-17+5$, and then taking the 5 away, the remainder is -17.

4. Lastly, suppose we have to subtract -5 from -12: then, instead of -12, writing its equivalent $-7-5$, and then taking the -5 away, the remainder is -7.

Having thus taken all the possible varieties, as to the signs, of the two numbers—which numbers have, of course, been taken at random—we safely infer the truth of the following results of subtraction:—

From	12	12	-12	-12
Subtract	5	-5	5	-5
There remains	7	17	-17	-7

and we moreover see that the very same results would have been obtained if we had *changed the sign* of each number to be subtracted, and had then *added*.*

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* If we use general symbols instead of figures, the following results, namely,

From	a	a	$-a$	$-a$
Subtract	b	$-b$	b	$-b$
Remainder	$a-b$	$a+b$	$-a-b$	$-a+b$

are shown to be true, by substituting for a the following equivalents, namely:—

$a=a-b+b$, $a=a+b-b$, $-a=-a-b+b$, $-a=-a+b-b$.

We conclude therefore that algebraic subtraction may always be converted into algebraic addition, by simply changing the sign of the quantity to be subtracted. Hence the following rule.

14. RULE.—Conceive the sign of every quantity that is to be subtracted, to be *changed:* and then, with this supposed change of signs, proceed as if the operation were that of *addition* instead of subtraction: the result will be the *remainder.*

	(1)	(2)	(3)
From	$7ax+9by$	$5xy-8az+2$	$-3axy+bz-2x$
Subtract	$3ax-5by$	$-3xy+2az-1$	$5axy+7bz-3x-6$
Remainder	$4ax+14by$	$8xy-10az+3$	$-8axy-6bz+x+6$

NOTE.—It must be carefully borne in mind that when terms occur that have no like, those in the upper row must be annexed to the remainder with their proper signs, and those in the lower row with their signs *changed.*

(1) From $4ax-2by+4$ subtract $2ax-6by-3$.
(2) From $2abx+3ay-2z$ subtract $5abx-3ay+2z$.
(3) From $-7xyz+5xy+6$ subtract $-2xyz-3xy$.
(4) From $-3mxy-6nyz+2a$ subtract $2mxy+nyz+8$.
(5) From $8az-4by+3x$ subtract $-6x+2by-7az+1$.
(6) From $2\frac{1}{2}ax-3by+17$ subtract $23-1\frac{1}{2}by-\frac{1}{2}x+2z$.
(7) From $6\frac{1}{4}py-9az-m$ subtract $17az-\frac{3}{4}py+7n$.
(8) From $-5cxz-7ey+kz+a$ subtract $2cy-3kz+cxz-b+4$.

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15. **Brackets—Vincula.**—The signs of operation hitherto employed have been prefixed to simple quantities only: when we intend them to apply to compound quantities—or to quantities consisting of two or more simple terms—it is necessary to inclose the terms within *brackets*, such as ( ), or { }, or [ ]; or else to cover all with a *vinculum*, or bar ——, to imply that the several terms thus tied together are to be treated as one whole.

For instance, by writing or tying together the terms in this way, we may express the subtractive operations above thus:—

$7ax+9by-(3ax-5by)=4ax+14by$, $5xy-8az+2-\{-3xy+2az-1\}=8xy-10az+3$, $-3axy+bz-2x-[5axy+7bz-3x-6]=-8axy-6bz+x+6$.

16. The *minus* sign before each of these bracketed quantities implies that the sign of every inclosed term is to be changed on the removal of the brackets: when a bracketed quantity is connected to other quantities by a *plus* sign, the removal of the brackets leaves the signs of the quantities undisturbed. The following instances of the management of bracketed quantities will be easily understood:—

$(a+b)-(a-b)=a+b-a+b=2b$, $a-(b-c)-(b+c)=a-b+c-b-c=a-2b$.
$a+(a-b)-\{c-(a-c)\}=a+a-b-c+(a-c)=3a-b-2c$.
$2ax-\{ax-(2y-3ax)\}=2ax-ax+2y-3ax=2y-2ax$.

17. A multiplier, or coefficient, placed before a bracketed quantity, implies that the compound whole—or, which is the same—that each individual term is to be multiplied by that coefficient. thus, $4(a+b)=4a+4b$, $5(a-b)=5a-5b$, $3(4a-b+2)-6(a+b-8)=12a-3b+6-6a-6b+48=6a-9b+54$.

18. The equivalence of the following expressions the learner is left to prove for himself—

(1) $4\{3a-(b-a)\}=4(4a-b)$. (2) $2\{2x-(x+y)-1\}=2(x-y-1)$. (3) $5\{a+x-2(x-a)\}=5(3a-x)$. (4) $3\{(x-4)-2(3x+2)-5(1-x)\}=-39$. (5) $6\{(a-2x)-3(4x-2a)\}-3\{(4x+a)-(9x-a)\}=36a-69x$.

19. **Simple Equations.**—An *equation* is merely a declaration, expressed in the characters of algebra, combined or not, with those of arithmetic, that two quantities are *equal*. Thus: if x be such a number that $x-3$ is equal to 8, then the algebraic statement of this equality, namely, $x-3=8$ is an *equation*. It is plain that to satisfy this equation, x must represent the number 11. In like manner, if x be such a number that $4x-3$ is equal to $2x+7$, we shall have the equation $4x-3=2x+7$. The finding the value of the unknown quantity—in this case x—that is, the discovering what the unknown quantity really stands for, is called the *solution of the equation*.

20. By help of the few principles established in the preceding pages, the more easy kinds of simple equations may be readily solved. It will be only necessary to observe the following particulars.

1. Transposition.—Any term may be taken from one side of an equation and carried over to the other side, provided that, when thus transposed, its *sign* be changed. For by thus transposing a term, the balance, or equality of the two sides of the equation, remains undisturbed: the result is still an *equation*. Suppose, for instance, the term $3ab$ occurs on one side of an equation, and that we wish to remove it from that side without disturbing the equality of the two sides. All we have to do is to *add* $-3ab$ to *both* sides, which addition, it is plain, removes the $3ab$ from the one side, and transfers it, *with changed sign*, to the other. In like manner, if the term $-5x$, standing on either side of the sign of equality, is to be removed, we have only to *add* $5x$ to both sides, which addition merely transposes the $-5x$ from one side to the other, on which other it re-appears with changed sign. And it is plain that if the two sides were equal before the transposition, they must be equal afterwards.

For example, take the equation above; namely, $4x-3=2x+7$. By transposing the 3, we have $4x=2x+7+3$: and by transposing the $2x$, $4x-2x=7+3$, that is, $2x=10$: consequently $x=5$, which is the *solution* of the proposed equation $4x-3=2x+7$, as it is easy to see, for $4x$, that is, 4 times 5, is 20; and $2x$, that is, twice 5, is 10: each side of the equation is therefore 17.

2. Clearing Fractions.—Whenever a fractional term occurs in an equation, we may free the equation from the fraction by simply multiplying *both* sides by the denominator of that fraction. And in this way may fractions be removed one after another: the final result still being an equation. For example: if the equation be $\frac{2x}{3}+4x=7$, by multiplying each side by 3, we convert it into the equation $2x+12x=21$, that is, $14x=21$, so that $x=\frac{21}{14}=\frac{3}{2}$, or $1\frac{1}{2}$.

21. In fact, the principles brought into operation in the solution of a simple equation are all justified by the following axiom; namely, that whether we equally increase or equally diminish, equally multiply or

equally divide the two sides of an equation, the result is still an equation; or, generally—whatever operation we perform on one side of an equation, if we perform the same on the other side, the result must be an equation.

22. **To solve a Simple Equation containing only one unknown Quantity.**—The symbolical representation of an unknown quantity is usually the letter x, or y, or z: the earlier letters of the alphabet are employed, almost exclusively, to represent *known* quantities. A question may be proposed in which it is declared that certain quantities are known, although their actual values may not be specified: in such circumstances, we should represent the known quantities by a, b, or c, etc. The rule for solving a simple equation, in which all the quantities but one are known, is as follows:—

Rule 1. If a fraction occur in the equation, clear it away by multiplying both sides of the equation by the denominator.

2. By transposition, bring all the terms containing the unknown quantity to one side of the equation, placing the known terms on the other, so as to get an equation in which the quantities on one side are *all* unknown, and those on the other, *all* known.

3. Collect the quantities on each side into a single term: there will then be a single unknown quantity on one side of the equation, and a single known quantity on the other.

4. If the x (or the y, or the z, as the case may be) in this unknown quantity, have a co-efficient, other than unity, divide each side by that coefficient: then the x will stand alone on one side of the equation, and a known quantity, which is its interpretation, on the other.

23. Examples.—(1) Find the value of x in the equation $9x-5=3x+19$. As there are here no fractions to be cleared, we commence with the second precept of the rule, and *transpose:* we thus get $9x-3x=19+5$. Collecting the terms, $6x=24$. Dividing by 6, $x=4$: hence the quantity x, at first unknown, is found to be 4: we see that 9 times 4 minus 5 *is* equal to 3 times 4 plus 19, as the equation affirms.

(2) Find the value of x in $4-\frac{5x}{6}+2x=11$. In order to clear the fraction, we multiply by 6, and thus get $24-5x+12x=66$
This, by transposition, becomes $-5x+12x=66-24$
which, collecting the terms, is $7x=42$
so that, dividing by the co-efficient 7, we have finally $x=6$.

(3) $\frac{x}{2}-\frac{x}{3}+5=x-5$

Trans., $\frac{x}{2}-\frac{x}{3}=x-10$

Mult. by 2, $x-\frac{2x}{3}=2x-20$

Mult. by 3, $3x-2x=6x-60$
Trans., $3x-2x-6x=-60$
Collecting, $-5x=-60$
or Trans., $60=5x$
$\div 5$, $12=x$.

(4) $\frac{x-1}{2}+\frac{x}{6}-\frac{x}{4}=0$

Mult. by 2, $x-1+\frac{x}{3}-\frac{x}{2}=0$

by 3, $3x-3+x-\frac{3x}{2}=0$

by 2, $6x-6+2x-3x=0$
Trans., $5x=6$, therefore $x=1\frac{1}{5}$.

(5) $\frac{3(x-2)}{2}+6=3(x-3)$

Mult. by 2, $3(x-2)+12=6(x-3)$
that is, $3x-6+12=6x-18$
Trans., $3x-6x=-18+6-12$
that is, $-3x=-24$
or Trans., $24=3x$
$\div 3$, $8=x$.

24. Examples for Exercise.

(1) $8-3x+12=30-5x+4$.

(2) $4x+3=3(x+4)$.

(3) $2(x-3)=3(6-x)+2$.

(4) $\frac{x+3}{7}-\frac{2(x-1)}{3}+1=0$.

(5) $\frac{2x}{3}+\frac{3x}{4}-\frac{x}{5}=73$.

(6) $x+\frac{x}{2}-\frac{x}{3}=4x-17$.

(7) $\frac{x}{3}-\frac{x}{4}+\frac{x}{5}+\frac{1}{2}=0$.

(8) $ax-b=cx+m$.

(9) $(a+b)x=bx+4$.

(10) $\frac{2(21-y)}{3}=\frac{48-3y}{2}$.

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25. Solution of Questions by Simple Equations with one unknown.—Questions implying but one unknown quantity are solved by Algebra by representing that unknown quantity by a letter, as x, by then expressing the conditions of the question in the form of an *equation*, and solving that equation as in the foregoing examples.

Note.—To avoid the frequent repetition of the word *therefore*, in the steps of an algebraic operation, the symbol $\therefore$ is used to stand for it: this symbol reversed (namely $\because$) is frequently used for the word *because*.

Examples.—(1) What number is that whose third part exceeds its fourth part by 8?

Let x be the number, then by the conditions of the question,

$$\frac{x}{3}-\frac{x}{4}=8 \therefore \text{Mult. by 3, } x-\frac{3x}{4}=24.$$

And, mult. by 4, $4x-3x=96$, that is, $x=96$. The third part of this is 32, and the fourth part 24.

(2) Divide the number 48 into two parts, such that one part may be three-fourths of the other.

Let x be one part, then $\frac{3x}{4}$ is the other; so that

$$x+\frac{3x}{4}=48. \quad \text{Mult. by 4, } 4x+3x=192 \therefore 7x=192 \therefore x=27\tfrac{3}{7}.$$

Hence one part is $27\frac{3}{7}$, and the other is therefore $48-27\frac{3}{7}=20\frac{4}{7}$.

(3) If A can finish a piece of work in 8 days, and B can do it in 10 days, in what time will they finish it both working together?

Suppose in x days. Now the part of the *whole*, done in 1 day, is, by the question, $\frac{1}{8}+\frac{1}{10}$, that is, $\frac{9}{40}$: hence, in x days, $\frac{9x}{40}$ is done: but this is 1 whole.

$$\therefore \frac{9x}{40}=1. \quad \text{Mult. by 40, } 9x=40 \therefore x=4\tfrac{4}{9} \text{ days.}$$

(4) A vessel can be filled by four taps, running separately, in 2 hours, 3 hours, 4 hours, and 5 hours: in what time will it be filled if all run together?

Let x be the number of hours. The part of the whole filled in 1 hour is $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ or $\frac{77}{60}$. Consequently the part filled in x hours is $\frac{77x}{60}$. But this is 1 whole.

$$\therefore \frac{77x}{60}=1 \therefore 77x=60 \therefore x=\frac{60}{77} \text{ hours.}$$

Hence the vessel will be filled in the $\frac{60}{77}$ part of an hour.

(5) A child being asked the time by the clock, answered that the hands were both together between 5 and 6: required the exact time.

Suppose the hands are at x minute-spaces past the 5, then, while the short hand has moved over these x minute-spaces, the long hand has moved up to it from the 12, and therefore over $25+x$ such spaces. But the long hand moves 12 times as fast as the short hand,

$$\therefore 12x=25+x \therefore 11x=25 \therefore x=2\tfrac{3}{11} \therefore 25+x=27\tfrac{3}{11},$$

the number of minutes past 5 o'clock.

(6) What number is that of which the fifth part exceeds the sixth part by 7?

(7) What number is that which when increased by one-half and two-fifths of itself, the sum may be 76?

(8) Divide the number 30 into two parts, such that one part may exceed the other by 13.

(9) Divide £100 among A, B, and C, so that A may have £20 more than B, and B £10 more than C.

(10) From two towns, 144 miles apart, two persons set out at the same time to meet each other; one travels 16 miles a day, and the other 20: in how many days will they meet?

(11) A can finish a piece of work in 15 days, which B can do in 12 days: in what time can they finish it, both working together?

(12) A railway-train leaves London at 12 o'clock to run to York, a distance of 200 miles; another leaves York at the same time for London: the former train goes at the rate of 25 miles an hour, and the latter, at the rate of 35 miles an hour. At what o'clock will they pass each other?

(13) A detachment of soldiers march from a certain place at the rate of $2\frac{1}{2}$ miles an hour, and two hours afterwards another detachment, marching at the rate of $3\frac{1}{2}$ miles an hour, is sent to overtake them: how far must they march to do so?

(14) A garrison of 1000 men is victualled for 25 days; but after 9 days 250 men are withdrawn: how long will the provisions last those who remain?

(15) Out of a cask of wine, which had lost $\frac{1}{4}$ of the whole by leakage, 30 gallons were drawn: it was then found to be half full: how much did it hold?

(16) How much tea at 4*s*. 6*d*. per lb., must be mixed with 50 lb. at 6*s*. per lb., so that the mixture may be worth 5*s*. per lb.?

(17) A vessel holding 120 gallons is partly filled by a spout which delivers 14 gallons in a minute; this is then turned off, and from a second spout, delivering 9 gallons in a minute, the filling of the vessel is completed: the whole time occupied is 10 minutes: how long did each spout run?

(18) A market woman had a certain number of apples at 2 a penny, and as many more at 3 a penny; but resolved to sell the whole at the rate of 5 for twopence; and, contrary to her expectation, found she lost 3*d*. How many apples had she?

(19) A person pays a bill of £2 16*s*. 6*d*. with half-crowns and shillings; the number of pieces was 40: how many were there of each?

(20) A person has three debtors, A, B, C, whose debts he only so far remembers that A'*s* and B'*s* together amount to £60, A'*s* and C'*s* to £80, and B'*s* and C'*s* to £92; required the debt of each?

(Additional examples in simple equations will be given further on).

26. **Definitions.**—*Powers.* The product of any number of *equal* factors is called a *power* of the factor thus repeated. It is the *second* power when produced from two equal factors, the *third* power when produced from *three* equal factors, and so on. For example, since $2\times2=4$, 4 is the second power or the *square* of 2, and since $2\times2\times2=8$, 8 is the third power or *cube* of 2. In like manner, $2\times2\times2\times2$ or 16 is the fourth power of 2, 32 is the fifth power of 2, and so on. Similarly with letters, aa is the square or second power of a, aaa is the cube or third power, $aaaa$ is the fourth power of a, and so on.

27. *Exponents* or *Indices.* The inconvenience of thus repeating the equal factors in expressing powers is avoided by writing the factor only *once*, and marking the *number* of factors—that is, the power intended—by a small figure placed over the right-hand corner: thus, $2^2=4$, $2^3=8$, $2^4=16$, $2^5=32$; a^2, a^3, a^4, a^5, &c,

These small corner figures are called *exponents*, or *indices*. Conformably to this notation, the ninth power of x is x^9.

28. *Roots.* The number or quantity which, by repeated multiplication by itself, produces a power, is called a *root* of that power; thus, 2 is the square root of 4, 3 is the cube root of 27, 2 is the fourth root of 16, &c. And generally any quantity of which the square, cube, fourth power, &c., produces another quantity, is the square root, cube root, fourth root, &c., of that other quantity. The notation for roots is similar to that for powers: thus, $9^{\frac{1}{2}}$, $27^{\frac{1}{3}}$, $16^{\frac{1}{4}}$, &c., denote respectively the square root of 9 or 3, the cube root of 27, which is also 3, the fourth root of 16, which is 2, &c. But there is another way of indicating roots, namely, by means of the *radical sign*, $\surd$, thus: $9^{\frac{1}{2}}$, $27^{\frac{1}{3}}$, $16^{\frac{1}{4}}$, $a^{\frac{1}{5}}$, $x^{\frac{1}{7}}$, &c., are the same as $\surd 9$, $\sqrt[3]{27}$, $\sqrt[4]{16}$, $\sqrt[5]{a}$, $\sqrt[7]{x}$, &c., where for the *square root*, the small index-figure, 2, is suppressed.

29. To indicate powers or roots of *compound* quantities, the several simple terms must be inclosed in brackets, or united by a vinculum: thus the square of the expression ax^3-bx^2+c would be indicated in one or other of the following ways, namely:—

$$(ax^3-bx^2+c)^2,\ \text{or}\ [ax^3-bx^2+c]^2,\ \text{or}\ \{ax^3-bx^2+c\}^2,\ \text{or}\ \overline{ax^3-bx^2+c}^2$$

and the square root by—

$$(ax^3-bx^2+c)^{\frac{1}{2}},\ \text{or}\ [ax^3-bx^2+c]^{\frac{1}{2}},\ \text{or}\ \{ax^3-bx^2+c\}^{\frac{1}{2}},\ \text{or}\ \overline{ax^3-bx^2+c}^{\frac{1}{2}}$$

$$\text{Or}\ \surd(ax^3-bx^2+c),\ \surd[ax^3-bx^2+c],\ \surd\{ax^3-bx^2+c\},\ \surd\overline{ax^3-bx^2+c}$$

30. The operations of addition and subtraction, hitherto applied to simple quantities only, may be easily extended to more complicated expressions, such as these. So long as the expressions are *like*, they may be just as readily incorporated, whether they are simple or compound: it has been sufficiently seen that it is with the *coefficients* only of these like quantities that we deal. Any multiplier prefixed to an expression is the coefficient of that expression; and this term *coefficient* is extended to mean the prefixed multiplier, whether it be *numeral* or *literal*. Thus in $3(x^2-2y+z)$, $a(x^2-2y+z)$, $(a+b)(x^2-2y+z)$, $(2a-5)(x^2-2y+z)$, the multipliers 3, a, $a+b$, $2a-5$, are the respective coefficients of (x^2-2y+z). If the above four quantities are to be added together, then proceeding with these coefficients, as in the margin, we should find the sum to be $(4a+b-2)(x^2-2y+z)$.

3
a
$a+b$
$2a-5$
$4a+b-2$

EXAMPLES.—Add together the following expressions:—

1. $7(x+y)$, $(a+b)(x+y)$, $(2a-b)(x+y)$, $(3b-6)(x+y)$.
2. $4-2\surd(x-a)$, $5\surd\overline{x-a}+1$, $3(x-a)^{\frac{1}{2}}-6$, $2-[x-a]^{\frac{1}{2}}$.
3. $\sqrt[3]{(x-y)}-\surd(x+y)$, $a\surd(x+y)+5\sqrt[3]{(x-y)}$, $(3a+2)\sqrt[3]{(x-y)}-2(x+y)^{\frac{1}{3}}$.
4. Subtract $3z+4x^2-2y^2+b$ from $2x^2-5y^2+9z+a$.
5. Subtract $7(x-y)-(a-b)xy+6z$ from $3a(x-y)-4xy+2z$.
6. Subtract $-3(x+y)+b(x^2-y^2)-(b+c)z$ from $6(x+y)-a(x^2-y^2)-(b+c)z$.
7. Subtract $-6a\surd y+3b\surd x+a+d$ from $4ax^{\frac{1}{2}}-5by^{\frac{1}{2}}+a-b$.

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31. **Multiplication.**—In writing down the product of two algebraic factors *three* things must be attended to: first the *sign*, then the *coefficient*, and, lastly, the *letters*.

If the signs of the factors be *like*, that is, both *plus* or both *minus*, the sign of the product is PLUS.

If the signs of the factors are *unlike*, the sign of the product is MINUS. This may be shown as follows:—

Take any two numbers at random for factors—say 8 and 3—then all possible varieties, as respects the signs of these factors, will be comprehended in the four following,—

8	− 8	8	− 8
3	3	− 3	− 3
24	−24	−24	24

The first, being the case of common arithmetic, requires no remark. In the second case the product must be −24, because multiplication by 3 means repeating the thing multiplied three times. To discover the correct product in the third case, we may proceed thus: add 4 to the multiplier; then it is plain, that by operating with this altered multiplier, the product we shall get will be 8×4, or 32 too great.

The altered multiplier will be $-3+4$, that is 1; and $8\times1=8$. Subtracting, then, the 32, we have $8-32=-24$ for the correct product.

In like manner in the fourth case, the increased multiplier being $-3+4=1$, and the product by it $-8\times1=-8$, we have by subtracting -8×4 or −32, 24 for the correct product.*

I. *When the factors are simple quantities.*—RULE. 1. If the signs of

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* The proof is the same when letters are used instead of numbers; thus:—

b	$-b$	b	$-b$
a	a	$-a$	$-a$
ab	$-ab$	$-ab$	ab

The first and second results are obviously true. Taking the third, add $a+1$ to the multiplier, then to correct the product, we must subtract from it $a+1$ times b, that is, a times b and *once* b, which is $ab+b$. Now the product from the new multiplier is once b or b; and subtracting $ab+b$, the result is $-ab$. Increasing the last multiplier in the same way, the product is $-b$, and subtracting $a+1$ times $-b$, that is, $-ab-b$ from this, the result is ab. To explain the meaning of an isolated negative quantity, the learner may be reminded that + and − stand in opposite relation to each other. If +£4 represent a *gain*, then −£4 represents a *loss*. If $+4^0$ express the elevation of an object *above* the horizon, then -4^0 expresses its depression *below* the horizon; if $+a$ denote time *after* any epoch, then $-a$ denotes the same time *before* that epoch; and so on.

the two factors are like, put + for the sign of the product; if they are unlike, put −.

2. After the sign write the product of the coefficients.

3. To this product annex the product of the letters, that is, write the letters in both factors one after another, without any intervening sign.

NOTE.—The letters of the product are usually arranged in the order in which they follow in the alphabet; but when the same letter recurs, the notation for *powers* is used; thus, instead of aaa, we write a^3, instead of x^2x^3, or which is the same, $xx \times xxx$, we write x^5, the *sum* of the exponents in the factors being the exponent in the product: thus, generally, $x^m \times x^n = x^{m+n}$, whatever integers m and n may be.

	(1)	(2)	(3)	(4)
Multiply	$5ax$	$-8x^2y$	$7abx^3$	$-6ab^2x^2y^4$
by	$3ab$	$2ay$	$-4ax^2$	$-4a^3xy^2$
Product	$15a^2bx$	$-16ax^2y^2$	$-28a^2bx^5$	$24a^4b^2x^3y^6$

(5) $9cx^4y^6 \times -2acxy - = -18ac^2x^5y^7$. (6) $-\frac{1}{4}x \surd y \times \frac{1}{3}ax^3 = -\frac{1}{12}ax^4 \surd y$. (7) $-abxy^{\frac{1}{3}} \times -5a^2b^3x^2 = 5a^3b^4x^3y^{\frac{1}{3}}$. (8) $8abx^3 \times -7a^7b^2x = -56a^4b^3x^4$. (9) $-4x^{\frac{1}{2}}y^{\frac{1}{2}} \times -3ax^{\frac{1}{2}}y^{\frac{1}{2}} = 12axy$.

EXAMPLES FOR EXERCISE.

(1) $3a^2x \times 6a^3bxy$. (2) $-4ay^3 \times 7ab^2xy^4$. (3) $-5ab^2xy^4 \times -3a^2bxy$. (4) $7axy \times 2a^2xy^2 \times -bxy$. (5) $-bxy^2 \times 3ab^2y^3 \times -2y^2 \times 5$. (6) $-\frac{1}{2}ax^2y \times -\frac{3}{4}by^2 \times -2xyz$.

II. *When one of the factors is a Compound quantity.*—RULE. Multiply *each* term of the compound quantity by the simple factor, commencing the operation on the left hand; the several partial products, connected together with their proper signs, will be the complete product.

	(1)	(2)
Multiply	$2ax^2-3by$	$6a^2x^3-4bx^2y^2-3cz^4$
by	$4a^2x^3$	$-4axy$
Product	$8a^3x^5-12a^2bx^3y$	$-24a^3x^4y+16abx^3y^3+12acxyz^4$.

EXAMPLES FOR EXERCISE.

(1) $(4abx^2-2xy^2) \times 3a^2bx$.
(2) $(6ax^2y^2+13bz^3-5) \times -2abxz$.
(3) $(2ax-3by+4cz) \times ax \times -bc$.
(4) $\{ax+(2by+3)\} \times 4axy$.
(5) $\frac{3}{4}ax^2\{4bx^3-(3ax+1)\}$.
(6) $(6ab^2x-5cy) \times -\frac{1}{2}ax \times \frac{1}{3}by$.

III. *When both factors are compound quantities.*—RULE 1. Multiply every term of the multiplicand by *each* term of the multiplier. 2. Arrange the several *like* products one under another, and find their sum by Addition—

	(1)	(2)	(3)
Multiply	$a+b$	$a-b$	$a+b$
by	$a+b$	$a-b$	$a-b$
	a^2+ab	a^2-ab	a^2+ab
	$ab+b^2$	$-ab+b^2$	$-ab-b^2$
Product	$a^2+2ab+b^2$	$a^2-2ab+b^2$	a^2-b^2.

Hence, whatever be a and b, we see that $(a+b)^2=a^2+2ab+b^2$, $(a-b)^2=a^2-2ab+b^2$, and $(a+b)(a-b)=a^2-b^2$, general truths which should be borne in remembrance. They show that the square of the *sum* of any two quantities is equal to the sum of the squares of the quantities themselves, *plus* twice their product; that the square of the *difference* of any two quantities is equal to the sum of the squares of those quantities, *minus* twice their product; and that the sum of two quantities multiplied by their difference is equal to the difference of their squares, and, *consequently*, that if the difference of the squares of two quantities be divided by the sum of those quantities, the quotient must be the difference of the quantities; and if divided by the difference, the quotient must be the sum.

(4)

$$\begin{array}{l} x^2+xy+y^2 \\ x-y \\ \hline x^3+x^2y+xy^2 \\ \quad -x^2y-xy^2-y^3 \\ \hline x^3-y^3. \end{array}$$

(5)

$$\begin{array}{l} 2x^2-3x \\ 4x-5 \\ \hline 8x^3-12x^2 \\ \quad -10x^2+15x \\ \hline 8x^3-22x^2+15x. \end{array}$$

(6)

$$\begin{array}{l} ax^2+bx+c \\ 2x-b \\ \hline 2ax^3+2bx^2+2cx \\ \quad -abx^2-b^2x-bc \\ \hline 2ax^3+(2b-ab)x^2+(2c-b^2)x-bc. \end{array}$$

The product in Ex. 6 may be written thus: $2ax^3+(2-a)bx^2+(2c-b^2)x-bc$.

(7)

$$\begin{array}{l} x^2-xy+y^3 \\ x+y \\ \hline x^3-x^2y+xy^2 \\ \quad x^2y-xy^2+y^3 \\ \hline x^3+y^3. \end{array}$$

(8)

$$\begin{array}{l} 12ay-5ab+3 \\ ay-2ab+2 \\ \hline 12a^2y^2-5a^2by+3ay \\ \quad -24a^2by+10a^2b^2-6ab \\ \quad 24ay-10ab+6 \\ \hline 12a^2y^2-29a^2by+27ay+10a^2b^2-16ab+6. \end{array}$$

When, as in examples (1) and (2) above, the factors multiplied together are all alike, the operation is usually called INVOLUTION: the result of the involution is, of course, a *power* of the quantity thus multiplied into itself. Although special examples in involution are not absolutely necessary, we shall here exhibit two for the sake of the inferences to be drawn from them, and which will be made use of hereafter.

(8) Required the cube of $a+b$.

$$\begin{array}{l} (a+b)^2=a^2+2ab+b^2 \\ \quad a+b \\ \hline a^3+2a^2b+ab^2 \\ \quad a^2b+2ab^2+b^3 \\ \hline (a+b)^3=a^3+3a^2b+3ab^2+b^3 \end{array}$$

(9) Required the cube of $a-b$.

$$\begin{array}{l} (a-b)^2=a^2-2ab+b^2 \\ \quad a-b \\ \hline a^3-2a^2b+ab^2 \\ \quad -a^2b+2ab^2-b^3 \\ \hline (a-b)^3=a^3-3a^2b+3ab^2-b^3 \end{array}$$

These two results may evidently be written as follows; namely,—

$$(a+b)^3=a^3+b^3+3(a+b)ab, \text{ or } a^3+\{3a^2+3ab+b^2\}b.$$
$$(a-b)^3=a^3-b^3-3(a-b)ab, \text{ or } a^3-\{3a^2-3ab+b^2\}b.$$

and in these forms it will be advisable for the learner to recollect them; the symbols a, b, may of course be replaced by any quantities whatever.

Examples for Exercise.

(1) Multiply $x^3+2ax+a^2$ by $x^3-2ax+a^2$

(2) $(x^4+x^2+1)(x^2-1)=$

(3) $(x^4+a^2x^2+a^4)(x^2-a^2)=$

(4) $(x+1)(x^2-\frac{1}{4}x+1)=$

(5) $(3x-4)(2x^2-5x+2)=$

(6) $(x-y)(x^3+x^2y+xy^2+y^3)=$

(7) $(x^2-2x+1)(2x^2-\frac{1}{2}x+2)=$

(8) $(3x-2)(2x+3)(3ax-1)=$

(9) $(ax^m+bx^n)(bx^m+ax^n)=$

(10) $(\frac{1}{3}x+2)(x^2-\frac{1}{2}x+\frac{2}{3})=$

(11) $(x^2-\frac{1}{2}x)(x^2-\frac{2}{3}x+1)=$

(12) $(ax+by)(ax-by)=$ [See page 15.]

(13) $(3ax-7by)^2=$ [See page 15.]

32. **Division.**—As in multiplication, *three* things are to be attended to: the *sign* of the quotient, the *coefficient* of the letters, and the *letters* themselves. Whether in multiplication or in division, *like* signs give *plus*, and *unlike* signs *minus*.

I. *When dividend and divisor are both simple quantities.*—Rule 1. Write the sign of the quotient. 2. Then the quotient of the co-efficients. 3. And finally, the quotient of the letters:—these being such that, when joined to the letters of the divisor, they make up those of the dividend.

Note.—Should letters appear in the divisor which are not found also in the dividend, the division by *them* cannot of course be actually performed; and, as in arithmetic, after the quotient by the other quantities is obtained, the unemployed factors of the divisor must be written under it, to imply a further division which has not been executed.

(1) Divide $-12a^3x^4y^2$ by $3ax^2yz$. Here the coefficient of the quotient is -4, and to this we are to annex such letters, that when they are united to (multiplied by) those in the divisor, they may make up the letters of the dividend. The factor z in the divisor, not occurring in the dividend, must be reserved: omitting this, we see that the letters wanting in the divisor to make up those in the dividend are a^2x^2y; hence the quotient is $\frac{-4a^2x^2y}{z}$, that is to say, $\frac{-12a^3x^4y^2}{3ax^2yz}=\frac{-4a^2x^2y}{z}$, or $-\frac{4a^2x^2y}{z}$.

(2) $18a^6b^4x^5y\div-9a^3by^2=-\frac{2a^3b^3x^3}{y}$. (3) $-14ab^5x^3y^2\div-2b^4xy=7abx^2y$.

(4) $\frac{8a^7b^5c^6x^2y^2}{5a^2b^3c^2x^4y^5}=\frac{8a^5b^2c^4}{5x^2y^3}$.

Note.—As these illustrations sufficiently show, when powers of the same letter enter both dividend and divisor, the quotient, so far as that letter is concerned, is got by simply *subtracting the lower exponent from the upper:* the remainder is the exponent to be given to that letter in the quotient. In the fourth example, for instance, $\frac{a^4}{a^2}=a^2$, $\frac{b^5}{b^3}=b^2$, $\frac{c^6}{c^2}=c^4$. But when the lower exponent *exceeds* the upper, then only the part of the lower exponent, equal to the upper, is subtracted, and the lower letter, with its exponent thus diminished, is still retained as a divisor; since thus much of the division is still unperformed. In the above example, for instance, $\frac{x^2}{x^4}=\frac{1}{x^2}$, $\frac{y^2}{y^5}=\frac{1}{y^3}$; the learner thus perceives that when only powers of the same letter are concerned, multiplication merely implies the addition of the exponents, and division the subtraction of them.

Examples for Exercise.

(1) Divide $-15a^3x^3y^4$ by $5a^2xy^3$.

(2) $18a^3b^2x^2y^3 \div -6ab^2xy =$.

(3) $-27ax^3y^5\sqrt{z} \div -3axy^2 =$.

(4) $2ab^3x^4y^3 \div a^2bx^2y^3 =$.

(5) $-21a^5b^4c^3x^6y \div 9a^2b^5c^4x^3 =$.

(6) $\dfrac{14a^3bx^4y^3}{-8ab^2x^3y^4} =$.

(7) $\dfrac{6a^{2n}b^{2m}x^4}{2a^nb^mx^3} =$.

(8) $\dfrac{2a^6bx^5\sqrt{z}}{\frac{1}{2}a^2b^4x\sqrt{z}} =$.

II. *When the dividend is a compound quantity, and the divisor a simple quantity.*—Rule. Find the quotient of the divisor and *each* term of the dividend, as in last case, and connect these several quotients together by their proper signs.

(1) $\dfrac{8a^2x^3y^5-4x^3y^4}{2x^2y^3} = 4a^2xy^2-2xy$. (2) $\dfrac{4ax^2z^3-8a^2xz^2-4xz}{4xz^2} = axz-2a^2-\dfrac{1}{z}$.

Examples for Exercise.

(1) $a^4x^2-3abx^3+5ax^4 \div ax^2$. (2) $\dfrac{36x^2-9ax^3+12}{3x}$. (3) $-12abx^2z+9b^2y^3z-6bz^5 \div -3bz$.

(4) $3\{2x-(16+x)+1\} \div 5$. (5) $-24a^2x^2y-3axy+6x^2y^3 \div -3xy$.

(6) $\frac{1}{4}abc\sqrt{x} \div \frac{1}{2}a^2b^3cx^{\frac{1}{2}}$.

III. *When dividend and divisor are both compound quantities.*—Rule. 1. Arrange the terms both of dividend and divisor, so that the *powers* of some letter common to both may follow each other in ascending or descending order.

2. Divide the first term of the dividend by that of the divisor; the result will be the first term of the quotient. Multiply the whole divisor by this first term, subtract the product from the like terms of the dividend, and to the remainder annex a new term of the dividend, or two terms if necessary; regarding the result as a new dividend.

3. Proceed with the divisor and this dividend as at first, and continue the operation till all the terms in the original dividend have been brought down. If there be a final remainder, it must be written, with the divisor underneath, and annexed to the quotient, as in arithmetic.

Note.—When divisor and dividend are seen to have a factor in common, this common factor may be cancelled from both before proceeding with the operation.

(1) Divide $12x^5-13x^4-34x^3+40x^2$ by $4x^2-7x$. Here it is seen that the factor x is common to both quantities: expunging this common factor, therefore, we proceed by the rule as follows, the terms being already arranged according to the descending powers of x:—

$$4x-7)12x^4-13x^3-34x^2+40x(3x^3+2x^2-5x+\frac{5x}{4x-7}$$

$$\begin{array}{l} 12x^4-21x^3 \\ \hline 8x^3-34x^2 \\ 8x^3-14x^2 \\ \hline -20x^2+40x \\ -20x^2+35x \\ \hline \text{Remainder} \quad 5x \end{array}$$

(2) Divide $4a^3x+51a^2x^2+10x^4-48ax^3-15a^4$ by $4ax-5x^2+3a^2$.

Arranging the terms in order, according to the powers of x, the operation will be as follows:—

$$\begin{array}{l}
-5x^2+4ax+3a^2)10x^4-48ax^3+51a^2x^2+4a^3x-15a^4(-2x^2+8ax-5a^2 \\
\quad 10x^4-\ 8ax^3-\ 6a^2x^2 \\
\quad\quad\quad

And in the case before us, the Remainder, being free from x, must continue unaltered whatever value we put for x in this equation. Let a be put for x, if the divisor be $x-a$, or $-a$ if it be $x+a$; then the above equation will be

$$\text{Quotient} \times 0 + \text{Remainder} = \text{Dividend}.$$

But any finite quantity multiplied by *nothing* produces nothing; hence the equation last written is simply Remainder=Dividend; that is, when the proposed substitution for x is made in the dividend, the changed expression is the same as the remainder. The following are illustrations of the application of this theorem:—(1) Is $x^4-3x^3-5x-14$ divisble by $x+2$? Putting -2 for x, the dividend becomes $16-12+10-14$, which is equal to 0. $\therefore$ Remainder $=0$, $\therefore$ the expression *is* divisible by $x+2$. (2) Is x^3-2x^2+3x-4 divisible by $x-2$? Putting 2 for x in the dividend, we have $8-8+6-4=2$, the rem. Hence the proposed expression is not divisible by $x-2$; the division leaves a remainder 2. The following also are useful inferences:—

Whenever n is *odd*
- x^n-a^n is divisible by $x-a$, but not by $x+a$
- x^n+a^n ,, $x+a$, but not by $x-a$

Whenever n is *even*
- x^n-a^n ,, both $x-a$ and $x+a$
- x^n+a^n ,, neither $x-a$ nor $x+a$

By the principle above, the remainder, in this last case, is $2a^n$.

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34. Algebraic Fractions.—Fractions in Algebra are treated exactly in the same way as those in Arithmetic: whether the symbols employed be figures or letters the rules of operation are just the same. This the learner will be prepared to expect, because algebra becomes converted into arithmetic so soon as the letters are replaced by numbers. The *Rules* therefore need here be but very briefly recapitulated.

Reduction of a mixed quantity to an improper (or single) fraction.—Rule. Multiply the integral part by the denom. of the fractional part: connect the product, by the proper sign, to the num., and write the denom. underneath: thus—

$$(1)\ a+\frac{3x}{b}=\frac{ab+3x}{b}. \quad (2)\ a+x-\frac{2b}{y}=\frac{(a+x)\,y-2b}{y}. \quad (3)\ \sqrt{x}+\frac{a-x}{\sqrt{x}}=\frac{x+a-x}{\sqrt{x}}=\frac{a}{\sqrt{x}}.$$

$$(4)\ \frac{a^2+b^2}{ab}-2=\frac{a^2-2ab+b^2}{ab}=\frac{(a-b)^2}{ab}.$$

Examples for Exercise.—Reduce each of the following mixed quantities to an improper fraction:—

$$(1)\ ax+\frac{b}{y}. \quad (2)\ 1-\frac{2a}{x}. \quad (3)\ 1-\frac{a-b}{a+b}. \quad (4)\ \frac{a^2+b^2}{ab}+2. \quad (5)\ x^2-3x+2+\frac{2x^2-5}{x-2}.$$

$$(6)\ x-a+\frac{x^2+a^2}{x+a}. \quad (7)\ x^2-a^2-\frac{x^4+a^4}{x^2+a^2}. \quad (8)\ \frac{(a+b+c)\,(a+b-c)}{2ab}-1.$$

Reduction of an improper fraction to a whole or mixed quantity.—Rule. Divide num. by den. If there be a remainder, annex it, with the denom. underneath, to the quotient, as in all cases of division.

$$(1)\ \frac{ab+3x}{b}=a+\frac{3x}{b}. \quad (2)\ \frac{(a+x)\,y-2b}{y}=a+x-\frac{b}{y}.$$

$$(3)\ \frac{4ax^2-8ax+1}{2ax}=2x-4+\frac{1}{2ax}=2\,(x-2)+\frac{1}{2ax}.$$

EXAMPLES FOR EXERCISE.—Reduce each of the following to a whole or mixed quantity:—

(1) $\frac{(a+b)^2}{a}$. (2) $\frac{x^2-y^2+3}{x-y}$. (3) $\frac{a^2}{a+b}$. (4) $\frac{9ax^2+6ax+2}{3ax}$. (5) $\frac{x^4-8x+7}{x^2-3x+2}$.

(6) $\frac{(a+x)^2-(a-x)^2}{4ax}$. (7) $\frac{12x^5-13x^4-14x^3+8x^2+1}{4x^2-7x}$.

Reduction of fractions to a common denominator.—RULE. 1. Multiply each numerator by the product of all the denominators except its own, the several new numerators will thus be obtained. 2. Multiply *all* the denominators together, the product will be the common denominator.

That the *values* of the fractions remain unaltered by this change in their forms is obvious; for, take whichever of the original fractions we may, we see that the change is effected by multiplying its num. and den. by the same thing; namely, the product of the denominators of all the *other* fractions. The common denom. found as above is, of course, a common multiple of all the original denominators: if it be not the *least* common multiple, the changed fractions will not be in their most simple forms. By glancing at the original denominators, the least common multiple of them may often be readily found:—if common factors enter two or more of the denominators, they should be retained in but one:—the repetitions should be struck out; the product of the denominators, thus deprived of common factors, will be the *L.C.M.* (least common multiple). As to the numerators, we have only to multiply each by the quotient arising from dividing the *L.C.M.* by the corresponding denominator, so that, as before, the values of the changed fractions remain undisturbed.

The object of this reduction of fractions to a com. den. is merely to fit them for addition and subtraction, the rules for these operations being as follows.

Addition.—RULE. Reduce the fractions to a common den., and write this com. den. under the sum of the changed numerators.

Subtraction.—RULE. Write the com. den. under the difference of the changed numerators.

(1) Required the sum of $\frac{2}{3ax}, \frac{b}{2x}, \frac{c}{4}$. Applying the rule, the new numerators are $2\times 8x$, $b\times 12ax$, $c\times 6ax^2$, and the com. den. is $3ax\times 2x\times 4$: hence the changed fractions are $\frac{16x}{24ax^2}, \frac{12abx}{24ax^2}, \frac{6acx^2}{24ax^2}$, in which we see that $2x$ is a superfluous factor in the num. and den. of each. We might have foreseen this by running the eye along the row of denominators:—there is a repetition of the factor x and of 2: suppressing, therefore, one x and one 2, we have $3ax\times 2\times 2=12ax$ for the $L.C.M.$ of the denominators. Hence multiplying the numerators in order by 4, $6a$, $3ax$, and adding the results, we have $\frac{8+6ab+3acx}{12ax}$ for the sum.

(2) $\frac{2x+3}{4}+\frac{x}{5}+\frac{1}{3}$. Here the denominators have no com. factor; hence by the rule, $\frac{30x+45+12x+20}{60}=\frac{42x+65}{60}$ is the sum.

(3) $\frac{2x-3}{4}-\frac{3x+4}{7}+\frac{5-2x}{28}$. Here 28 is the $L.C.M.$ of the denominators.

$\therefore \frac{14x-21-12x-16+5-2x}{28}=-\frac{32}{28}=-\frac{8}{7}$, the sum.

(4) $\frac{x-y}{x+y}+\frac{x^2+y^2}{x^2-y^2}=\frac{(x-y)(x-y)}{(x+y)(x-y)}+\frac{x^2+y^2}{x^2-y^2}=\frac{(x-y)^2+x^2+y^2}{x^2-y^2}=\frac{2(x^2-xy+y^2)}{x^2-y^2}$.

EXAMPLES FOR EXERCISE.

(1) $\frac{a+x}{3}+\frac{a-x}{2}=$.

(2) $3x+\frac{2x-5}{7}-\frac{1}{14}=$.

(3) $\frac{a+b}{a-b}+\frac{a-b}{a+b}+2=$.

(4) $\frac{3+2x}{1-x^2}-\frac{2}{1-x}=$.

(5) $\frac{a+x}{a^2-x^2}+\frac{a-x}{a+x}-\frac{1}{a-x}=$.

(6) Prove that $\frac{x}{x+y}+\frac{y}{x-y}=\frac{x}{x-y}-\frac{y}{x+y}$.

(7) Prove that $\frac{x}{x+z}-\frac{y}{y+z}=\frac{z}{y+z}-\frac{z}{x+z}$.

Multiplication and Division.—For MULTIPLICATION.—RULE. 1. Multiply the numerators together: the result will be the num. of the product. 2. Multiply the denominators together: the result will be the den. of the product.

For DIVISION.—RULE. Invert the terms of the *divisor;* that is, turn the fraction upside down, and then proceed as for multiplication. The truth of these rules may be proved thus:—Let $\frac{a}{b}$, $\frac{c}{d}$ be two fractions, which represent by x, y respectively: then

$$x=\frac{a}{b}, \text{ and } y=\frac{c}{d} \therefore bx=a, \text{ and } dy=c \therefore bdxy=ac$$

$$\therefore xy=\frac{ac}{bd}, \text{ that is, } \frac{a}{b}\times\frac{c}{d}=\frac{a}{bd}.$$

Again: resuming the equations $bx=a$, and $dy=c$, we have

$$\frac{bx}{dy}=\frac{a}{c} \therefore \frac{x}{y}=\frac{ad}{bc}; \text{ that is, } \frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}.$$

These results suggest the foregoing rules.

NOTE.—Before performing either operation, simplify the fractions by cancelling whatever factors may be seen to be common to both num. and den. of either. If the num. of one fraction and the den. of another have a factor in common, then, before multiplying those fractions, the common factor may be suppressed.

(1) $\frac{5x}{3a}\times\frac{2x}{a}\times\frac{1}{7}=\frac{10x^2}{21a^2}$. (2) $\frac{3x}{4a^2}\times\frac{2y}{5}\times\frac{2a}{7}=\frac{3x\times y\times 1}{a\times 5\times 7}=\frac{3xy}{35a}$.

(3) $\frac{a+b}{2a+c}\div\frac{b}{2a}=\frac{a+b}{2a+c}\times\frac{2a}{b}=\frac{2a(a+b)}{b(2a+c)}$.

(4) $\left(3a-\frac{3b^2}{a}\right)\div\left(\frac{a^2}{b}-a\right)=\frac{3(a^2-b^2)}{a}\div\frac{a(a-b)}{b}=\frac{3(a^2-b^2)}{a}\times\frac{b}{a(a-b)}=\frac{3b(a+b)}{a^2}$.

EXAMPLES FOR EXERCISE.

(1) $\frac{2x-3}{5}\times\frac{3x}{4y}=$.

(2) $\frac{6ax^2}{5b^2y}\times\frac{2cx}{3ay}\times b^2=$.

(3) $\frac{2\sqrt{x}}{a}\times\frac{a}{\sqrt{x}}=$.

(4) $\frac{a^2-x^2}{b^2-y^2}\div\frac{a+x}{b+y}=$.

(5) $\frac{x^2-a^2}{x+2a}\div\frac{x-a}{4x+8a}=$.

(6) $\frac{x^2-y^2}{x}\times\frac{x}{x+y}\times\frac{1}{x-y}=$.

(7) $12 \div \left\{\frac{(a+x)^2}{x} - 2a\right\} =$.

(8) $\left(\frac{1}{x}+x\right)\left(\frac{1}{y}-y\right) =$.

(9) $\frac{x}{x^2+y^2} \div \frac{(x-y)^2}{x^4-y^4} =$.

(10) $\left(5-\frac{x}{4}\right) \div \left(x+\frac{4}{3}\right) =$.

(11) $\frac{3\sqrt{(a+x)}}{7} \div \frac{4(a+x)}{5} =$.

(12) $\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{a+b}{a}+\frac{a-b}{b}\right) \div \left(\frac{a}{b^2}-\frac{b}{a^2}\right)\left(\frac{1}{a}+\frac{1}{b}\right) =$.

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35. **Greatest Common Measure.**—A common measure of two or more quantities is any expression that will divide them all; and it is the *greatest* common measure when it is made up of *all* the factors which are common to the quantities measured: the symbols for the greatest common measure are *G.C.M.**

Let $A\delta$, $B\delta$, be two quantities in which δ is the greatest factor common to both, so that A and B have no factor in common; and $A\delta$ being taken for the greater of the two quantities, let the successive divisions be carried on, as below, till the work stops for want of a remainder. Let the last of the quotients be d, then since Quotient × Divisor + Rem. = Dividend, we shall have the following series of equations on the right of the operation.

$$
\begin{array}{l|l}
B\delta)\ A\delta(a & \\
\quad aB\delta & \\
\overline{\quad C\delta)\ B\delta(b} & \\
\quad\quad bC\delta & \\
\overline{\quad\quad D\delta)\ C\delta(c} & \\
\quad\quad\quad cD\delta & \\
\overline{\quad\quad\quad E\delta)D\delta(d} & \\
\quad\quad\quad D\delta & \\
\quad\quad\quad\quad\quad & \therefore dE=D \\
 & cD+E=C \\
 & bC+D=B \\
 & aB+C=A
\end{array}
$$

From inspecting these equations in succession, it is seen from the first that E is a factor of D, and therefore, from the second, that it is a factor of C, and therefore, from the third, that it is a factor of B, and therefore, finally, that it is a factor of A. But, by the original condition, A and B have no factor in common, except unity; therefore $E=1$: hence the final divisor $E\delta$ is simply δ, the greatest common measure of the two proposed quantities. If this final divisor should happen to be $\delta=1$, we should then conclude that the quantities have no common measure, unity not being regarded as a measure.

The above, then, is the process for finding the *G.C.M.* of two quantities, whether they be numbers or algebraic expressions. And it is plain, from the general type of the operation, that what is the *G.C.M.* of the first dividend and divisor ($A\delta$, $B\delta$) is equally the *G.C.M.* of every other dividend and divisor to the end.

If, on arriving at any dividend and divisor, we find a factor in one which is not in the other, it may be expunged; or a factor may be intro-

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* It is very desirable that the old term *measure*, in reference to the present subject, should be abolished, and the term *divisor* always employed instead. The above symbols would then be replaced by *G.C.D.*, which are the more to be preferred, because *L.C.M.* already stands for least common *multiple*.

duced into either provided it be excluded from the other; for these changes cannot affect the *common measure* of the two.

(1) Required the *G.C.M.* of $7x^2-12x+5$, and x^2-6x+5

$$x^2-6x+5)7x^2-12x+5(7$$
$$7x^2-42x+35$$
$$30x-30$$

or expunging the factor 30,

$$x-1)x^2-6x+5(x-5$$
$$x^2-x$$
$$-5x+5$$
$$-5x+5$$

Hence $x-1$ is the *G.C.M.*

(2) Required the *G.C.M.* of $4x^3-2x^2-3x+1$ and $3x^2-2x-1$.

Multiply the first dividend by 3, then the second by 2, in order to avoid fractions.

$$3x^2-2x-1)3\times 4x^3-6x^2-9x+3(4x$$
$$-8x^2-4x$$
$$2x^2-5x+3)2\times 3x^2-4x-2(3$$
$$-15x+9$$
$$11x-11, \text{ or } x-1)2x^2-5x+3(2x-3$$
$$2x^2-5x+3$$

Hence $x-1$ is the *G.C.M.*

By finding in this way the *G.C.M.* of the num. and den. of a fraction, and then dividing the two by that *G.C.M.*, the fraction will be reduced to its lowest terms, so that no further simplification will be possible: thus, taking the two examples above, we have

$$\frac{x^2-6x+5}{7x^2-12x+5}=\frac{x-5}{7x-5}, \text{ and } \frac{3x^2-2x-1}{4x^3-2x^2-3x+1}=\frac{3x+1}{4x^2+2x-1},$$

the num. and den. of each of the fractions being divided by the *G.C.M.*, $x-1$.

36. It may be as well to notice here that algebraic expressions are distinguished from one another by certain designations which refer exclusively to the number of terms they contain, and to the highest power which the principal symbol in them reaches: thus ax is a *monomial* of the first degree, because it consists of but *one* term, and x occurs only in the *first* power; ax^2 is a monomial of the second degree, ax^3 one of the third, and so on. Again, $ax+b$ is a *binomial* of the first degree, ax^2+b a binomial of the second degree, ax^3+b one of the third degree, and so on. Similarly, $3x^2-2x-1$ is a *trinomial* of the second degree, $4x^3-2x^2-3x+1$ is a *quadrinomial* of the third degree, and so on. But instead of multiplying particular names for the number of terms, the word *polynomial* is usually employed to denote an expression of several terms, without explicitly implying how many.

It is worth remembering that when an expression of the second degree, or a *quadratic* expression, as it is sometimes called, is known to be divisible by a binomial of the first degree, the quotient may be at once obtained by simply dividing the first term of the dividend by the first term of the divisor, and the last by the last: thus referring to the quotients above, we see that $x-5$ may be got by dividing x^2 by the x, and 5 by the -1. In like manner, $3x+1$ may be got by dividing $3x^2$ by the x, and -1 by the

-1. Similarly, knowing, as we do, that $x^2-5x-24$ is divisible by $x+3$ (see 33) we may get the quotient by dividing x^2 by the x and -24 by the 3; this quotient being $x-8$.

If we wish to find the *G.C.M.* of *three* expressions, we find the *G.C.M.* of two, as above, and the *G.C.M.* of this and the third, and so on, if there be more quantities than three.

37. **Least Common Multiple.**—The least common multiple of two or more quantities is the least quantity, or the quantity of lowest degree, that is divisible by each of them. The symbols for the least common multiple are *L.C.M.*

If two quantities have no common measure their product is their *L.C.M.*; for, since the factors of the two quantities are all different, these factors must all be contained in every expression which is divisible by them all, the least expression thus divisible is therefore simply the product of all the factors. Let, then, the two quantities have δ for their *G.C.M.*: we may denote them by $A\delta$ and $B\delta$, where A, B, have no factor in common. It is plain that the *L.C.M.* will be $AB\delta$, that is, there cannot be any superfluous factor in $AB\delta$; for if this be divided by $A\delta$, the quotient is B, and if it be divided by $B\delta$, the quotient is A, and these quotients have no factor in common; so that no factor can be spared from $AB\delta$. Hence, to find the *L.C.M.* of two quantities, we divide their product by their *G.C.M.* And the *L.C.M.* of three quantities may be found by taking the *L.C.M.* of two of them, and then the *L.C.M.* of this and the third, and so on.

(1) What is *L.C.M.* of 12 and 18? Here the *G.C.M.* is evidently 6.

$$\therefore \frac{12\times 18}{6}=12\times 3=36, \text{ the } L.C.M.$$

(2) Find the *L. C. M.* of x^3-y^3 and x^2-y^2. Each of these is divisible by $x-y$ (art. 33): the quotient from the second is $x+y$ (page 15), by which quantity the first is not divisible (33). Hence the *G. C. M.* is $x-y$

$$\therefore (x^3-y^3)(x^2-y^2)\div(x-y)=(x^3-y^3)(x+y)=x^4+x^3y-xy^3-y^4, \text{ the } L.C.M.$$

In a similar way, we find that the

L.C.M. of x^3+1, and $(x+1)^2$ is x^4+x^3+x+1.
,, $2x^3+x^2-2x-1$, and $2x^3-x^2-2x+1$ is $4x^4-5x^2+1$.
,, x^2-1, $x-2$, and x^2-4 is x^4-5x^2+4.

---

38. Before leaving the subject of fractions, we would invite the special attention of the student to the following property of two *equal* fractions: he will find it of frequent application in the solution of equations.

Let there be two equal fractions, as $\frac{a}{b}=\frac{c}{d}$. Adding and subtracting unity, we have—

$$\frac{a}{b}+1=\frac{c}{d}+1, \text{ that is, } \frac{a+b}{b}=\frac{c+d}{d} \quad \ldots [1].$$

$$\frac{a}{b}-1=\frac{c}{d}-1, \quad \text{,,} \quad \frac{a-b}{b}=\frac{c-d}{d} \quad \ldots [2].$$

Dividing [1] by [2], we have the property alluded to; namely, that

$$\text{If } \frac{a}{b}=\frac{c}{d}, \text{ then also } \frac{a+b}{a-b}=\frac{c+d}{c-d}, \text{ and } \therefore \frac{a-b}{a+b}=\frac{c-d}{c+d}.$$

And it is plain that if, instead of the minus sign before the 1 in [2], we had used the sign $\sim$, implying *difference*, the inference would have been, that—

$$\text{If } \frac{a}{b}=\frac{c}{d}, \text{ then also, } \frac{a+b}{a\sim b}=\frac{c+d}{c\sim d}, \text{ and } \frac{a\sim b}{a+b}=\frac{c\sim d}{c+d}.$$

---

39. **Simple Equations in General.**—In a former page of this work, some illustrations were given of the application of the first four rules of Algebra to the solution of simple equations. These illustrations were proposed at that early stage of the subject, in order that the learner might gain an insight, as soon as possible, into the use and efficiency of the symbols of algebra in certain inquiries respecting numbers. We are now prepared to take a more enlarged view of the doctrine of simple equations, and to employ methods of solution preferable to some of those adopted in the former article (art. 22).

To solve a Simple Equation with one unknown Quantity.—Rule. 1. *Clear the equation of fractions*, if any enter. This is done thus: regard every integral term as a fraction with 1 for denominator, and then multiply each numerator by all the denominators except its own, exactly as in finding the new numerators in the operation for reducing a set of fractions to a common denominator (34). But here the com. den. is suppressed:—the results furnishing an equation without fractions. Or, the equation may be cleared by multiplying every term by a common multiple (the smaller the better) of all the denominators.

2. *Clear the equation of radical signs*, if any enter. This is done thus: by transposition (20) cause the radical that is to be removed to *stand alone* on one side of the equation, the other terms all occupying the other side: then perform on each side the operation which is the reverse of that indicated by the radical; thus, if the radical be $\surd$, *square* both sides; if it be $\sqrt[3]{}$, *cube* both sides, and so on. [It is plain that the square of $\surd k$ is k, that the cube of $\sqrt[3]{k}$, is k, and so on, whatever k may represent:—the reverse operation merely removing or *clearing* the radical.]

Note.—The order in which the general precepts for the solution of an equation may be most conveniently applied will be suggested by the example itself; but as fractions are more troublesome to deal with than integral quantities, it is usual to clear them away early, as in the following specimens.

But the learner is not to expect that the neatest and best method of solving every equation that may be proposed can be explained by written instructions. He can become practically familiar with the various artifices resorted to, in particular cases, only by observing the purposes effected by them in the examples worked out for his guidance and imitation. Thus, in ex. 7, page 26, the second step in the process of solution directs that 1 be added to each side of the equation: this step is suggested from observing that, if 1 be added, the denom. of the fraction is such that, upon reducing the mixed quantity to an improper fraction, the whole of the original numerator becomes cancelled, and that thus a considerable simplification is effected.

(1) $\frac{x}{2}-\frac{x}{3}+5=x-5$. (See p. 9.)

Trans. $\frac{x}{2}-\frac{x}{3}-x=-10$.

Clearing fractions (regarding $-x$, and -10, as $\frac{-x}{1}$ and $\frac{-10}{1}$)

$3x-2x-6x=-60$, or $-5x=-60$.

Dividing by -5, $x=12$.

(2) $\frac{x-1}{2}+\frac{x}{6}-\frac{x}{4}=0$. (See p. 9.)

Mult. by 12, the *L. C. M.* of 2, 6, 4,

$6x-6+2x-3x=0.$

Trans., $5x=6$. $\therefore x=1\frac{1}{5}$.

(3) $\frac{7x+8}{4}-\frac{9x-12}{8}=\frac{3x+1}{5}-\frac{29-8x}{10}$.

Multiply by 40, the *L. C. M.* of the den.

$70x+80-45x+60=24x+8-116+32x.$

Trans., $70x-45x-24x-32x=8-116-80-60$

$\therefore -31x=-248$. $\therefore x=8$.

(4) $\frac{3\surd x}{2}+5=8.$

Trans., and mult. by 2, $3\surd x=6$

$\therefore \surd x=2$; Squaring, $x=4$.

(5) $\surd(3+4x)+8=2x+9.$

Trans., $\surd(3+4x)=2x+1.$

Squaring, $3+4x=4x^2+4x+1.$

Trans., $2=4x^2$. $\therefore \frac{1}{2}=x^2$. $\therefore x=\surd\frac{1}{2}$.

(6) $\surd(4+x)+\surd x=4.$

Trans. the $\surd x$, in order that $\surd(4+x)$ may stand alone on one side,

$\surd(4+x)=4-\surd x.$

Squaring, $4+x=16-8\surd x+x.$

Trans., $8\surd x=12$. $\therefore 2\surd x=3.$

Squaring, $4x=9$. $\therefore x=2\frac{1}{4}$.

(7) $\frac{\surd(2ax-x^2)}{a-x}=b.$

Squaring, $\frac{2ax-x^2}{(a-x)^2}=b^2.$

Adding 1 to each side,

$\frac{2ax-x^2}{(a-x)^2}+1=b^2+1$, or $\frac{a^2}{(a-x)^2}=b^2+1$

$\therefore \frac{a}{a-x}=\surd(b^2+1)$. $\therefore \frac{a-x}{a}=\frac{1}{\surd(b^2+1)}$

$\therefore a-x=\frac{a}{\surd(b^2+1)}$. $\therefore x=a\left(1-\frac{1}{\surd(b^2+1)}\right)$

(8) $\frac{x+3}{x-3}=\frac{7}{5}.$

By the principle at (38),

$\frac{2x}{6}=\frac{12}{2}$. $\therefore \frac{x}{3}=6$. $\therefore x=18.$

(9) $\frac{1+x+\surd(2x+x^2)}{1+x-\surd(2x+x^2)}=4$ [or $=\frac{4}{1}$].

By the principle at (38),

$\frac{\surd(2x+x^2)}{1+x}=\frac{3}{5}$. $\therefore$ Squaring, $\frac{2x+x^2}{(1+x)^2}=\frac{9}{25}$.

Subtract each side from 1, then

$\frac{1}{(1+x)^2}=\frac{16}{25}$. $\therefore \frac{1}{1+x}=\frac{4}{5}$.

Reversing the terms of each fraction,

$\therefore 1+x=\frac{5}{4}$. $\therefore x=\frac{1}{4}$.

(10) $\frac{5x^2-9}{x\surd5+3}-\frac{x\surd5-3}{2}=1.$

Here the den. of the first fraction is the sum of two quantities, and the num. the difference of their squares; hence (p. 15) the equa. is the same as

$x\surd5-3-\frac{1}{2}(x\surd5-3)=1,$

that is, $\frac{1}{2}(x\surd5-3)=1$

$\therefore x\surd5-3=2$. $\therefore x\surd5=5$

$\therefore x=\frac{5}{\surd5}=\surd5.$

(11) $\frac{x+\frac{3}{4}}{3}-\frac{10\frac{3}{4}-x}{3\frac{2}{3}}=\frac{1}{4}.$

Multiplying by 4,

$\frac{4x+3}{3}-\frac{43-4x}{3\frac{2}{3}}=1$

that is, $\frac{4x}{3}+1-\frac{129-12x}{11}=1.$

(by mult. the terms of the second fraction by 3),

$\therefore \frac{4x}{3}-\frac{129-12x}{11}=0.$

Clearing fractions, $44x-387+36x=0.$

Trans., $80x=387$. $\therefore x=4\frac{67}{80}$.

(12) $a+x=\surd\{a^2+x\surd(b^2+x^2)\}.$

Squaring, $a^2+2ax+x^2=a^2+x\surd(b^2+x^2)$

Subtracting a^2, and dividing by x,

$2a+x=\surd(b^2+x^2).$

Squaring, $4a^2+4ax+x^2=b^2+x^2.$

Subtracting x^2, and transposing,

$4ax=b^2-4a^2$. $\therefore x=\frac{b^2-4a^2}{4a}=\frac{b^2}{4a}-a.$

(13) $\frac{\surd a+\surd(a-x)}{\surd a-\surd(a-x)}=b$ [or $=\frac{b}{1}$].

By art. (38), $\frac{\surd a}{\surd(a-x)}=\frac{b+1}{b-1}$.

Squaring, $\frac{a}{a-x}=\left(\frac{b+1}{b-1}\right)^2$

Reversing the terms of the fractions,

$\frac{a-x}{a}=\left(\frac{b-1}{b+1}\right)^2$. $\therefore a-x=a\left(\frac{b-1}{b+1}\right)^2$

$\therefore x=a-\frac{a(b-1)^2}{(b+1)^2}=\frac{a\{(b+1)^2-(b-1)^2\}}{(b+1)^2}$

$=\frac{4ab}{(b+1)^2}$

(14) $\sqrt[3]{(a+\sqrt{x})}+\sqrt[3]{(a-\sqrt{x})}=b$.

For brevity, put A for the first expression, and B for the second; then the equa. is $A+B=b$; also, $A^3+B^3=2a$.

Cubing (see page 15),

$$A^3+B^3+3(A+B)AB=b^3$$

that is, by substitution of $2a$ for A^3+B^3, and b for $A+B$,

$$2a+3bAB=b^3. \therefore AB=\frac{b^3-2a}{3b}.$$

Cubing, $A^3B^3=\left(\frac{b^3-2a}{3b}\right)^3$

that is, replacing the expressions for A, B,

$$(a+\sqrt{x})(a-\sqrt{x})=a^2-x=\left(\frac{b^3-2a}{3b}\right)^3$$

$$\therefore x=a^2-\left(\frac{b^3-2a}{3b}\right)^3$$

(15) $\sqrt{(x+9)}=\frac{x-9}{\sqrt{(x+9)}-3}-2$.

Add 3 to each side, then

$$\sqrt{(x+9)}+3=\frac{x-9}{\sqrt{(x+9)}-3}+1.$$

Clearing the fractions (see p. 15),

$$x+9-9=x-9+\sqrt{(x+9)}-3.$$

Trans., $12=\sqrt{(x+9)}$.

$\therefore$ Squaring, $x+9=144. \therefore x=135$.

NOTE.—The learner is recommended to study the above solutions carefully, before proceeding to the following examples.

EXAMPLES FOR EXERCISE.

(1) $\frac{x+3}{7}-\frac{2(x-1)}{3}+1=0$.

(2) $x-2-(x+1)=\frac{3x}{x-2}$.

(3) $\frac{x}{7}-\frac{2}{15}-\frac{1}{14}+\frac{x}{5}=\frac{43}{30}$.

(4) $\frac{x-5}{3}+\frac{x}{2}=12-\frac{x-10}{3}$.

(5) $\frac{\sqrt{3x}}{2}+7=10$.

(6) $\frac{1}{a^2-x^2}-\frac{a}{a+x}=\frac{ax}{a-x}+a$.

(7) $\frac{x-2}{\sqrt{x}}-\frac{2\sqrt{x}}{3}=0$.

(8) $\frac{\sqrt{x}+2}{\sqrt{x}-2}=1\frac{1}{2}$.

(9) $\frac{8x+5}{14}+\frac{7x-3}{6x+2}=\frac{16x+15}{28}+\frac{2\frac{1}{4}}{7}$.

(10) $x-\frac{^3x-\frac{1}{5}(x-4)+1}{11}=19+\frac{\frac{3}{4}x-2}{8}$.

(11) $\sqrt{(x+1)}+\sqrt{(x-1)}=\frac{2}{\sqrt{(x+1)}}$.

(12) $\sqrt{\left(\frac{1}{x}+1\right)}-\sqrt{\left(\frac{1}{x}-1\right)}=1$.

(13) $\frac{\sqrt{a}+\sqrt{(a-x)}}{\sqrt{a}-\sqrt{(a-x)}}=\frac{1}{a}$.

(14) $\frac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}=\sqrt{b}$.

(15) $\frac{x^2+2ax}{a^2}=b^2-1$.

(16) $\frac{\sqrt{(2ax-x^2)}}{a-x}+1=b$.

(17) $a+x=\sqrt{\{a^2+x\sqrt{(b^2+x^2)}\}}$.

(18) $\frac{1+x+x^2}{1-x+x^2}=\frac{62}{63}\cdot\frac{1+x}{1-x}$.

(19) $\frac{2x-x^2}{(1-x)^2}=a$.

(20) $\sqrt{(a+x)}+\sqrt{(a-x)}-2\sqrt{x}=0$.

(21) $\frac{\sqrt{x}+28}{\sqrt{x}+4}=\frac{\sqrt{x}+38}{\sqrt{x}+6}$.

(22) $(1-x)^3=1-\frac{3x}{1+x}$.

(23) $\sqrt[3]{(1+x)}+\sqrt[3]{(1-x)}=\sqrt[3]{2}$.

40. TO SOLVE A PAIR OF SIMPLE EQUATIONS WITH TWO UNKNOWN QUANTITIES. FIRST METHOD.—RULE. 1. From the first equation find an expression for one of the unknown quantities in terms of the other and of the known quantities; that is, solve the equation for that unknown.

2. Find, in like manner, an expression for *the same* unknown quantity from the second equation.

3. Put these two expressions for the same thing equal to each other; and an equation, with only one unknown quantity, will be exhibited. This equation being solved, the value of that unknown becomes determined, and this substituted in either of the preceding expressions for the *other*

unknown, completes the solution. For example, let the pair of equations be—

$$3x+5y=26 \therefore x=\frac{26-5y}{3}$$
$$7x-2y=6 \therefore x=\frac{6+2y}{7} \therefore \frac{26-5y}{3}=\frac{6+2y}{7}.$$

Clearing fractions, $182-35y=18+6y \therefore 182-18=6y+35y \therefore 41y=164 \therefore y=4$

$$\therefore x=\frac{26-5y}{3}=\frac{6}{3}=2.$$

SECOND METHOD.—RULE. 1. Find an expression for one of the unknown quantities from either of the equations.

2. Substitute this expression for that unknown in the other equation, and the result will be an equation containing only the other unknown; this being solved, the value of the latter becomes known, and thence, by substitution in the equation first deduced, the value of the former unknown becomes determined; as in the following example—

$3x+5y=26$, $7x-2y=6$ $\Big\}$ $\therefore x=\frac{26-5y}{3}$. Substituting this for x, the second equation is $\frac{7(26-5y)}{3}-2y=6$. Multiplying by 3, and removing brackets,

$182-35y-6y=18 \therefore 182-18=35y+6y \therefore 41y=164 \therefore y=4 \therefore x=\frac{26-5y}{3}=\frac{6}{3}=2.$

THIRD METHOD.—RULE. 1. Multiply the given equations by such factors—the smaller the better—as will cause the coefficients of *the same* unknown in the resulting equations to be *equal* in magnitude: whether they differ in sign or not is of no consequence.

2. If these equal coefficients have like signs, subtract one equation from the other; if they have unlike signs, add the two equations together; the terms with equal coefficients will thus disappear (or be *eliminated*), and the result will be an equation with only one unknown quantity. Thus, taking the pair of equations already considered, let us *eliminate* y, by multiplying the first equation by 2, the second by 5, and then adding.

$$6x+10y=52$$
$$35x-10y=30$$
$$41x \quad =82$$

$\therefore x=2$. This value, substituted in one of the given equations, the first for instance, gives $6+5y=26 \therefore 5y=20 \therefore y=4$.

Or, having eliminated y, we may eliminate x, in a similar manner, thus: Multiplying each equa. by the coefficient of x in the other, we have

$$21x+35y=182$$
$$21x-\ 6y=\ 18$$

Subtracting, $41y=164$

$$\therefore y=4$$

It is plain that the coefficients of the unknown to be eliminated may always be equalized by multiplying each by the other, as in this example: but sometimes smaller multipliers will serve the same purpose: the *smallest* are those which give for result the least common multiple of the coefficients.

It is worthy of notice that this third method never introduces fractions. As applied to a pair of equations expressed in general terms the process is as follows:—

The general equations are $a_1x+b_1y=c_1$, and $a_2x+b_2y=c_2$, the small figures being merely marks to distinguish different coefficients. Equalizing first the coefficients of x, and then the coefficients of y, the equations become

1st. $\begin{cases} a_1a_2x+a_2b_1y=a_2c_1 \\ a_1a_2x+a_1b_2y=a_1c_2 \end{cases}$

$\therefore (a_1b_2-a_2b_1)y=a_1c_2-a_2c_1$

$\therefore y=\dfrac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$

2nd. $\begin{cases} a_1b_2x+b_1b_2y=b_2c_1 \\ a_2b_1x+b_1b_2y=b_1c_2 \end{cases}$

$\therefore (a_1b_2-a_2b_1)x=b_2c_1-b_1c^2$

$\therefore x=\dfrac{b_2c_1-b_1c^2}{a_1b_2-a_2b_1}$

These general expressions for the unknown quantities comprehend, of course, every particular case. In any specified example, if the given coefficients be substituted for the general coefficients above, the solutions for that example will be obtained. And if either of the other methods be applied to the preceding equations, the same general values will be deduced.

It may be noticed here, that every such general expression for an unknown quantity is called a *formula*. The above are the two *formulæ* for the solution of a pair of *simultaneous* simple equations with two unknowns. Equations are said to be simultaneous when they coexist for the same values, in each, of the unknown quantities. Such equations, to admit of definite values for these unknown quantities, must be independent of one another; that is, each must imply conditions distinct from those implied in the other equations. If one of the equations be but a necessary consequence of some other of them, that one cannot be regarded as embodying a distinct condition, so that the number of independent equations will be fewer than the number of unknowns to be determined from them; and to fix the values of these unknowns another condition would be necessary.

We shall now give a few examples of the solution of a pair of simultaneous simple equations.

(1) $2x+5y=23$
$3x-2y=6$

Multiplying the first by 3 and the second by 2, to equalize the coefficients of x,

$6x+15y=69$
$6x-4y=12$

By subtracting, $19y=57 \therefore y=3$.

This value of y, put for that symbol in one of the given equations, determines x; thus, from the second equation,

$3x-6=6 \therefore 3x=12 \therefore x=4$.

(2) $\left.\begin{matrix} 3x+\frac{1}{3}y=36 \\ \frac{1}{4}(3y-x)=4 \end{matrix}\right\}$ or, $\begin{cases} 9x+y=108 \\ 3y-x=16 \end{cases}$

From the first, $y=108-9x$

Substituting this in the second,

$324-27x-x=16 \therefore 28x=308$

$\therefore x=11, \therefore y=108-9x=9$.

(3) $\left.\begin{matrix} \frac{1}{5}x+\frac{1}{4}y=6 \\ \frac{1}{4}x+\frac{1}{6}y=5\frac{2}{3} \end{matrix}\right\}$ or, $\begin{cases} \frac{1}{5}x+\frac{1}{2}y=12 \\ \frac{3}{4}x+\frac{1}{2}y=17 \end{cases}$

By subtracting, $(\frac{3}{4}-\frac{1}{5})x=5$.

Mult. by 12, $(9-4)x=60 \therefore x=12$

$\therefore \frac{1}{5}x+\frac{1}{2}y=4+\frac{1}{2}y=12 \therefore \frac{1}{2}y=8 \therefore y=16$.

(4) $\left.\begin{matrix} \frac{1}{x}+\frac{1}{y}=12 \\ \frac{3}{x}-\frac{4}{y}=1 \end{matrix}\right\}$ or, $\begin{cases} x'+y'=12 \\ 3x'-4y'=1 \end{cases}$

where x', y', are put for $\frac{1}{x}$, $\frac{1}{y}$.

Multiplying the first by 3, and subtracting, $7y'=35 \therefore y'=5 \therefore x'=12-y'=7$

$\therefore x=\frac{1}{x'}=\frac{1}{7}$, and $y=\frac{1}{y'}=\frac{1}{5}$.

(5) $\dfrac{3x+4y}{5}+\dfrac{x}{4}=10$

$\dfrac{6x-2y}{3}+\dfrac{y}{6}=14$

Clearing fractions, these become

$\left.\begin{matrix} 17x+16y=200 \\ 12x-3y=84 \end{matrix}\right\}$ or, $\begin{cases} 17x+16y=200 \\ 4x-y=28 \end{cases}$

From the second, $y=4x-28$; and by substi. in the first, $17x+64x-448=200$

$\therefore 81x=648 \therefore x=8$

$\therefore y=4x-28=4$.

Otherwise,

Multiplying the second equa. by 16, and adding to that above it,

$81x=200+448=648 \therefore x=8$

$\therefore y=4x-28=32-28=4.$

(6) $a(x-y)=b(x+y)$ . . . . [1]

$x^2-y^2=c$ . . . . . [2]

Mult. [1] by $x+y$, and [2] by a,

$a(x^2-y^2)=b(x+y)^2$

$a(x^2-y^2)=ac$

$\therefore \quad 0 \quad = \quad b(x+y)^2-ac$

$\therefore (x+y)^2=\frac{ac}{b} \therefore x+y=\sqrt{\frac{ac}{b}}$

Consequently, from [1],

$$x-y=\frac{b}{a}\sqrt{\frac{ac}{b}}.$$

Adding and subtracting,

$$2x=\left(1+\frac{b}{a}\right)\sqrt{\frac{ac}{b}}=\frac{a+b}{a}\sqrt{\frac{ac}{b}}$$

$$2y=\left(1-\frac{b}{a}\right)\sqrt{\frac{ac}{b}}=\frac{a-b}{a}\sqrt{\frac{ac}{b}}.$$

But $\sqrt{\frac{ac}{b}}=\sqrt{\frac{abc}{b^2}}=\frac{\sqrt{abc}}{b}$

$$\therefore x=\frac{a+b}{2ab}\sqrt{abc}$$

$$y=\frac{a-b}{2ab}\sqrt{abc}$$

Examples for Exercise.

(1) $\begin{cases}3x+4y=43\\7x-2y=21.\end{cases}$

(2) $\begin{cases}5x-3y=-13\\7x+2y=81.\end{cases}$

(3) $\begin{cases}3x+4y=29\\17x-3y=36.\end{cases}$

(4) $\begin{cases}\frac{1}{3}(x+2)+8y=31\\\frac{1}{4}(y+5)+10x=192.\end{cases}$

(5) $\begin{cases}\frac{x+y}{3}+1=6\\\frac{x-y}{7}+3=4.\end{cases}$

(6) $\begin{cases}\frac{2x-3}{2}+y=7\\5x-13y=33\frac{1}{2}.\end{cases}$

(7) $\begin{cases}\frac{3x-5y}{2}+3=\frac{2x+y}{5}\\8-\frac{x-2y}{4}=\frac{x}{2}+\frac{y}{3}.\end{cases}$

(8) $\begin{cases}\frac{3x-7y}{3}=\frac{2x+y+1}{5}\\\frac{x-y}{5}+6=8.\end{cases}$

(9) $\begin{cases}\frac{1}{x}+\frac{1}{y}=5\\\frac{5}{x}-\frac{3}{y}=1.\end{cases}$

(10) $\begin{cases}\frac{4}{x}+\frac{5}{y}=\frac{17}{63}\\\frac{19}{x}+\frac{6}{y}=1.\end{cases}$

(11) $\begin{cases}x+y=20\\x^2-y^2=120.\end{cases}$

(12) $\begin{cases}2x+\cdot4y=1\cdot2\\3\cdot4x-\cdot02y=\cdot01.\end{cases}$

(13) $\begin{cases}\frac{\sqrt{(4x+1)}+\sqrt{y}}{\sqrt{(4x+1)}-\sqrt{y}}=9\\9x-5y=5.\end{cases}$

(14) $\begin{cases}\frac{\sqrt{x}+6}{\sqrt{x}+4}=\frac{\sqrt{y}+2}{\sqrt{y}-2}\\2\sqrt{x}+3\sqrt{y}=2.\end{cases}$

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41. **To solve three Simple Equations with three Unknown Quantities.**—In solving a *pair* of simultaneous simple equations, it has been sufficiently seen that the first object is, by the elimination of one of the unknowns, to reduce the two equations to one, containing only the other unknown. In like manner, in treating *three* simultaneous equations, the aim is to reduce them to two, by the previous elimination of one of the unknowns; this is done as follows.

Rule. 1. Take any two of the proposed equations, eliminate from them one of the unknowns: the result will be an equation containing only the other two.

2. Take now the third equation, in connection with one of those already employed, and eliminate *the same* unknown from *them:* the result will be a second equation containing only the other two unknowns. We shall thus have reduced our three equations with three unknowns, to two

with two unknowns; and these may be solved as already explained; and thence the value of the third unknown found by substitution, as in the following examples:—

(1) $\begin{cases} 2x+4y-3z=22 \ldots [1] \\ 4x-2y+5z=18 \ldots [2] \\ 6x+7y-z=63 \ldots [3] \end{cases}$

Eliminate x from [1] and [2], thus

$$4x+8y-6z=44$$
$$4x-2y+5z=18$$
$$10y-11z=26 \ldots [A].$$

Eliminate the same from [1] and [3], thus

$$6x+12y-9z=66$$
$$6x+7y-z=63$$
$$5y-8z=3 \ldots [B].$$

We have thus two equations [A] and [B] with only two of the unknowns.

Multiplying [B] by 2, and subtracting, to eliminate y, we have $5z=20$

$$\therefore z=4,\ y=\frac{3+8z}{5}=7,\ x=\frac{22-4y+3z}{2}=3$$

where y is got by substitution in [B], and x by substitution in [1].

(2) $\begin{cases} 3x+2y-4z=8 \ldots [1] \\ 5x-3y+3z=33 \ldots [2] \\ 7x+y+5z=65 \ldots [3] \end{cases}$

Here it is plain that y may be most readily eliminated.

Subtract [1] from twice [3], then

$$11x+14z=122 \ldots [A]$$

Add [2] to three times [3], then

$$26x+18z=228$$

or, $13x+9z=114 \ldots [B]$

Mult. [A] by 13, and [B] by 11,

$$13.11x+182z=1586$$
$$11.13x+99z=1254$$
$$83z=332$$

$$\therefore z=4,\ x=\frac{122-14z}{11}=6, \text{ and [3]},$$
$$y=65-7x-5z=3.$$

Examples for Exercise.

(1) $\begin{cases} x+y+z=53 \\ x+2y+3z=105 \\ x+3y+4z=134. \end{cases}$

(2) $\begin{cases} x+\frac{1}{2}y+\frac{1}{3}z=27 \\ x+\frac{1}{3}y+\frac{1}{4}z=20 \\ x+\frac{1}{4}y+\frac{1}{5}z=16. \end{cases}$

(3) $\begin{cases} 3x-9y+8z=41 \\ 5x-4y-2z=20 \\ 11x-7y-6z=37. \end{cases}$

(4) $\begin{cases} x+\frac{1}{2}y+\frac{1}{3}z=32 \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15 \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12. \end{cases}$

(5) $\begin{cases} 2x+3y+4z=16 \\ 3x+2y-5z=8 \\ 5x-6y+3z=6. \end{cases}$

(6) $\begin{cases} 2x-3y+5z=27 \\ 3x+6y-4z=2 \\ 5x+4y+2z=40. \end{cases}$

(7) $\begin{cases} 5x-4y+2z=48 \\ 3x+3y-4z=24 \\ 2x-5y+3z=19. \end{cases}$

(8) $\begin{cases} \frac{a}{x}+\frac{b}{y}=1 \\ \frac{a}{x}+\frac{c}{z}=1 \\ \frac{b}{y}+\frac{c}{z}=1. \end{cases}$

It was observed (at p. 29), that in order that a set of simultaneous equations may admit of definite solutions, the equations must be independent of one another: they must, of course, be also compatible with one another; that is, no two of them may involve a contradiction. In the first of the following sets, the third equation is an obvious consequence of the first, so that the conditions are insufficient: in the second set, the conditions are contradictory, for if twice the first be added to the second, we shall have $8x-3y+2z=38$, whereas the third condition affirms that this is 35, so that $38=35$, which is absurd.

Insufficient Conditions.

$$2x-3y+4z=12$$
$$6x+2y-z=14$$
$$4x-6y+8z=24$$

Incompatible Conditions.

$$3x-2y+5z=14$$
$$2x+y-8z=10$$
$$8x-3y+2z=35$$

But we need never take the trouble actually to examine a set of simultaneous equations with a view to discovering whether they are independent and consistent or not: we may proceed at once with our eliminations. Indications, abundantly obvious, of insufficient or incongruous conditions, will spontaneously offer themselves. Thus, taking the first set above, if we eliminate z between the first and second, we shall have $26x+5y=68$, and if we then eliminate z between the second and third, we shall have *also* $26x+5y=68$; and we at once infer insufficiency of conditions. As to the other set of equations, we have already seen that they involve the absurdity of confounding 35 with 38.

By the process of successive elimination just explained, any set of simultaneous equations, however numerous, may be solved.

Questions to be solved by Simple Equations.

Note.—In solving questions by algebra, instead of invariably representing an unknown quantity by x or y, it will often be better to denote it by some multiple of the letter; that is, to prefix to it a coefficient. Whenever a fractional part of the unknown is to be taken, it will obviously save work and trouble if we prefix to the letter a coefficient, such that the prescribed fractional part will be a whole number (see examples 2 and 3 following).

(1) A labourer was engaged for 60 days, on condition that for every day he worked he should receive 15*d.*, and for every day he idled he should forfeit 5*d.* At the end of the 60 days he received 20*s.*: how many days did he work?

1*st Solution.* Suppose he worked x days, and idled y days, then by the question

$$x+y=60$$

and $15x-5y=240$, pence

5 times 1st equa., $5x+5y=300$

$$\therefore 20x=540 \therefore x=27$$

Hence he worked 27 days.

2*nd Solution.* Suppose he worked x days, then he idled $60-x$ days, and by the question

$$15x-5(60-x)=240, \text{ pence,}$$

that is, $15x-300+5x=240$

Trans., $20x=540 \therefore x=27$

Hence he worked 27 days, and consequently he idled 33 days.

Note.—In the solution of problems, it is usually preferable to employ as many unknown quantities as there are values to be found.

(2) A gamester at play lost $\frac{1}{3}$ of his money, and then won 10*s.*: he afterwards lost $\frac{1}{3}$ of what he then had, and won 3*s.*, leaving off with 3 guineas. How much had he at first?

Suppose he had $5x$ shillings at first: then he lost x shillings; so that after winning 10*s.* he had $4x+10$; of this sum, he had at last only two-thirds and 3*s.*, which by the question was 63*s.*

$$\therefore \tfrac{2}{3}(4x+10)+3=63 \therefore \tfrac{2}{3}(4x+10)=60$$

$$\therefore 8x+20=180 \therefore 8x=160 \therefore x=20$$

$$5x=100.$$

Hence he began with 100*s.*, or £5.

(3) Two persons, A and B, have the same annual income: A saves the fifth part of his; but B, who spends £80 a year more than A, at the end of 4 years finds himself £220 in debt: what is the annual income and expenditure of each?

Let $5x$ represent the income, then A spends $4x$, and B spends $4x+80$ yearly, $\therefore$ in 4 years B spends $16x+320$, while his income in that time is only $20x$: hence by the question, $16x+320=20x+220$.

$\therefore$ Trans., $100=4x \therefore x=25 \therefore 5x=125$.

The annual income is $\therefore$ £125.

Also, $4x=$£100, A's expenditure,

$\therefore$ £100+£80=£180, B's expenditure.

(4) There is a certain number consisting of two figures, the sum of which is 5; and if 9 be added to the number the figures will be reversed: what is the number?

Let x, y represent the figures, then the number is $10x+y$; but if the figures be reversed the number is $10y+x$: hence by the question,

$$x+y=5 \therefore y=5-x$$
$$10x+y+9=10y+x.$$

Substituting the above value of y,

$$10x+5-x+9=50-10x+x.$$

Trans. $18x=36 \therefore x=2 \therefore y=5-x=3$.

Hence the number is 23.

(5) There are two sorts of spirits, one worth 20*s*. a gallon and the other worth 12*s*.: how much of each must be taken to make 1 gallon worth 14*s*.?

Suppose x gal. at 20*s*. and y gal. at 12*s*.: then by the question,

$$x+y=1 \therefore y=1-x.$$

Also, the worth of the compound is

$$20x+12y=14,$$

$\therefore$ by substitution,

$$20x+12-12x=14.$$

Trans., $8x=2 \therefore x=\frac{1}{4} \therefore y=1-\frac{1}{4}=\frac{3}{4}$, so that there must be $\frac{1}{4}$ gal. at 20*s*., and $\frac{3}{4}$ gal. at 12*s*.

(6) Spirits at 9*s*. 6*d*. per gallon were mixed with some at 13*s*. 6*d*. There were 104 gals. in the mixture, and they were worth £56: how much of each sort was used?

Suppose x gals. at 9*s*. 6*d*., and y at 13*s*. 6*d*.: then, in *sixpences* the worth of the mixture is

$$19x+27y=2240 \text{ (by quest.)}$$

Also, $x+y=104 \therefore x=104-y$

$$\therefore 19x+19y=1976$$

$$\therefore \quad 8y=264 \therefore y=33 \therefore x=71.$$

(7) A labourer is hired for a specified number of days, on condition that for each day he works he shall receive a certain sum, but for each day he is idle he shall pay a certain sum for his board. At the end of the time the proper balance is paid to him: how is the number of days he worked to be found?

Let a=the price of each day's work,
b= ,, ,, board,
c=the amount received,
d=the whole number of days,
Also x=the number of days' work.

Then the price of the work done is ax, and that of the board b $(d-x)$.

By the question, the difference of these is

$$ax-b(d-x)=c,$$

or $ax+bx-bd=c \therefore (a+b)x=c+bd$

$$\therefore x=\frac{c+bd}{a+b}, \text{ and } \therefore d-x=\frac{ad-c}{a+b},$$

the former value expressing the number of working days, and the latter the number of idle days.

The above is a general solution of the problem of which ex. 1 is a particular case. Putting in the preceding formulæ $a=15$, $b=5$, $c=240$, and $d=60$, we have

$$x=\frac{240+5.60}{15+5}=\frac{540}{20}=27, \text{ and } d-27=33.$$

The second formula above enables us to find the number of idle days, without first finding the number of working days.

(8) If A and B can finish a piece of work in 8 days, A and C in 9 days, and B and C in 10 days: in how many days can each, alone, finish it?

Let x, y, z represent the number of days occupied by A, B, C, *alone*; then in one day, the parts of the whole they can do respectively will be $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$, which, for brevity, write x', y', and z'.

Now, since A and B together do the whole in 8 days, A and C, in 9 days, and B and C in 10 days,

$$\therefore 8x'+8y'=1 \text{ (}one\text{ whole work)}$$
$$9x'+9z'=1 \therefore x'+z'=\frac{1}{9}$$
$$10y'+10z'=1.$$

Mult. 1st by 5, 3rd by 4, and subtract,

$$\therefore 40x'-40z'=1 \therefore x'-z'=\frac{1}{40}.$$

Add and subtract the two equations on the right,

$$\therefore 2x'=\frac{1}{9}+\frac{1}{40}, \quad 2z'=\frac{1}{9}-\frac{1}{40}$$
$$\therefore x'=\frac{49}{720}, \quad z'=\frac{31}{720}.$$

Substitute these in the third equation,

$$\therefore 10y'=1-\frac{31}{72}=\frac{41}{72}\therefore y'=\frac{41}{720}.$$

$$\therefore x=\frac{720}{49}=14\frac{34}{49} \text{ days required by } A.$$

$$y=\frac{720}{41}=17\frac{23}{41} \quad ,, \quad ,, \quad B.$$

$$z=\frac{720}{31}=23\frac{7}{31} \quad ,, \quad ,, \quad C.$$

(9) It is recorded that Hiero, King of Syracuse, ordered a crown of solid gold to be fabricated for Jupiter: its weight was to be 10 lbs. (avoir.), which weight the crown was found exactly to balance. But, suspecting that there was an alloy of silver in it, he directed Archimedes to ascertain the extent of the fraud. The philosopher found that a specimen of the gold supplied, when weighed in water, lost $\frac{52}{1000}$ of its weight; and that silver lost $\frac{99}{1000}$ of its weight in water; also, that the crown itself lost 10 oz. of its weight. How much of each metal was there in its composition?

Let x be the ounces of gold, and y the ounces of silver, then $x+y=160$; also, $\frac{52x}{1000}$ = what the x oz. of gold loses in water, and $\frac{99y}{1000}$ = what the y oz. of silver loses in water; and the sum of these is the whole loss of weight of the crown in water:

$$\therefore \frac{52x}{1000}+\frac{99y}{1000}=10 \therefore 52x+99y=10000.$$

From the first equation, $x=160-y$.

Hence by substitution,

$$8320-52y+99y=10000.$$

$$\therefore 47y=1680 \therefore y=\frac{1680}{47}=35\frac{35}{47}\text{oz.}$$

$$\therefore x=160-y=124\frac{12}{47}\text{oz.}$$

Hence the crown was composed of $124\frac{12}{47}$oz. of gold, and $35\frac{35}{47}$oz. of silver.

Questions for Exercise in Simple Equations with One, Two, and Three Unknown Quantities.

(1) Divide £100 among A, B, C, so that A may have £20 more than B, and B £10 more than C.

(2) A farmer has wheat at 4*s*. 6*d*. a bushel, with which he wishes to mix rye at 3*s*. 6*d*. a bushel, and barley at 3*s*. a bushel, so as to make 136 bushels that shall be worth 4*s*. a bushel: what quantity of rye and barley must he take?

(3) What fraction is that to the numerator of which, if 1 be added, the value will be $\frac{1}{2}$; but, if 1 be added to the denominator the value will be $\frac{1}{3}$?

(4) A man and his wife usually drank a barrel of beer in 15 days, but, after drinking together from a fresh barrel for 6 days, the man left home, when the woman alone drank the remainder in 30 days. In what time could either alone finish the barrel?

(5) A person is desirous of relieving some poor persons by giving them half-a-crown each, but finds he has not money enough by 3*s*.: he therefore gave them 2*s*. each, and had 4*s*. to spare. How much had he at first, and how many did he relieve?

(6) Divide 90 into four parts, such that if the first part be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the four results may all be equal.

(7) The trinomial expression ax^2+bx+c is such that when 4 is put for x, its value is 42; when 3 is put for x, its value is 22; and when 2 is put for x, its value is 8. Find the values of the coefficients a, b, c.

(8) A hare is 80 of her leaps in advance of a pursuing greyhound, and she takes three leaps to the greyhound's two; but one leap of the latter is equal to two leaps of the former. How many leaps will the hare have taken when she is caught?

(9) In what time would the work alluded to in ex. 8, p. 33, be done by A, B, and C all working together?

(10) Gunpowder is to be conveyed into a garrison in *full* casks: more than 2 cwt. cannot be carried in each. There are two sorts of powder: a cask full of one sort weighs 230 lbs., a cask full of the other weighs 182 lbs. How much of each sort must be sent to make a cask full of the extreme weight?

(11) A cistern can be filled by the pipes A and B in 70 minutes, by the pipes A and C in 84 minutes, and by the pipes B and C in 140 minutes. What time will each pipe take to fill it, and in what time will it be filled if all the pipes flow together?

(12) A composition of copper and tin, containing 100 cubic inches, weighed 505 oz. How much of each metal did it contain, supposing a cubic inch of copper to weigh $5\frac{1}{4}$ oz. and a cubic inch of tin to weigh $4\frac{1}{4}$ oz.?

(13) 37 lbs. of tin is found to lose 5 lbs. when weighed in water; and 23 lbs. of lead to lose 2 lbs. when weighed in water; a composition of tin and lead, weighing 120 lbs., is found to lose 14 lbs. when weighed in water. How much of each metal is there in the composition?

(14) A number consists of three figures, such that the difference between the first and second is the same as the difference between the second and third. If the number be divided by the sums of the figures the quotient will be 26, and if 198 be added to the number, the figures of it will be reversed. Required the number.

(15) Three ingots, A, B, C, are composed of different metals: 1 lb. of A contains 7 oz. of silver, 3 oz. of copper, and 6 oz. of pewter; 1 lb. of B contains 12 oz. of silver, 3 oz. of copper, and 1 oz. of pewter; and 1 lb. of C contains 4 oz. of silver, 7 oz. of copper, and 5 oz. of pewter. How much must be taken from each ingot to form 1 lb., which shall contain 8 oz. of silver, $3\frac{3}{4}$ oz. of copper, and $4\frac{1}{4}$ oz. of pewter?

(16) Standard gold for coinage is £3 17*s.* $10\frac{1}{2}d.$ per oz. What is the smallest integral number of ounces that can be coined into sovereigns, and how many sovereigns will there be?

(17) The hands of a clock are together at 12 o'clock. State all the times at which they are together from 12 o'clock at noon till 12 o'clock at night.

(18) Three vessels together contain 18 gallons: half the first is poured into the second, a third of what is then in the second is poured into the third, and, lastly, a fourth of what the third then contains is poured into the first; the liquid is then found to be equally divided among the three vessels. What did each contain at first?

(19) A, B, C have certain sums of money, such that if A's were increased by half what B and C together have, he would have £a; if B's were increased by one-third of what A and C have, he would have £b; and if C's were increased by one-fourth of what A and B have, he would possess £c. What sum has each?

(20) If from a vessel of wine containing a gallons, b gallons be drawn off, and the vessel then filled up with water; then b gallons of the mixture be drawn off and the vessel again filled up with water, and so on;

and that this operation be repeated n times successively: what will be the quantity of *wine* left after these n drawings?

(21) Two gamesters play on the following conditions: whichever loses the first game shall double the money of his opponent; whichever loses the second game shall triple the money of his opponent; and whichever loses the third game shall quadruple the money of his opponent. They lose alternately, and at the end of the third game each has the same sum, namely, 48*s*. How much had each at first?

(22) Five persons engage in play on condition that he who loses shall give to each of the others as much as that other already has. All lose in turn, and yet at the end of the fifth game they all have the same sum, namely, £32. How many pounds did each begin with?

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42. **Theory of Exponents.**—It has already been explained (27, 28) that when n is a positive integer a^n denotes the nth power of a, and $a^{\frac{1}{n}}$ the nth root; we are now to extend this notation, and to give a consistent meaning to a quantity with any exponent, whether whole or fractional, positive or negative.

Negative Exponents.—The extension to the case of a negative exponent is suggested from the following considerations:—

By division (32) $\frac{a^1}{a^1} = a^{1-1} = a^0$; But $\frac{a^1}{a^1} = 1 \therefore a^0 = 1$, whatever be a

$\frac{a^1}{a^2} = a^{1-2} = a^{-1}$; but $\frac{a^1}{a^2} = \frac{1}{a} \therefore a^{-1} = \frac{1}{a}$

$\frac{a^1}{a^3} = a^{1-3} = a^{-2}$; but $\frac{a^1}{a^3} = \frac{1}{a^2} \therefore a^{-2} = \frac{1}{a^2}$

And generally $\frac{a^1}{a^{1+n}} = a^{1-1-n} = a^{-n}$; but $\frac{a^1}{a^{1+n}} = \frac{1}{a^n} \therefore a^{-n} = \frac{1}{a^n}$

Consequently, whatever integer n may be, a^{-n} is the same as $\frac{1}{a^n}$; 1, divided by any quantity, is called the *reciprocal* of that quantity, so that, not only powers, but also the reciprocals of powers, may be indicated by exponents: in the former case the exponents are *positive*, in the latter, they are *negative*.

43. **Fractional Exponents.**—From what has already been laid down, we know that $(a^m)^{\frac{1}{n}}$ means the nth root of the mth power of a (27, 28). Instead of this cumbersome notation, the following more compact form is used, $a^{\frac{m}{n}}$; it being agreed that the numerator of the exponent shall denote the power, and the denominator the root of that power. It is matter of indifference, however, whether we call the expression just written the nth root of the mth power of a, or the mth power of the nth root of a; for in whichever order the operations are performed upon any number or quantity, the final result is the same; that is to say, $(a^m)^{\frac{1}{n}}$ and $\left(a^{\frac{1}{n}}\right)^m$ are the same in value. For instance,

suppose $m=3$ and $n=5$; then we have $(a^3)^{\frac{1}{5}}=(aaa)^{\frac{1}{5}}=a^{\frac{1}{5}}a^{\frac{1}{5}}a^{\frac{1}{5}}$, the 5th power of each being aaa; and since $(a^{\frac{1}{5}})^3=a^{\frac{1}{5}}a^{\frac{1}{5}}a^{\frac{1}{5}}$, consequently, $(a^3)^{\frac{1}{5}}=(a^{\frac{1}{5}})^3$.

44. The following principles require to be established before we can safely apply the fundamental rules of operation to quantities with exponents:—

1. $x^m\times y^m=(xy)^m$. 2. $x^m\div y^m=\left(\frac{x}{y}\right)^m$. 3. $x^{\frac{m}{n}}\times x^{\frac{r}{s}}=x^{\frac{m}{n}+\frac{r}{s}}$. 4. $x^{\frac{m}{n}}\div x^{\frac{r}{s}}=x^{\frac{m}{n}-\frac{r}{s}}$.

1. $xxx\ldots$to m factors$\times yyy\ldots$to m factors$=xy.xy.xy\ldots$.to m factors.

2. $\frac{xxx\ldots\text{to } m \text{ factors}}{yyy\ldots\text{to } m \text{ factors}}=\frac{x}{y}.\frac{x}{y}.\frac{x}{y}.\ldots$ to m factors.

Consequently, $x^m\times y^m=(xy)^m$, and $\frac{x^m}{y^m}=\left(\frac{x}{y}\right)^m$.

3. Put the single symbols a and b, for $x^{\frac{m}{n}}$ and $x^{\frac{r}{s}}$; then $\left(x^{\frac{m}{n}}\right)^n=a^n$, and $\left(x^{\frac{r}{s}}\right)^s=b^s$; that is, $x^m=a^n$, and $x^r=b^s\therefore x^{ms}=a^{ns}$, and $x^{nr}=b^{ns}$.....[A]

$\therefore x^{ms}\times x^{nr}=a^{ns}\times b^{ns}$, that is, $x^{ms+nr}=(ab)^{ns}\therefore ab=x^{\frac{ms+nr}{ns}}$,

that is, replacing the symbols, a, b, by their values above, $x^{\frac{m}{n}}\times x^{\frac{r}{s}}=x^{\frac{ms+nr}{ns}}=x^{\frac{m}{n}+\frac{r}{s}}$.

4. From the equations marked [A], $x^{ms}\div x^{nr}=a^{ns}\div b^{ns}=\left(\frac{a}{b}\right)^{ns}$; that is, $x^{ms-nr}=(a\div b)^{ns}\therefore a\div b=x^{\frac{ms-nr}{ns}}$, or replacing the symbols a, b, by their values,

$$x^{\frac{m}{n}}\div x^{\frac{r}{s}}=x^{\frac{ms-nr}{ns}}=x^{\frac{m}{n}-\frac{r}{s}}.$$

Hence, whatever be the exponents attached to equal quantities, the following are the general rules for multiplying and dividing.

45. For Multiplication.—Add the exponents.

For Division.—Subtract the exponent of the divisor from the exponent of the dividend.

As to Powers, we merely multiply the exponent by the index of the power; thus $(x^m)^n=x^{mn}$, because each form equally represents x^m repeated n times.

For Roots, the rule is the same; that is, $(x^m)^{\frac{p}{n}}=x^{\frac{mp}{n}}$; for $(x^m)^{\frac{p}{n}}=(x^{mp})^{\frac{1}{n}}$, by last case, and by (43), this is the same as $x^{\frac{mp}{n}}$.

Examples.

(1) $a^{\frac{3}{4}}\times a^{\frac{1}{2}}=a^{\frac{3}{4}+\frac{1}{2}}=a^{\frac{5}{4}}$. (2) $a^4\times a^{-\frac{1}{2}}=a^{4\,-\frac{1}{2}}=a^{\frac{7}{2}}=a^3a^{\frac{1}{2}}=a^3\sqrt{a}$.

(3) $a^{-\frac{1}{2}}\div a^4=a^{-\frac{1}{2}-4}=a^{-\frac{9}{2}}=\frac{1}{a^{\frac{9}{2}}}=\frac{1}{\sqrt{a^9}}$. (4) $\left(a^{\frac{1}{6}}\right)^{\frac{1}{2}}=a^{\frac{1}{12}}$.

(5) $\left(a^{-\frac{1}{2}}\right)^{\frac{3}{4}}=a^{-\frac{3}{8}}=\frac{1}{a^{\frac{3}{8}}}=\frac{1}{\sqrt[8]{a^3}}$. (6) $\left(a^{-3}\right)^{-\frac{1}{2}}=a^{\frac{3}{2}}=\sqrt{a^3}$.

46. **Square Root of a Polynomial.**—In this article we propose to show how the square root of any expression of even degree of the

form $A+Bx+Cx^2+Dx^3+Ex^4+\ldots$*, may be extracted. We shall find, by observing the constitution of a polynomial, known to be the square of a given root, that the reverse operation of evolving the root from the square will be suggested.

1. The polynomial $x^2+2ax+a^2$ is known to be the square of $x+a$, and it is easy to see that this root may be evolved from its square thus; the square-root of the leading term is x, the first term of the sought root; and x^2 being subtracted from the proposed trinomial, leaves $2ax+a^2$ for remainder, the leading term of which, if divided by *twice* the first term x of the root, already obtained, will give the second term a. We may regard $2x$ as a trial or partial divisor of the remainder $2ax+a^2$, and place it against this remainder, as in the margin; and having from the trial divisor got a, we have only to connect this a with the trial divisor to get the true or complete divisor due to the quotient a.

$$\begin{array}{l|l} x^2+2ax+a^2 & x+a \\ \underline{x^2} & \\ 2x+a & 2ax+a^2 \\ \underline{\quad} & \underline{2ax+a^2} \\ & \end{array}$$

2. The polynomial $x^4+2ax^3+(a^2+2b)x^2+2abx+b^2$ is the square of x^2+ax+b. And proceeding in imitation of the process above, doubling what may already be in the root-place, for the trial divisor expected to give the next term of the root, we find the complete root as before, thus:—

$$\begin{array}{l|l} x^4+2ax^3+(a^2+2b)x^2+2abx+b^2 & x^2+ax+b \\ \underline{x^4} & \\ 2x^2+ax & 2ax^3+(a^2+2b)x^2 \\ & \underline{2ax^3+a^2x^2} \\ 2x^2+2ax+b & 2bx^2+2abx+b^2 \\ & \underline{2bx^2+2abx+b^2} \\ & \end{array}$$

47. There was scarcely any necessity to exhibit this last operation; the efficiency of it might have been inferred from the preceding one; for since $(x^2+ax+b)^2=\{(x^2+ax)+b\}^2=(x^2+ax)^2+2b(x^2+ax)+b^2$ (p. 15), the proposed square is no other than the trinomial just written. The root of the first term, that is the portion x^2+ax, of the whole root is got as previously explained, and thence the remaining portion b. And by the same uniform process is the root of the polynomial furnished by $(x^3+ax^2+bx+c)^2$ or $(x^3+ax^2+bx)^2+2c(x^3+ax^2+bx)+c^2$ obtained; and so on: the rule for the operation is, therefore, this:—

Rule. 1. Arrange the polynomial as if for division.

2. The square root of the first term, will be the leading term of the sought root; place the square of it under the first term, to which it is of course equal, and, having drawn a line under it, bring down the next *two* terms of the polynomial;† regard these as a dividend, and for a trial divisor of it, write twice the root-term just found.

3. Find the quotient of the leading term of the dividend by this trial divisor, and connect it, with its proper sign, both to the root-term, and to the trial divisor; the divisor will then be completed.

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* It is scarcely necessary to premise here that x may be anything—even 1.

† When terms are absent from the arranged polynomial, their places should be supplied by *zeros:* thus x^4+6x^2+2x+1 should be written $x^4+0x^3+6x^2+2x+1$.

4. Multiply the complete divisor by this second root-term; subtract the product from the dividend, and connect with the remainder two more terms of the polynomial for a second dividend.

5. Proceed now exactly as before, taking twice the part of the root already found as a trial divisor, for finding the third term of that root, which third term, when thus obtained, completes the divisor; and so on till all the terms of the polynomial have been brought down.

If after this, there be still a remainder, we may be sure that the polynomial is not a complete square.

(1) $4x^4-4x^3+13x^2-6x+9 \mid 2x^2-x+3$

$4x^4$

$4x^2-x \mid -4x^3+13x^2$

$-4x^3+x^2$

$4x^2-2x+3 \mid 12x^2-6x+9$

$12x^2-6x+9$

(2) $4x^4-12x^3+25x^2-24x+18 \mid 2x^2-3x+4$

$4x^4$

$4x^2-3x \mid -12x^3+25x^2$

$-12x^3+9x^2$

$4x^2-6x+4 \mid 16x^2-24x+18$

$16x^2-24x+16$

2

We infer from there being a remainder in this second example, that the proposed polynomial is not a complete square.

Examples for Exercise.

(1) $4x^4-4x^3-3x^2+2x+1$. (2) $9x^4-6ax^2+a^2$.

(3) $9x^6-12x^5+10x^4-28x^3+17x^2-8x+16$. (4) $4x^6+12x^5+5x^4-2x^3+7x^2-2x+1$.

(5) $4x^2y^4-12x^3y^3+17x^4y^2-12x^5y+4x^6$. (6) $\frac{x^2}{y^2}+\frac{y^2}{x^2}+2\left(\frac{x}{y}+\frac{y}{x}\right)+3$.

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48. **Cube Root of a Polynomial.**—Examining the constitution of a cube, we find that

$$(x+a)^3=x^3+\{3x^2+3ax+a^2\}a \quad \text{[See page 15].}$$

$$\therefore (x^2+ax+b)^3=\{(x^2+ax)+b\}^3=(x^2+ax)^3+\{3(x^2+ax)^2+3(b.\overline{x^2+ax}+b^2\}b.$$

&c. &c.

Taking the first of these cubes namely, $x^3+3ax^2+3a^2x+a^3$, the process in the margin, for evolving the root, is suggested.

$x^3+3ax^2+3a^2x+a^3 \mid x+a$

x^3

$3x^2+3ax \quad a^2 \mid 3ax^2+3a^2x+a^3$

$3ax^2+3a^2x+a^3$

The first term x of the root being found, and the dividend, consisting of *three* terms, brought down, we write 3 times the square of the root-term as a trial divisor for finding a; we then complete the divisor by annexing *three* times the product of the last found term and the preceding, and *also* the square of the last found term. Hence the ollowing rule.

Rule. 1. The terms being arranged as in the square root, find the root of the first term, place the cube of it under that term and bring down the three next terms for a dividend, the trial divisor of which will be three times the square of the preceding part of the root.

2. By aid of the trial divisor find the second term of the root, and complete the divisor by annexing thrice the product of the last root term and what precedes it, as also the square of this last root term.

3. Multiply the divisor by the new root term, subtract the product from the dividend, and to the remainder annex three new terms from the polynomial. And proceed, as before, to find a third root term; and so on.

Ex. $27x^6-54x^5+63x^4-44x^3+21x^2-6x+1 \mid 3x^2-2x+1$

$27x^6$

$27x^4-18x^3+4x^2 \mid -54x^5+63x^4-44x^3$

$-54x^5+36x^4-8x^3$

$27x^4-36x^3+21x^2-6x+1 \mid 27x^4-36x^3+21x^2-6x+1$

$=3(3x^2-2x)^2-3(3x^2-2x)+1$ $27x^4-36x^3+21x^2-6x+1$

EXAMPLES FOR EXERCISE.

(1) $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$. (2) $x^6+6x^5-40x^3+96x-64$.

(3) $8x^3+36x^2+54x+27$. (4) $x^6-3ax^5+6a^2x^4-7a^3x^3+6a^4x^2-3a^5x+a^6$.

From the preceding general rules are derived the methods for extracting the square and cube roots of numbers, as given in most books on Arithmetic. But as respects the extraction of the cube root of a number, a much more simple and convenient process will be explained hereafter.

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49. **Surds.**—When a quantity having the radical sign, or a fractional exponent attached to it, is such that the indicated root cannot be accurately extracted, the expression is called a *surd*, and sometimes an *irrational quantity*. Quantities which are not surds are called *rational quantities*. The following are all surds, namely, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{8}$, &c., &c., because there is no number whose square is accurately 2, or 5, or 7, or 8, &c.; so also $\sqrt[3]{6}$, $\sqrt[4]{8}$, &c., are surds, because there does not exist any three equal numbers which, multiplied together, will accurately make 6, nor four equal numbers which, multiplied together, will make exactly 8, &c. The above are arithmetical surds: the following are algebraic surds; $\sqrt{a}$, $\sqrt[3]{a^2}$, $\sqrt[4]{a^3}$, $(a+b)^{\frac{1}{2}}$, &c., because no letter, or letters, free from radical signs, can, by the reverse operation of squaring, cubing, &c., produce a, a^2, &c.

On the contrary, $\sqrt{4}$, $\sqrt{9}$, $\sqrt[3]{8}$, $\sqrt[3]{-27}$, $\sqrt[3]{a^6}$, $\sqrt{a^8}$, $\sqrt{(a+b)^4}$, &c., are all rational quantities: the indicated operations can be accurately executed, and the results correctly exhibited: they are 2, 3, 2, -3, a^2, a^4, $(a+b)^2$, &c.

50. It must be noticed, however, that the result of a *square root* operation upon a positive quantity is, in strictness, of a twofold character: it is either $+$, or $-$: thus $\sqrt{4}$ is as much -2 as $+2$, because, whichever of these we multiply by itself, the result is still 4: hence, employing the *double sign*, we are justified in saying that $\sqrt{4}=\pm 2$, that is, that the square root of 4 is *either plus or minus* 2. In like manner, $\sqrt{9}=\pm 3$, $\sqrt{36}=\pm 6$, and so on. And this twofold form equally belongs to every *even* root of a positive quantity, because an even number of factors, whether they be all positive, or all negative, equally gives a *plus* product; thus, $\sqrt[4]{16}=\pm 2$, because $(2)^4$, and $(-2)^4$ equally give 16. Any *odd* root of a quantity is, however, unambiguous, because any odd number of factors, with the same sign, produces that sign: thus $\sqrt[3]{8}=2$, $\sqrt[3]{-8}=-2$, &c., because $2^3=8$, $(-2)^3=-8$, &c.: and it is plain that no other real values of $\sqrt[3]{8}$, $\sqrt[3]{-8}$, &c., exist.

51. **Imaginary or Impossible Quantities.**—But when an even root of a *negative* quantity is indicated, there is implied an impossibility; such root is called an *imaginary* root: no even power of a number, whether the number be positive or negative, can ever produce a *negative* result: thus, such operations as $\sqrt{-4}$, $\sqrt[4]{-16}$, $\sqrt{-a}$, &c., are all of

impossible performance: there is not only no number which, multiplied by itself, will produce -4, but there is no number which, squared, can even approximate to -4. It is true that no number, squared, can produce 2, or 5; but we can assign a number which, when squared, shall approach as near to either of these as we please, as arithmetic shows: thus, $\surd 2 = 1{\cdot}4142\ldots$, and $\surd 5 = 2{\cdot}23606\ldots$ But imaginaries cannot even be approximated to. All that can at present be said, as to the interpretation of imaginary quantities, is, that whenever they necessarily enter into the result of any problem, that problem may be safely inferred to be of impossible solution, or to involve impossible conditions.

52. It is desirable to notice that every single imaginary may be always represented by two factors, one of which is real, though not always rational, and the other the imaginary, $\surd -1$ or $\sqrt[4]{-1}$, &c.:* thus $\surd -4 = \surd(4\times -1) = 2\surd -1$, $\sqrt[4]{-16} = 2\sqrt[4]{-1}$, $\surd -5 = \surd 5 \surd -1$, $\surd -a = \surd a \surd -1$, &c.

Especial care, too, must be taken not to regard such expressions as $\surd(-5)^2$, $\sqrt[4]{(-2)^4}$, $\surd(-a)^2$, &c., as imaginary: in these, the operations indicated by the radical and the exponent, merely neutralize one another, and the quantity to which they are attached remains unaffected by their joint influence: thus, $\surd(-5)^2 = -5$, $\sqrt[4]{(-2)^4} = -2$, $\surd(-a)^2 = -a$, &c. In fact, these expressions are the same as $(-5)^{\frac{2}{2}}$, $(-2)^{\frac{4}{4}}$, $(-a)^{\frac{2}{2}}$, &c. (43).

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53. **Reduction of Surds.**—A surd may be reduced to another of simpler form, whenever the quantity under the radical has a factor upon which the extraction of the root indicated can be actually performed: thus, since $\surd 18 = \surd(9\times 2)$ $\therefore \surd 18 = 3\surd 2$. In like manner, since $\surd(a^3 - a^2x) = \surd\{a^2(a-x)\}$ $\therefore \surd(a^3 - a^2x) = a\surd(a-x)$. If the surd be *fractional*, it can be reduced to an *integral* form by multiplying numerator and denominator by such a quantity as will make the new denominator a complete power corresponding to the root: thus,—

$$\surd\tfrac{2}{3} = \sqrt{\frac{2.3}{3.3}} = \tfrac{1}{3}\surd 6, \text{ also, } \sqrt[3]{\frac{16}{81}} = \sqrt[3]{\frac{8.2}{9.9}} = \sqrt[3]{\frac{8.2.9}{9.9.9}} = \tfrac{2}{9}\sqrt[3]{18}.$$

In like manner, $\sqrt{\dfrac{8x^3}{75a^2y^5}} = \sqrt{\dfrac{4.2x^2x}{25.3a^2y^4y}} = \dfrac{2x}{5ay^2}\sqrt{\dfrac{2x}{3y}} = \dfrac{2x}{5ay^2}\sqrt{\dfrac{6xy}{9y^2}} = \dfrac{2x}{15ay^3}\surd 6xy.$

EXAMPLES FOR EXERCISE.

(1) $\sqrt{\frac{2}{7}} =$.

(2) $\surd 60 =$.

(3) $\surd -54 =$.

(4) $\surd 12a^2x^4y^6 =$.

(5) $\sqrt{\frac{x^2}{y^3}} =$.

(6) $\sqrt{\frac{2a^3}{3b^5}} =$.

(7) $(8a^2b^3)^{-\frac{1}{2}} =$.

(8) $\sqrt[3]{(ax^3 + bx^4)} =$.

(9) $\surd(3a^2x + 6abx + 3b^2x) =$.

(10) $(a+x)\sqrt{\frac{a-x}{a+x}} =$.

54. Besides reducing a surd to its simplest form, it is necessary, when two surds are to be multiplied or divided, the one by the other, that both be reduced to a common surd-index. This is done by using fractional exponents instead of the radical sign, and then bringing the fractions to a common denominator, which denominator is, of course, the surd-index common to both expressions: thus, the surd-indices in $\surd a$ and $\sqrt[3]{b}$, are

* It may be proved, though not conveniently in this place, that any root of -1 is reducible to a form involving no imaginary, but $\surd -1$.

different, namely, 2 and 3: to change the surds into equivalent ones with the same index, we change the fractions $\frac{1}{2}$, $\frac{1}{3}$, in $a^{\frac{1}{2}}$, $b^{\frac{1}{3}}$, into their equivalents $\frac{3}{6}$, $\frac{2}{6}$: the expressions are then $a^{\frac{3}{6}}$, $b^{\frac{2}{6}}$, or $\sqrt[6]{a^3}$, $\sqrt[6]{b^2}$, so that $\sqrt{a}$, $\sqrt[3]{b}$, are thus reduced to equivalent surds with a common surd-index, namely, the index 6.

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55. **Addition and Subtraction of Surds.**—RULE. 1. Reduce the surds to their simplest forms. 2. If the surds be *like* or *similar*, add or subtract the *coefficients*, and annex the common surd part to the result.

But if the surds are not similar surds, the addition or subtraction can only be indicated—

(1) $\sqrt{72}+\sqrt{128}=\sqrt{(36\times2)}+\sqrt{(64\times2)}=6\sqrt{2}+8\sqrt{2}=14\sqrt{2}$.

(2) $9\sqrt[3]{4}-\sqrt[3]{108}=9\sqrt[3]{4}-\sqrt[3]{(27\times4)}=9\sqrt[3]{4}-3\sqrt[3]{4}=6\sqrt[3]{4}$.

(3) $4\sqrt{\frac{2}{5}}-3\sqrt{\frac{5}{27}}=\frac{4}{5}\sqrt{15}-\frac{3}{9}\sqrt{\frac{4}{3}}=\frac{4}{5}\sqrt{15}-\frac{1}{9}\sqrt{15}=\frac{7}{15}\sqrt{15}$.

(4) Add together $\sqrt[3]{24}$, $3\sqrt[3]{32}$, and $\sqrt[3]{192}$. Here $\sqrt[3]{24}=\sqrt[3]{(8\times3)}=2\sqrt[3]{3}$, and $3\sqrt[3]{32}=3\sqrt[3]{(8\times4)}=6\sqrt[3]{4}$, also $\sqrt[3]{192}=\sqrt[3]{(64\times3)}=4\sqrt[3]{3}$: hence the sum is $2\sqrt[3]{3}+6\sqrt[3]{4}+4\sqrt[3]{3}=6(\sqrt[3]{3}+\sqrt[3]{4})$.

EXAMPLES FOR EXERCISE.

(1) $\sqrt{32}+\sqrt{72}=$.
(2) $3\sqrt{\frac{2}{5}}-2\sqrt{\frac{1}{10}}=$.
(3) $\sqrt{27a^4x}+\sqrt{3a^4x}=$
(4) $2\sqrt{18}-3\sqrt{8}+5\sqrt{50}=$.
(5) $\sqrt{a^2x}-\sqrt{x^3}=$.
(6) $\sqrt{a^3b}+\sqrt{ab^3}-b\sqrt{ab}=$.
(7) $\frac{1}{2}\sqrt{\frac{5}{3}}-\frac{1}{4}\sqrt{\frac{3}{5}}+\sqrt{15}=$.
(8) $\sqrt{(a^2-b^2)(a-b)}+b\sqrt{(a+b)}=$.
(9) $3\sqrt{-16}-5\sqrt{-4}=$.
(10) $2\sqrt{72}+\sqrt{24}+a\sqrt{bx^2}=$.

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56. **Multiplication and Division of Surds.**—RULE. Reduce the surds to equivalent ones expressing the same root (54): then multiply or divide, as required—

(1) $\sqrt{2}\times\sqrt[3]{3}=2^{\frac{1}{2}}\times3^{\frac{1}{3}}=2^{\frac{3}{6}}\times3^{\frac{2}{6}}=\sqrt[6]{(2^3\times3^2)}=\sqrt[6]{72}$.

(2) $\sqrt{8}\times\sqrt[3]{16}=8^{\frac{1}{2}}\times16^{\frac{1}{3}}=2.2^{\frac{1}{2}}\times2.2^{\frac{1}{3}}=4\times2^{\frac{3}{6}}\times2^{\frac{2}{6}}=4.2^{\frac{5}{6}}=4\sqrt[6]{32}$.

(3) $(3\sqrt{2}-5\sqrt[3]{2})\div\sqrt{3}=3\sqrt{\frac{2}{3}}-5.2^{\frac{2}{6}}\div3^{\frac{3}{6}}=\sqrt{6}-5\sqrt[6]{\frac{4}{3^3}}\sqrt{6}-\frac{5}{3}\sqrt[6]{108}$.

EXAMPLES FOR EXERCISE.

(1) $2\sqrt{3}\times3\sqrt[3]{4}=$.
(2) $\sqrt[3]{4}\times7\sqrt[3]{6}\times\sqrt[3]{\frac{5}{8}}=$.
(3) $\left(\frac{1}{2}\sqrt{3}\right)^3=$.
(4) $(2\sqrt[3]{2})^4=$.
(5) $\sqrt{12}\div\sqrt[3]{24}=$.
(6) $2\sqrt[3]{\frac{2}{3}}\times\sqrt[3]{\frac{5}{6}}\times\sqrt[3]{16}\div\sqrt[6]{32}=$.
(7) $(3\sqrt{2}-5\sqrt[3]{2})\div\sqrt{3}=$.
(8) $\sqrt[4]{\frac{a}{b}}\div\sqrt{\frac{a}{b}}=$.
(9) $\sqrt[n]{a}\div\sqrt[m]{a}=$.
(10) $\left(\frac{1}{2}\sqrt[6]{18}+\frac{1}{4}\sqrt[4]{8}\right)\times\sqrt{2}=$.

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57. **Binomial Surds.**—In the foregoing examples, every *divisor* has been a monomial surd: when it is a *binomial* surd, and of one or other of the forms $\sqrt{a}\pm\sqrt{b}$, $\sqrt[3]{a}\pm\sqrt[3]{b}$, a multiplier of it may always be found that will render the product rational, so that when a fraction occurs with a denominator of such a form, that den. may be rationalized by multiplying both terms of the fraction by the suitable multiplier. When the form of the binomial surd is $\sqrt{a}\pm\sqrt{b}$, the rationalizing multiplier at once suggests itself, from the property, that the sum

multiplied by the difference of two quantities, gives the difference of their squares:—it is always either $\sqrt{a}-\sqrt{b}$, or $\sqrt{a}+\sqrt{b}$, thus:—
$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$, and $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})=a-b$.....[A]

But when the binomial consists of two *cube* roots, the rationalizing multiplier is not so obvious: in this case it is *trinomial*, and consists of the squares of the two surd terms, and of their product with changed sign. This will be sufficiently seen by inspecting examples (4) and (7) at page 15. As in those examples, x and y may each be anything, we may replace them by $\sqrt[3]{a}$ and $\sqrt[3]{b}$, when we shall obviously have $(\sqrt[3]{a}-\sqrt[3]{b})(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2})=a-b$, and $(\sqrt[3]{a}+\sqrt[3]{b})(\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2})=a+b$...[B].

These expressions [A] and [B] furnish all needful directions for rationalizing binomial surds of the form $\sqrt{a}\pm\sqrt{b}$, or $\sqrt[3]{a}\pm\sqrt[3]{b}$.

(1) $\dfrac{3}{2\sqrt{5}-3\sqrt{7}}=\dfrac{3(2\sqrt{5}+3\sqrt{7})}{(2\sqrt{5}-3\sqrt{7})(2\sqrt{5}+3\sqrt{7})}=\dfrac{3(2\sqrt{5}+3\sqrt{7})}{20-63}=-\dfrac{3}{43}(2\sqrt{5}+3\sqrt{7}).$

(2) $\dfrac{2}{3\sqrt{6}+\sqrt{5}}=\dfrac{2(3\sqrt{6}-\sqrt{5})}{54-5}=\dfrac{2}{49}(3\sqrt{6}-\sqrt{5}).$

(3) $\dfrac{1}{\sqrt[4]{4}+\sqrt[4]{2}}=\dfrac{\sqrt[4]{4}-\sqrt[4]{2}}{\sqrt{4}-\sqrt{2}}=\dfrac{(\sqrt[4]{4}-\sqrt[4]{2})(2+\sqrt{2})}{2}.$

(4) $\dfrac{5}{\sqrt[3]{7}-\sqrt[3]{5}}=\dfrac{5(\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25})}{7-5}=\dfrac{5}{2}(\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25}).$

Examples for Exercise.

(1) $3\div(2\sqrt{7}-3\sqrt{5})=$.
(2) $(2+\sqrt{3})\div(3+\sqrt{3})=$.
(3) $\sqrt{6}\div(\sqrt{8}+\sqrt{3})=$.
(4) $x\div(a-\sqrt{x})=$.
(5) $2\sqrt{5}-3\sqrt{2})\div(2\sqrt{2}+3\sqrt{5})=$.
(6) $3\div(\sqrt[4]{5}+\sqrt[4]{3})=$.
(7) $2\div(\sqrt[3]{3}-\sqrt[3]{2})=$.
(8) $\dfrac{a}{\sqrt[3]{x}+\sqrt[3]{y}}=$.
(9) $\dfrac{1+\sqrt{\frac{1}{3}}}{1-\sqrt{\frac{1}{3}}}-\dfrac{1-\sqrt{\frac{1}{3}}}{1+\sqrt{\frac{1}{3}}}=$.

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58. *To extract the square root of a binomial, one of whose terms is rational, and the other a quadratic surd.*

Put $a+\sqrt{b}=(x+y)^2$, and $a-\sqrt{b}=(x-y)^2$, then $a=x^2+y^2$...[1]
Also, $(a+\sqrt{b})(a-\sqrt{b})$, $a^2-b=(x^2-y^2)^2$ $\therefore$ $\sqrt{(a^2-b)}=x^2-y^2$...[2].
From [1] and [2], by addition and subtraction,

$$x^2=\frac{a+\sqrt{(a^2-b)}}{2},\quad y^2=\frac{a-\sqrt{(a^2-b)}}{2}$$

$$\therefore\ x+y=\sqrt{(a+\sqrt{b})}=\sqrt{\frac{a+\sqrt{(a^2-b)}}{2}}+\sqrt{\frac{a-\sqrt{(a^2-b)}}{2}}\quad\ldots\quad[3]$$

$$x-y=\sqrt{(a-\sqrt{b})}=\sqrt{\frac{a+\sqrt{(a^2-b)}}{2}}-\sqrt{\frac{a-\sqrt{(a^2-b)}}{2}}\quad\ldots\quad[4]$$

Consequently, whenever a^2-b happens to be a square, the complex surd $\sqrt{(a\pm\sqrt{b})}$ may be expressed by the sum or difference of two simple surds, but not under other circumstances.

59. When in any particular example the above-mentioned condition has place, these general formulæ will enable us at once to write down the desired simplified expression for the square root of the binomial. But it is nearly as easy to go through the process by which the general results have been obtained, thus:—

Required the square root of $16-\sqrt{87}$. Here the necessary condition is fulfilled, for $16^2-87=169=13^2$. Put then $16-\sqrt{87}=(x-y)^2$...[1], and $16+\sqrt{87}=(x+y)^2$...[2].

Multiplying these together, $256-87=(x^2-y^2)^2 \therefore 13=x^2-y^2$
and adding and subtracting $16=x^2+y^2$

$\therefore 29=2x^2$, and $3=2y^2 \therefore x=\sqrt{\frac{29}{2}}=\frac{1}{2}\sqrt{58}$, and $y=\sqrt{\frac{3}{2}}=\frac{1}{2}\sqrt{6}$.

$$\therefore x-y=\sqrt{(16-\sqrt{87})}=\tfrac{1}{2}(\sqrt{58}-\sqrt{6}).$$

EXAMPLES FOR EXERCISE.

(1) $\sqrt{(2+\sqrt{3})}=$.
(2) $\sqrt{(8+\sqrt{39})}=$.
(3) $\sqrt{(10-\sqrt{96})}=$.
(4) $\sqrt{(7-2\sqrt{10})}=$.

(5) $\sqrt{(8+2\sqrt{7})}=$.
(6) $\sqrt{\left(42+3\sqrt{174\tfrac{2}{9}}\right)}=$.
(7) $\sqrt[4]{(6+\sqrt{20})}=$.

60. The following are interesting properties of quadratic surds.

1. The product of two dissimilar quadratic surds cannot be rational. If possible let $\sqrt{a}+\sqrt{b}=$ a rational quantity $=p$, then $ab=p^4$.

$$\therefore b=\frac{p^2}{a}=\frac{p^2}{a^2}a, \text{ and } \therefore \sqrt{b}=\frac{p}{a}\sqrt{a}.$$

so that $\sqrt{a}$ and $\sqrt{b}$ are really *similar* surds, having the same surd-factor $\sqrt{a}$.

2. A quadratic surd can never be equal to the sum or difference of a quadratic surd and a rational quantity.

If possible, let $\sqrt{a}=b\pm\sqrt{c} \therefore a=b^2\pm 2b\sqrt{c}+c \therefore \pm\sqrt{c}=\dfrac{a-c-b^2}{2b}$,

so that the surd $\sqrt{c}$ is equal to a rational quantity, which is absurd.

3. In every equation of the form $a+\sqrt{b}=x+\sqrt{y}$, if $\sqrt{b}$ and $\sqrt{y}$ are surds, then necessarily, $a=x$, and $\therefore b=y$.

For since $a+\sqrt{b}=x+\sqrt{y} \therefore \sqrt{b}=x-a+\sqrt{y} \therefore x-a=0$, or $x=a$, otherwise $\sqrt{b}$ would be $=$ a rational quantity and a surd; $\therefore$ also $b=y$. If $a+\sqrt{b}=0$, then, necessarily, $a=0$, and $b=0$.

4. If $\sqrt{(a+\sqrt{b})}=x+y$, (x, y being one or both quadratic surds) then also must $\sqrt{(a-\sqrt{b})}=x-y$, provided that $\sqrt{b}$ be a surd. For since $a+\sqrt{b}=x^2+2xy+y^2$, we must have, equating the rational and surd parts, $a=x^2+y^2$, $\sqrt{b}=2xy \therefore a-\sqrt{b}=x^2-2xy+y^2=(x-y)^2 \therefore \sqrt{(a-\sqrt{b})}=x-y$.

NOTE.—*Any* two quantities may always be expressed by the sum and difference of two others; for, assuming this sum and difference to be $x+y$ and $x-y$, we can always arrive at determinate values for x and y. The above, therefore, is only a particular case of a perfectly general theorem. It is from the proof of the property in this single case that the expressions [3], [4] at art. (58) are deduced in all the books; so that they are limited to the condition of $\sqrt{b}$ being a *surd*. But they are entirely unrestricted, being absolute identities; and the investigation of them, given at (58), is quite independent of the condition that $\sqrt{b}$ is a surd. Introducing this condition, however, the process for getting $\sqrt{(a\pm\sqrt{b})}$ may be conducted a little differently; thus, taking the example worked at (59) we may proceed thus:—

Put $\sqrt{(16+\sqrt{87})}=\sqrt{x}+\sqrt{y}$, one of these, at least, being a surd, then $16+\sqrt{87}=x+y+2\sqrt{xy} \therefore x+y=16$, and $2\sqrt{xy}=\sqrt{87} \therefore 4xy=87$

$\therefore (x+y)^2-4xy=16^2-87=(x-y)^2 \therefore x-y=13$: hence, $x=\dfrac{29}{2}$, $y=\dfrac{3}{2}$.

$$\therefore \sqrt{(16-\sqrt{87})}=\sqrt{\frac{29}{2}}-\sqrt{\frac{3}{2}}=\frac{1}{2}(\sqrt{58}-\sqrt{6}).$$

The objection to investigating the formulæ [3], [4] upon this plan may be removed by referring, not to the principle 4, above, to prove that $\pm 2\sqrt{xy} = \pm\sqrt{b}$, but to the fact of the double sign of the radical, whether $\sqrt{b}$ be a surd or not.

61. **Imaginary Quantities.**—All the preceding rules of operation equally apply, whether the surds be real or imaginary; but in dealing with the latter, it will be convenient to replace the imaginary $\sqrt{-1}$ by the symbol i; that is, to put $-1 = i^2$, in order to avoid all confusion as to signs. Thus,—

(1) $(4+\sqrt{-3})(3+\sqrt{-5}) = (4+\sqrt{3}\sqrt{-1})(3+\sqrt{5}\sqrt{-1}) = (4+i\sqrt{3})(3+i\sqrt{5}) =$
$12+3i\sqrt{3}+4i\sqrt{5}+i^2\sqrt{15} = 12-\sqrt{15}+(3\sqrt{3}+4\sqrt{5})\sqrt{-1}.$

(2) $(2+\sqrt{-2})(3-\sqrt{-4}) = (2+i\sqrt{2})(3-2i) = 6+3i\sqrt{2}-4i-2i^2\sqrt{2} =$
$6+2\sqrt{2}+(3\sqrt{2}-4)\sqrt{-1}.$

(3) $\dfrac{2+\sqrt{-2}}{3-\sqrt{-3}} = \dfrac{2+i\sqrt{2}}{3-i\sqrt{3}} = \dfrac{(2+i\sqrt{2})(3+i\sqrt{3})}{9-3i^2} = \dfrac{1}{12}(6+3i\sqrt{2}+2i\sqrt{3}+i^2\sqrt{6}) =$
$\dfrac{1}{2}-\dfrac{1}{12}\sqrt{6}+\left(\dfrac{1}{4}\sqrt{2}+\dfrac{1}{6}\sqrt{3}\right)\sqrt{-1}.$

(4) $\dfrac{4+\sqrt{-2}}{2+\sqrt{-3}} = \dfrac{4+i\sqrt{2}}{2+i\sqrt{3}} = \dfrac{(4+i\sqrt{2})(2-i\sqrt{3})}{4-3i^2} =$
$(8+\sqrt{6}+2(\sqrt{2}-2\sqrt{3})\sqrt{-1}) \div 7$

(5) $(a+bi)(c+di) = (ac-bd)+(ad+bc)i.$

(6) $\dfrac{a+bi}{c+di} = \dfrac{(a+bi)(c-di)}{(c+di)(c-di)} = \dfrac{ac+bd}{c^2+d^2} + \dfrac{bc-ad}{c^2+d^2}i.$

NOTE.—If expressions of the form $a+b\sqrt{-1}$ be united together by the additive or subtractive signs, or by both, it is plain that the result of the combination will always be of the same form, namely, $A+B\sqrt{-1}$. And from the last two examples, it appears that when such expressions are combined by multiplication or division, or by both, the results are still of the same form.

EXAMPLES FOR EXERCISE.

(1) $(2-\sqrt{-3})\sqrt{-7} =$.

(2) $(2-\sqrt{-5})(3+\sqrt{-2}) =$.

(3) $\left(7+\dfrac{1}{2}\sqrt{-1}\right) \div (1+2\sqrt{-1}) =$.

(4) $(2+3\sqrt{-2})(3-2\sqrt{-1}) =$.

(5) $\dfrac{3}{\sqrt{-5}+\sqrt{-6}} =$.

(6) $\dfrac{2+\sqrt{-1}}{3\sqrt{-2}-2\sqrt{-3}} =$.

(7) $(3-2\sqrt{-4})^3 =$.

(8) $\sqrt{(a \pm b\sqrt{-1})} =$.*

62. We shall conclude this subject by giving an interesting example of the use of imaginary quantities in investigating certain properties of *real* quantities. [As above, i is put for $\sqrt{-1}$, and $\therefore i^2 = -1$.]

To prove that the sum of two squares multiplied by the sum of two squares, always gives the sum of two squares for the product.

Let the factors be a^2+b^2, and $a'^2+b'^2$. Each of these may be produced from a pair of imaginary factors; for they arise from $(a+bi)(a-bi)$, and $(a'+b'i)(a'-b'i)$, so that the product of the original factors is the product of the four factors $a+bi$, $a-bi$, $a'+b'i$, $a'-b'i$. Now, by actually multiplying the first and third of these, and then the second and fourth,

* The result will be found to be of the same form: hence the square root of that result will be of the same form, and so on. (See foot-note, page 41.)

we have the following pair, namely, $(aa'-bb')+(ab'+ba')i$, $(aa'-bb')-(ab'+ba')i$, of which the product is $(aa'-bb')^2+(ab'+ba')^2$;
$\therefore (a^2+b^2)(a'^2+b'^2)=(aa'-bb')^2+(ab'+ba')^2$, which proves the theorem.

For example: let $a=5$, $b=2$, $a'=3$, $b'=4$: then $(5^2+2^2)(3^2+4^2)=(5.3-2.4)^2+(5.4+2.3)^2=7^2+26^2=725$. Let now a and b be interchanged, that is, put $a=2$, $b=5$; the first side remains unaltered, but the second side is $=(2.3-5.4)^2+(2.4+5.3)^2=14^2+23^2=725$. If a' and b' had been interchanged, we should have got the same results. There are thus two ways in which the product of the sum of two squares may be expressed by the sum of two squares: in the instance here adduced, $(5^2+2^2)(3^2+4^2)=7^2+26^2=14^2+23^2$.*

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63. **Quadratic Equations.**—A quadratic is an equation into which the *square* of the unknown quantity enters, and enters in such a way as to be removable only by the application of a special rule for the purpose. In simple equations, as has already been seen, the square of the unknown sometimes occurs; but then its elimination could always be effected by transposition, division, or some other of the ordinary operations performed upon simple equations. The general form of a quadratic equation with one unknown quantity is this, namely, $ax^2+bx=c$; if the second term be absent, the form is $ax^2=c$, which is called a *pure* quadratic, the more general one, $ax^2+bx=c$, being an *adfected* quadratic. A pure quadratic requires no special rule for its solution; for from $ax^2=c$, we at once get $x=\sqrt{\frac{c}{a}}$.

64. The rule for solving an adfected quadratic is deduced from the following considerations. We already know that $x^2+2ax+a^2=(x+a)^2$, and that $x^2-2ax+a^2=(x-a)^2$ whatever be represented by a; and we see that the *third* term, in the first member of each of these *identities*, is nothing but the square of half the coefficient of the second term. Consequently every expression consisting of two terms of the form x^2+mx, or x^2-mx, can always be made a complete square by merely adding $\left(\frac{m}{2}\right)^2$ for a third term. Thus the expressions x^2+6x, x^2-8x, x^2+3x, x^2-5x, &c., become in this way converted into the complete squares $x^2+6x+9=(x+3)^2$; $x^2-8x+16=(x-4)^2$; $x^2+3x+(\frac{3}{2})^2=(x+\frac{3}{2})^2$; $x^2-5x+(\frac{5}{2})^2=(x-\frac{5}{2})^2$, &c. The *root* of each square consisting of two terms, namely, the root of the first term (x^2), and half the coeff. of the second term taken with its proper sign. Hence the following rule.

65. **To solve a Quadratic with one unknown.**—Rule.

1. Bring the unknown terms to one side of the equation, and the known terms to the other.

2. If the unknown square have a coefficient, other than unity, divide by that coefficient.

3. Add the square of half the coefficient of the simple unknown to each side of the equation; the unknown side will then be a complete square.

4. Take now the root of each side, affixing the double sign $\pm$ to that

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* The property in the text may be more briefly proved as follows:—

$$(a^2+b^2)(a'^2+b'^2)=(aa')^2+(bb')^2+(ab')^2+(ba')^2$$
$$=\{(aa')^2-2aa'bb'+(bb')^2\}+\{(ab')^2+2aa'bb'+(ba')^2\}$$
$$=(aa'-bb')^2+(ab'+ba')^2.$$

of the *known* side (50), and the quadratic will thus be reduced to a simple equation.

(1) Given $3x^2=12x+15$ to find the values of x.

Trans. and dividing by 3, $x^2-4x=5$.

Completing the square, by adding 2^2 to each side, $x^2-4x+4=9$.

Extracting the root (that is, writing for the first side, $x-2$), we have $x-2=\pm 3$

$$\therefore x=2+3, \text{ or } x=2-3,$$

that is, $x=5$, or -1.

If we substitute 5 for x in the given equation, there results $75=60+15$; and if we substitute -1 for x, there results $3=-12+15$; the equation being true for either value of x.

(2) $2x^2+4x-3=11$.

Trans., $2x^2+4x=14$.

$\div 2$, $x^2+2x=7$.

Completing the square,

$$x^2+2x+1=8.$$

Extracting the root,

$$x+1=\sqrt{8}=2\sqrt{2}.$$

$$\therefore x=-1+2\sqrt{2}, \text{ or } -1-2\sqrt{2}.$$

(3) $3x^2-14x+15=0$.

Trans., $3x^2-14x=-15$.

$$\div 3,\ x^2-\frac{14}{3}x=-5.$$

Completing the square,

$$x^2-\frac{14}{3}x+\left(\frac{7}{3}\right)^2=\frac{49}{9}-5=\frac{4}{9}.$$

Extracting the root,

$$x-\frac{7}{3}=\pm\frac{2}{3}\therefore x=\frac{7\pm 2}{3}=3, \text{ or } 1\frac{2}{3}.$$

(4) $\dfrac{x}{3}-\dfrac{\sqrt{x}}{2}=\dfrac{3}{2}$.

Multiply by 6, the *L.C.M.* of den.,

$$2x-3\sqrt{x}=9.$$

Put y for $\sqrt{x}$, $\therefore 2y^2-3y=9$.

$$\therefore y^2-\frac{3}{2}y=\frac{9}{2}.$$

Completing the square,

$$y^2-\frac{3}{2}y+\frac{9}{16}=\frac{9}{2}+\frac{9}{16}=\frac{81}{16}.$$

Extracting the root,

$$y-\frac{3}{4}=\pm\frac{9}{4}\therefore y=\frac{3}{4}\pm\frac{9}{4}=3, \text{ or } -\frac{3}{2}.$$

$$\therefore x=y^2=9, \text{ or } \frac{9}{4}.$$

NOTE.—Every equation of the form

$$x^{2n}+ax^n=b \quad . \quad . \quad . \quad [A]$$

may be solved as a quadratic by putting y for x^n: in the above example $n=\frac{1}{2}$. By this change

[A] becomes $y^2+ay=b$ . . . [B]

(5) $(x-12)(x+2)=0$.

By actually multiplying, the equation is the quadratic

$$x^2-10x-24=0,$$

which may be solved as above. But the factors of the first member being given, the equation is satisfied by equating either factor to 0; thus, from

$x-12=0$, we get $x=12$, and from

$x+2=0$, we get $x=-2$, which are the two values of x. Whenever we know the factors of an expression, if that expression be $=0$, the equation may always be solved by equating each factor to 0, since the whole can become 0 only by one or other of the factors becoming 0.

(6) $x+\sqrt{(5x+10)}=8$.

Trans., $\sqrt{(5x+10)}=8-x$.

Squaring, $5x+10=64-16x+x^2$.

Trans., $x^2-21x=-54$.

Completing the square,

$$x^2-21x+\left(\frac{21}{2}\right)^2=-54+\frac{441}{4}=\frac{225}{4}.$$

Extracting the root,

$$x-\frac{21}{2}=\pm\frac{15}{2}\therefore x=\frac{21\pm 15}{2}=18, \text{ or } 3.$$

(7) $\dfrac{x}{2}\left(\dfrac{x^2}{4}-1\right)=\dfrac{x-2}{4}$.

Put y for $\dfrac{x}{2}$, then the equation is

$$y(y^2-1)=\frac{1}{2}(y-1).$$

Dividing by $y-1$, it becomes

$$y(y+1)=\frac{1}{2}, \text{ or } y^2+y=\frac{1}{2}.$$

Completing the square,

$$y^2+y+\frac{1}{4}=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}.$$

Extracting the root,

$$y+\frac{1}{2}=\sqrt{\frac{3}{4}}=\frac{1}{2}\sqrt{3}\therefore y=-\frac{1}{2}\pm\frac{1}{2}\sqrt{3}$$

$$\therefore x=2y=-1\pm\sqrt{3}.$$

(8) $x^{-4}-9x^{-2}=-20$. Put $x^{-2}=y$

$$\therefore y^2-9y=-20.$$

Completing the square,

$$y^2-9y+\left(\frac{9}{2}\right)^2=-20+\frac{81}{4}=\frac{1}{4}.$$

Extracting the root,

$$y-\frac{9}{2}=\pm\frac{1}{2}\therefore y=\frac{9\pm1}{2}=5,\text{ or }4,$$

that is, $\frac{1}{x^2}=5$, or $4\therefore x^2=\frac{1}{5}$, or $\frac{1}{4}$,

$$\therefore x=\pm\sqrt{\frac{1}{5}},\text{ or }\pm\frac{1}{2}$$

$$=\pm\frac{1}{5}\sqrt{5},\text{ or }\pm\frac{1}{2}.$$

There are thus four values for x, either of which will satisfy the proposed equation, which is in fact of the *fourth* degree.

(9) $$\frac{a+x}{\sqrt{a+\sqrt{(a-x)}}}=\frac{a-x}{\sqrt{a-\sqrt{(a-x)}}}$$

$$\therefore\frac{a+x}{a-x}=\frac{\sqrt{a+\sqrt{(a-x)}}}{\sqrt{a-\sqrt{(a-x)}}}.$$

Applying the principle at (38)

$$\frac{a}{x}=\frac{\sqrt{a}}{\sqrt{(a-x)}}\therefore\frac{a^2}{x^2}=\frac{a}{a-x}.$$

Clearing fractions, $a^3-a^2x=ax^2$.

Dividing by a and trans., $x^2+ax=a^2$.

Completing the square,

$$x^2+ax+\frac{a^2}{4}=a^2+\frac{a^2}{4}=\frac{5a^2}{4}.$$

Extracting the root,

$$x+\frac{a}{2}=\pm\frac{a}{2}\sqrt{5}\therefore x=\frac{-a\pm a\sqrt{5}}{2},$$

$$\text{or } x=\frac{a}{2}(-1\pm\sqrt{5}).$$

(10) $x^2-2x+6\sqrt{(x^2-2x+5)}=11$.

Add 5 to each side, then

$(x^2-2x+5)+6\sqrt{(x^2-2x+5)}=16$,

or, putting y^2 for x^2-2x+5, $y^2+6y=16$.

Completing the square, $y^2+6y+9=25$.

Extracting the root,

$y+3=\pm5\therefore y=2$, or -8

$\therefore y^2=x^2-2x+5=4$ . . . (1).

or $y^2=x^2-2x+5=64$ . . . (2).

From (1), $x^2-2x=-1$.

Completing the square, $x^2-2x+1=0$.

Extracting the root, $x-1=0\therefore x=1$.

From (2), $x^2-2x=59$.

Completing the square,

$x^2-2x+1=60$.

Extracting the root,

$x-1=\pm\sqrt{60}=\pm2\sqrt{15}\therefore x=1\pm2\sqrt{15}$.

(11) $ax^2+bx=c$.

$\div a$, $x^2+\frac{b}{a}x=\frac{c}{a}$.

Completing the square,

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{c}{a}+\frac{b^2}{4a^2}.$$

Extracting the root,

$$x+\frac{b}{2a}=\sqrt{\frac{4ac+b^2}{4a^2}}=\frac{\sqrt{(4ac+b^2)}}{2a}$$

$$\therefore x=\frac{-b\pm\sqrt{(4ac+b^2)}}{2a}.$$

This is a general formula for the solution of every quadratic equation, a, b, c, being any values whatever.

(12) $$x-1=2\left(1+\frac{1}{\sqrt{x}}\right).$$

Subtract 1 from each side, then

$$x-2=1+\frac{2}{\sqrt{x}}$$

Add now $\frac{1}{x}$ to each side,

$$\therefore x-2+\frac{1}{x}=1+\frac{2}{\sqrt{x}}+\frac{1}{x}.$$

It is easy to see that each side of this equation is a complete square, the equa. being the same as

$$\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2=\left(1+\frac{1}{\sqrt{x}}\right)^2$$

$$\therefore\sqrt{x}-\frac{1}{\sqrt{x}}=1+\frac{1}{\sqrt{x}}.$$

$$\therefore\sqrt{x}=1+\frac{2}{\sqrt{x}}.$$

$$\therefore x=\sqrt{x}+2,$$

or, putting y for $\sqrt{x}$, and trans.,

$$y^2-y=2.$$

Completing the square,

$$y^2-y+\frac{1}{4}=2+\frac{1}{4}=\frac{9}{4}.$$

Extracting,

$$y-\frac{1}{2}=\pm\frac{3}{2}\therefore y=2,\text{ or }-1$$

$$\therefore x=y^2=4,\text{ or }1.$$

66. The learner may, perhaps feel disposed to inquire: how is it that the double sign is prefixed to the square root of the known side of the equation, and not to that of the unknown? The answer is, that it is *because* one side is unknown that no sign is prefixed: we look to the known side for the *complete* interpretation of the unknown: this interpretation furnishing sign as well as value: in the absence of it, we know nothing about the unknown side, and therefore suppress all prefix to it.

EXAMPLES FOR EXERCISE.

(1) $x^2+21x=100$.
(2) $x^2-26x+105=0$.
(3) $3x^2+52x=256$.
(4) $5x^2-12x+2=11$.
(5) $16x^2-72x+17=0$.
(6) $x+\sqrt{(10x+6)}=9$.
(7) $x+\sqrt{(5x-10)}=8$.
(8) $\sqrt{x}+2=\sqrt{(7+2x)}$.
(9) $\frac{x+1}{x-1}-\frac{x-1}{x+1}=1$.
(10) $x^4-8x^2=9$.
(11) $x^2-7x+\sqrt{(x^2-7x+18)}=24$.
(12) $3x^2-9x-4\sqrt{(x^2-3x+5)}+11=0$.
(13) $4\left(x+\frac{3}{x}\right)^2-5\left(x+\frac{3}{x}\right)=44$.
(14) $\sqrt{\frac{3x}{x+1}}-\sqrt{\frac{x+1}{3x}}=2$.
(15) $\sqrt{x^3}-2\sqrt{x}-x=0$.
(16) $\frac{1}{x^3}+\frac{1}{\sqrt{x^3}}=2$.
(17) $\sqrt[4]{(x+21)}+\sqrt{(x+21)}=12$.
(18) $\frac{x+\sqrt{(x^2-1)}}{x-\sqrt{(x^2-1)}}=x$.
(19) $\sqrt{x^5}+\sqrt{x^3}=6\sqrt{x}$.
(20) $x^4-2x^3+x=6$.
(21) $x^2+\frac{1}{x^2}+x+\frac{1}{x}=4$.
(22) $x^2+\frac{1}{x^2}+4\left(x+\frac{1}{x}\right)=-3$.

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67. **Second Method of Solving a Quadratic.**—When an equation, by the necessary preliminary operations, is reduced to a quadratic of the common form, the preceding method of solving it, by completing the square, frequently introduces *fractions*, though none occur in the equation itself. The method now to be explained is free from this objection, and is, therefore, well deserving of the student's attention.

If each side of the general quadratic $ax^2+bx=c$, be multiplied by $4a$, the result may evidently be written thus:—$(2ax)^2+2b(2ax)=4ac$, the first member of which becomes a complete square, without fractions, by adding to it b^2 for the third term: we thus have $(2ax)^2+2b(2ax)+b^2=4ac+b^2$, or, $(2ax+b)^2=4ac+b^2$, $\therefore 2ax+b=\sqrt{(4ac+b^2)}$. And in this way may any quadratic be reduced to a simple equation at a single step, and at the same time the introduction of fractions be avoided. The formula expressed in words is as foll·ws:—

RULE II. 1. Double the coefficient of x^2 in the proposed quadratic: this will give the coefficient of x in the reduced simple equation.

2. To the first term of the simple equation thus found, connect, with its own sign, the coef. of x in the quadratic: the first side of the simple equation will then be complete.

3. For the second side, multiply the second side of the quadratic (namely c) by 4 times the coef. of x^2, add the square of the next coef. to the result, and prefix to the two terms thus found the radical sign.

NOTE.—If the first and second coefficients, or the second and third (b and c), be both divisible by 2 or by 4, the division had better be performed; for though a fraction may thus be introduced into the other term, it will disappear from the resulting simple equation.

This rule merely describes how from $ax^2+bx=c$ to derive $2ax+b=\sqrt{(4ac+b^2)}$. Take for instance, the equation $6x^2+5x=1$, then by the rule, or by the above formula, $12x+5=\sqrt{(24+25)}=\sqrt{49}=\pm 7$.

$$\therefore x=\frac{-5\pm 7}{12}=\frac{1}{6}, \text{ or } -1.$$

Again: let the equation be $\frac{1}{2}x^2-3x=\frac{7}{2}$ $\therefore x-3=\sqrt{(7+9)}=\pm 4$

$\therefore x=3\pm 4=7$ or -1. Lastly, let the equation be $5x^2-7x=-3$.

$$\therefore 10x-7=\surd(-60+49)=\surd-11 \therefore x=\frac{7\pm\surd-11}{10}$$

68. **General Theory of Quadratic Equations.**—In discussing the general theory of equations, it is found convenient to have all the significant terms on one side of the equation, and zero, or 0 on the other side: we shall, therefore, write the general quadratic thus: $ax^2+bx+c=0$, where a, b, c, are any positive or negative values whatever. For the solution of this equation the formula is (see ex. 11, p. 48, changing the sign of c)

$$x=\frac{-b\pm\surd(b^2-4ac)}{2a}\ldots\ldots(A)$$

from which we deduce the following inferences, in the expression for which it will be convenient to use two signs not hitherto employed, namely, the signs of *inequality*. These are $>$ and $<$, the former of which placed between two quantities implies that the first is *greater* than the second, the latter, that the first is *less* than the second.

If $b^2=4ac$, the roots are real and equal, for then $x=-\frac{b}{2a}\pm0$, or $-\frac{b}{2a}$, and $-\frac{b}{2a}$.

$b^2>4ac$, ,, ,, real and unequal, for then b^2-4ac is positive.
[This condition is *always* fulfilled when c is negative.]

$b^2<4ac$, ,, ,, imaginary, for then b^2-4ac is negative,

consequently, the character of the roots of a quadratic may always be ascertained without solving the equation. Thus, taking the three equations just solved, namely, $6x^2+5x-1=0$, $\frac{1}{2}x^2-3x-\frac{7}{2}=0$, $5x^2-7x+3=0$, we see that the first two necessarily fulfil the second condition, c being negative, but in the third $7^2<4.5.3$ $\therefore$ the roots are imaginary. The sum of the two values of x in (A) is $-\frac{b}{a}$, and their product is $\frac{c}{a}$: we infer, therefore, that when any quadratic is divided by the leading coefficient, and is thus in the form $x^2+\frac{b}{a}x+\frac{c}{a}=0$, that the coef. of the second term *with its sign changed*, is always equal to the sum of the roots, and that the third term, without change of sign, is always equal to the product of the roots.* We may, therefore, readily frame a quadratic that shall have an assigned pair of roots: thus, let the assigned roots be 3 and 5, then the quadratic to which these roots belong will be $x^2-8x+15=0$. Let the roots be -3 and 5, then the quadratic is $x^2-2x-15=0$. Let them be -3 and -5, then the equation is $x^2+8x+15=0$. Generally, let the roots be r, r', then the quadratic to which they belong is $x^2-(r+r')x+rr'=0$, and as the first member of this is the product $(x-r)(x-r')$, it follows that every quadratic expression of the form $x^2+\frac{b}{a}x+\frac{c}{a}$, or x^2+mx+n, is compounded of two simple factors, found thus. Equate the expression to 0: find the roots r, r' of the equation, then the simple factors are $(x-r)$, $(x-r')$ $\therefore$ $ax^2+bx+c=a(x-r)(x-r')$.

NOTE.—From the formation of a square, it is obvious that $(x-a)^2$ may always be converted into $(x+a)^2$ by adding $4ax$; and that $(x+a)^2$ will

* In speaking of the roots of an equation, it is always understood that the leading coefficient of the equation, when all the terms are arranged on one side, is positive.

become $(x-a)^2$ by subtracting $4ax$; and from this fact we may easily solve an interesting problem, namely:—To find two integral squares whose sum shall be an integral square. The solution is

$$x^2+\left(\frac{x^2-1}{2}\right)^2=\left(\frac{x^2+1}{2}\right)^2$$

where x is any *odd* integer whatever. When x is odd, x^2 must be odd, and, therefore, $x^2\pm1$ must be divisible by 2. Suppose $x=7$, then we have $7^2+24^2=25^2$. Suppose $x=9$, then $9^2+40^2=41^2$. The root of the third square always exceeds that of the second by unity, and the sum of these roots is always the first square.

69. *Questions producing Quadratic Equations with one unknown Quantity.* (1) A company of travellers engaged a conveyance for 8*l.* 15*s.*; but at the end of the journey two found themselves without money; in consequence of which each of the others had 10*s.* more to pay: how many persons were there in company? Let x be the number, then $\frac{175}{x}$ shillings is the fair share of each, and $\frac{175}{x-2}$ is what those who contributed actually paid: hence, by the question,

$$\frac{175}{x-2}=\frac{175}{x}+10 \therefore 175x=175x-350+10x^2-20x \therefore 10x^2-20x=350 \therefore x^2-2x=35.$$

Completing the square, $x^2-2x+1=36 \therefore x-1=\pm6 \therefore x=7$, or -5. Consequently there were *seven* persons. The solution $x=-5$ is, of course, inadmissible: either value of x satisfies the *equation*, but as the *question* is such as to restrict the answer to a positive number, the negative solution must, of course, be rejected. And similar remarks apply to the values of x in the next question.

(2) Find a number, such, that if 3 times its square root be taken from 5 times its fourth root, the remainder may be $\frac{4}{3}$.

Let x be the number, then $5\sqrt[4]{x}-3\sqrt{x}=\frac{4}{3}$, or putting y for $\sqrt[4]{x}$,

$$15y-9y^2=4 \therefore 9y^2-15y=-4.$$

$\therefore$ Rule II. Page 49, $18y-15=\sqrt{(-144+225)}=\sqrt{81}=\pm9$

$$\therefore y=\frac{15\pm9}{18}=\frac{4}{3}, \text{ or } \frac{1}{3} \therefore y^4=x=\frac{256}{81}, \text{ or } \frac{1}{81}, \text{ that is, } 3\tfrac{13}{81}, \text{ or, } \frac{1}{81}.$$

(3) A and B leave the same place, at the same time, to travel a distance of 150 miles. A, by travelling 3 miles an hour faster than B, completes the distance 8h. 20m. before B; at what rate per hour did A and B travel? Suppose B goes x miles an hour, then A goes $x+3$: the time occupied by A is, therefore, $\frac{150}{x+3}$ hours, and by B, $\frac{150}{x}$ hours; and by the question,

$$\frac{150}{x+3}+8\tfrac{1}{3}=\frac{150}{x}, \text{ that is, } \frac{150}{x+3}+\frac{25}{3}=\frac{150}{x}, \text{ or } \div25, \frac{6}{x+3}+\frac{1}{3}=\frac{6}{x},$$

clearing fractions, $18x+x^2+3x=18x+54 \therefore x^2+3x=54.$

$$\therefore \text{Rule II., } 2x+3=\sqrt{(216+9)}=\sqrt{225}=\pm15 \therefore x=\frac{-3\pm15}{2}=6, \text{ or } -9.$$

Hence B goes 6 miles an hour, and A goes $x+3=9$ miles an hour.

(1) Two travellers, A and B, start from the same place, at the same time, on a journey of 90 miles. A rides one mile an hour more than B, and arrives at his destination an hour before him: at what rate per hour did each travel?

(2) A wine-merchant sold 7 dozen of sherry and 12 dozen of claret for 50*l.*: he sells 3 dozen more of sherry for 10*l.* than of claret for 6*l.* Required the price per dozen of each.

(3) Divide 20 into two parts, such that the product of the whole and one of the parts may be equal to the square of the other part.

(4) Is it possible to divide 14 into two parts, such that their product may be 50?

(5) In a certain right-angled triangle the difference between the base and hypotenuse is 8 inches, and the difference between the perpendicular and hypotenuse is 4 inches: required the three sides.

(6) Divide 11 into two parts, so that the sum of the cubes of those parts may be 407.

(7) Twenty work-people, men and women, receive 2*l.* 8*s.* for a day's work: the men 24*s.* and the women, 24*s.*; but the men receive 1*s.* each more than the women: how many men were there?

(8) A and B are employed to dig a trench: A alone digs half, and leaves the other half to B; the work is finished in 25 hours. They are afterwards employed to dig a similar trench, when, both working together, it is dug in 12 hours: in how many hours could each alone dig such a trench?

(9) What two numbers are those whose sum is 39, and the sum of their cubes 17199?

[Additional questions, solvable by quadratics with one unknown, will be found at page 62.]

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70. **Simultaneous Equations.** *When one is a simple Equation and the other a Quadratic.* RULE. 1. Solve the simple equation for one of the unknown quantities. 2. Substitute the expression for this unknown in the quadratic, from which that unknown is thus eliminated, and then solve the equation for the other unknown.

(1) $\left.\begin{matrix} 2x-3y=1 \\ 2x^2+xy-5y^2=20 \end{matrix}\right\}$ From the first, $x=\frac{1+3y}{2}$. Substituting in the second,

$$\frac{(1+3y)^2}{2}+\frac{1+3y}{2}y-5y^2=20 \therefore 1+6y+9y^2+y+3y^2-10y^2=40.$$

$$\therefore 2y^2+7y=39 \therefore \text{(Rule II. p. 49.)}\quad 4y+7=\sqrt{(312+49)}=\pm 19.$$

$$\therefore y=\frac{-7\pm 19}{4}=3, \text{ or } -6\tfrac{1}{2} \therefore x=\frac{1+3y}{2}=5, \text{ or } -9\tfrac{1}{2}.$$

Hence the pairs of values are either $x=5$, $y=3$; or else $x=-9\frac{1}{2}$, $y=-6\frac{1}{2}$.

If both the equations are quadratics, then the elimination of one of the unknowns will frequently lead to an equation of higher degree than the second, the solution of which would require more advanced principles. But there are two classes of simultaneous quadratics, which may always be successfully treated by the rules already established. They are respectively *homogeneous* quadratics and *symmetrical* quadratics.

71. **Homogeneous Quadratics.**—A pair of equations is said to be homogeneous when every unknown term is of the same *dimensions*, that is, when each term has the same number of simple unknown factors. For instance, the terms $4x^3$, $3xy^2$, $5x^2y$, $7y^3$, are each of *three* dimensions,

because each contains *three* simple unknown factors, namely, xxx, xyy, xxy, yyy. In homogeneous quadratics each term is of *two* dimensions.

RULE. 1. For either of the unknowns put an unknown multiple of the other; that is for x put zy, or for y put zx, then the *square* of this other will necessarily enter every unknown term. 2. Deduce an expression for this square from each equation; equate the two expressions, from which all but z will be eliminated, and solve the resulting quadratic in z, and thence deduce the values of x and y.

(2) $\left.\begin{matrix}2x^2-3xy+y^2=4\\ x^2-2xy+3y^2=9\end{matrix}\right\}$ Put zx for y, then the equa. are $\left\{\begin{matrix}2x^2-3zx^2+z^2x^2=4\\ x^2-2zx^2+3z^2x^2=9.\end{matrix}\right.$

$$\therefore x^2=\frac{4}{2-3z+z^2}=\frac{9}{1-2z+3z^2}\therefore 4-8z+12z^2=18-27z+9z^2.$$

$$\therefore 3z^2+19z=14\ \therefore \text{(Rule II. p. 49)}\ 6z+19=\sqrt{(12.14+19^2)}=\pm 23.$$

$$\therefore z=\frac{-19\pm 23}{6}=\frac{2}{3},\ \text{or}\ -7\therefore x^2=\frac{4}{2-3z+z^2}=9,\ \text{or}\ \frac{1}{18}\therefore x=\pm 3,\ \text{or}\ \pm\frac{1}{6}\sqrt{2},$$

$$\text{and } y=zx=\pm 2,\ \text{or}\ \mp\frac{7}{6}\sqrt{2}.$$

The values of x and y, properly paired, are therefore as follow:—

$$\left.\begin{matrix}x=3\\ y=2\end{matrix}\right\}\ \text{or}\ \left\{\begin{matrix}x=-3\\ y=-2\end{matrix}\right\}\ \text{or}\ \left\{\begin{matrix}x=\frac{1}{6}\sqrt{2}\\ y=-\frac{7}{6}\sqrt{2}\end{matrix}\right\}\ \text{or}\ \left\{\begin{matrix}x=-\frac{1}{6}\sqrt{2}\\ y=\frac{7}{6}\sqrt{2}\end{matrix}\right.$$

72. And it may here be noticed that, as there are always *two* unknown factors in each term of the proposed equations, whatever pair of values is discovered for them, there must always be a second pair arising from simply changing the signs of the former pair. We have spoken above of the values of x and y being *properly paired*. The student must be careful to observe consistency in this respect. Whatever value of z gives x, that same value of z must be employed to get the corresponding y, as above.

73. The above method may always be employed with success in the solution of a pair of homogeneous quadratics, but it is not always the shortest method. Rules, perfectly general in themselves, may often be advantageously departed from in particular instances. The exercise of a little independent thought and ingenuity will often suggest facilitating expedients which cannot be embodied in formal rules. These should in general be resorted to only when the penetration and skill of the algebraist fail to discover to him easier processes.

74. **Symmetrical Quadratics.**—An equation is said to be symmetrical when we may change the places of the unknowns without altering the equation or interfering with the condition expressed by it. Thus, the following are symmetrical equations, namely, $x+y=a$, $ax^2-bxy+ay^2=c$, $axy-x-y=b$, &c.; for, although we change every x into y, and every y into x, the equation itself remains substantially unchanged. The first and second of these equations are homogeneous as well as symmetrical. The rule for solving a pair of symmetrical quadratics—which rule, indeed, will often succeed for symmetrical equations of higher degrees—is as follows:—

RULE. Substitute for x and y the sum and difference of two other unknowns; that is, put $u+v$ for x, and $u-v$ for y, in each equation: the result will be a pair of equations with the new unknowns u and v, which pair will always present themselves in a solvable form.

(3) $x^2+y^2-x-y=18$
$xy+x+y=19.$

Putting $u+v=x$, and $u-v=y$, and remembering that $(u+v)^2+(u-v)^2=2(u^2+v^2)$, and that $(u+v)(u-v)=u^2-v^2$, we have

$u^2+v^2-u = 9$ . . . . [1]
$u^2-v^2+2u=19$ . . . . [2]

$\therefore \quad 2u^2+u=28$

$\therefore$ Rule II., $4u+1=\sqrt{225}=\pm 15$

$\therefore u=\frac{7}{2}$, or -4.

$\therefore$ [1], $v^2=9+u-u^2=\frac{1}{4}$, or -11

$\therefore v=\pm\frac{1}{2}$, or $\pm\sqrt{-11}$

$\therefore x=u+v=\frac{7\pm 1}{2}$, or $-4\pm\sqrt{-11}$

$\therefore x=4$, 3, or $-4\pm\sqrt{-11}$

$y=u-v=\frac{7\mp 1}{2}$, or $-4\mp\sqrt{-11}$

$\therefore y=3$, 4, or $-4\mp\sqrt{-11}$.

The values of y need not have been worked out; because of the symmetry of the equations, if $x=a$, $y=b$, be one pair of values, then $x=b$, $y=a$, must be another pair; hence from $x=a$, $x=b$, &c. we deduce $y=b$, $y=a$, &c.

(4) $x^2+y^2+x+y=18$
$xy=6$

Putting $u+v=x$, and $u-v=y$,

$u^2+v^2+u= 9$ . . . . [1]
$u^2-v^2 \quad = 6$ . . . . [2]

$\therefore 2u^2+u=15$

$\therefore$ Rule II., $4u+1=\sqrt{121}=\pm 11$

$\therefore u=\frac{5}{2}$, or -3.

$\therefore$ [2], $v^2=u^2-6=\frac{1}{4}$, or 3.

$\therefore v=\pm\frac{1}{2}$, or $\pm\sqrt{3}$

$\therefore x=u+v=\frac{5\pm 1}{2}$, or $-3\pm\sqrt{3}$,

that is, $x=2$, 3, or $-3\pm\sqrt{3}$, and *therefore*, $y=3$, 2, or $-3\mp\sqrt{3}$.

In the following miscellaneous examples we shall show how both these last may be worked without *Rules*.

75. *Miscellaneous Examples in Simultaneous Quadratics, &c.*

(1) $x^2+y^2-x-y=18$ . . . . [1]
$xy+x+y=19$ . . . . [2]

1. Add [1] to twice [2], then
$x^2+2xy+y^2+x+y=56$
that is, $(x+y)^2+(x+y)=56$
$\therefore$ Rule II., $2(x+y)+1=\sqrt{225}=\pm 15$
$\therefore x+y=\frac{-1\pm 15}{2}=7$, or -8 . . . [A]

2. Subtract twice [2] from [1], then
$x^2-2xy+y^2=3(x+y)-20=1$, or -44.
$\therefore x-y=\pm 1$, or $\pm 2\sqrt{-11}$ . . . [B]

Adding and subtracting [A], [B],
$2x=7\pm 1$, or $-8\pm 2\sqrt{-11}$
$2y=7\mp 1$, or $-8\mp 2\sqrt{-11}$
$\therefore x=4$, 3, or $-4\pm\sqrt{-11}$
$y=3$, 4, or $-4\mp\sqrt{-11}$

(2) $x^2+y^2+x+y=18$ . . . . [1]
$xy= 6$ . . . . [2]

Adding twice [2] to [1],
$(x+y)^2+(x+y)=30$ . . . [A]
Subtracting twice [2] from [1]
$(x-y)^2+(x+y)=6$ . . . [B]

From [A], $x+y=\frac{-1\pm 11}{2}=5$, or -6

$\therefore$ From [B], $(x-y)^2=1$, or 12
$\therefore x-y=\pm 1$, or $\pm 2\sqrt{3}$
$\therefore 2x=5\pm 1$, or $-6\pm 2\sqrt{3}$
$2y=5\mp 1$, or $-6\mp 2\sqrt{3}$
$\therefore x=3$, 2, or $-3\pm\sqrt{3}$
$y=2$, 3, or $-3\mp\sqrt{3}$

(3) $\frac{x^2}{y}-\frac{y^2}{x}+1=0.$
$x-y-1=0.$

Trans. and clearing fractions,
$x^3-y^3=-xy$ . . . . [1]
$x-y=1$ . . . . [2]

Dividing [1] by [2], (See ex. 4, p. 15).
$x^2+xy+y^2=-xy$
$\therefore x^2+2xy+y^2=0$
$\therefore x+y=0$ . . . [3]

Adding and subtracting [2], [3],
$x=\frac{1}{2}$, $y=-\frac{1}{2}$.

Or thus, $x=y+1$, from 2nd equa.
$\therefore$ from 1st, $(y+1)^3-y^3+y^2+y=0$
that is, $3y^2+3y+1+y^2+y=0$
that is, $4y^2+4y+1=0$
or, $(2y+1)^2=0$
$\therefore 2y+1=0$ $\therefore y=-\frac{1}{2}$ $\therefore x=\frac{1}{2}$.

(4) $x-y=1$, $x^3-y^3=19$.

Cubing the first (p. 15),

$$x^3-y^3-3xy(x-y)=1$$

that is, $x^3-y^3-3xy=1$.

Subtracting this from the second,

$$3xy=18 \therefore 4xy=24,$$

Also from 1st, $x^2-2xy+y^2=1$

$$\therefore x^2+2xy+y^2=25$$

$\therefore x+y=\pm 5$, and since $x-y=1$,

$\therefore x=3$, or -2 ; $y=2$, or -3.

(5) $x^3+y^3=189$, $x^2y+xy^2=180$.

Adding 3 times second to first,

$$x^3+y^3+3xy(x+y)=729.$$

that is (p. 15), $(x+y)^3=729$

$\therefore x+y=9 \therefore (x+y)^2=81$ . . . . [1]

But since $(x+y)xy=180 \therefore xy=20$

$$\therefore 4xy=80 \text{ . . . . [2]}$$

Subt. [2] from [1], $(x-y)^2=1$ . . [3]

$\therefore$ [1], [3], $x+y=9$, $x-y=\pm 1$

$\therefore x=5$, or 4 ; $y=4$, or 5.

(6) $3x^2-7y^2=-1$, $xy=6$.

Put $x=vy \therefore 3v^2y^2-7y^2=-1$, $vy^2=6$

$$\therefore y^2=\frac{1}{7-3v^2}=\frac{6}{v}.$$

$$\therefore v=42-18v^2 \therefore 18v^2+v=42$$

$$\therefore 36v+1=\sqrt{3025}=\pm 55 \therefore v=\frac{-1\pm 55}{36}$$

$$\therefore v=\frac{3}{2}, \text{ or } -\frac{14}{9} \therefore y=\sqrt{\frac{6}{v}}=\pm 2,$$

$$\text{or } \pm\sqrt{-\frac{27}{7}}.$$

$$\therefore x=vy=\pm 3, \text{ or } \mp\frac{14}{9}\sqrt{-\frac{27}{7}}.$$

(7) $x+y=a$, $x^5+y^5=b$.

Put $x=u+v, y=u-v$, then $u=\frac{a}{2}$.

$$x^5=u^5+5u^4v+10u^3v^2+10u^2v^3+5uv^4+v^5$$
$$y^5=u^5-5u^4v+10u^3v^2-10u^2v^3+5uv^4-v^5$$

$$x^5+y^5=2u^5+20u^3v^2+10uv^4=b$$

$$\therefore 10uv^4+20u^3v^2=b-2u^5$$

$$\therefore v^4+2u^2v^2=\frac{b-2u^5}{10u}$$

$$\therefore v^4+2u^2v^2+u^4=\frac{b+8u^5}{10u}$$

$$\therefore v^2=-u^2\pm\sqrt{\frac{b+8u^5}{10u}}$$

But $u=\frac{a}{2}$,

$$\therefore x=\frac{a}{2}+v=\frac{a}{2}\pm\sqrt{\left\{-\frac{a^2}{4}\pm\sqrt{\frac{4b+a^5}{20a}}\right\}}$$

$$y=\frac{a}{2}-v=\frac{a}{2}\mp\sqrt{\left\{-\frac{a^2}{4}\pm\sqrt{\frac{4b+a^5}{20a}}\right\}}$$

(8) Find a pair of values for x and y that will satisfy the equations

$$x^2y^4+y^2=10$$
$$xy^4+y=4$$

From the first, $x^2y^4=10-y^2$.

From the second, $x^2y^4=\frac{(4-y)^2}{y^4}$

$$\therefore 10-y^2=\frac{16-8y+y^2}{y^4}$$

$$\therefore y^6-10y^4+y^2-8y+16=0,$$

or, $(y^6-2y^4+y^2)-8(y^4+y-2)=0$

which may be put in the form

$$(y^3-y)^2-8\{(y^4-1)+(y-1)\}=0$$

or, $y^2(y^2-1)^2-8\{(y^4-1)+(y-1)\}=0$.

This is evidently divisible by $y-1$; that is, $y-1$ is one of the factors of the expression on the left; hence that expression becomes 0 for $y-1=0$; in other words, $y=1$ satisfies the equation,

$$\therefore x=\frac{4-y}{y^4}=3.$$

Solutions may often be obtained by thus decomposing the expression equated to 0 into parts, each of which are seen to be divisible by a known factor.

(9) $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\frac{3}{2}\left(\frac{x}{y}+\frac{y}{x}\right)=2\frac{3}{50}$, $4x-5y=10$.

Adding 2 to each side of the first,

$$\left(\frac{x}{y}+\frac{y}{x}\right)^2-\frac{3}{2}\left(\frac{x}{y}+\frac{y}{x}\right)=4\frac{3}{50},$$

or, $z^2-\frac{3}{2}z=\frac{203}{50}$

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* Equations of this form are called *reciprocal* equations, because the roots, $\frac{x}{y}$, are the reciprocals of each other; that is, if r be one root, then $\frac{1}{r}$ must be another. If, as in ex. 10, the coefficients of the terms, when all arranged on one side, are the same when read in order from right to left, as when read from left to right, the equation is always a reciprocal equation. Thus, the equation referred to is $x^4-2x^3+\left(2-\frac{25}{144}\right)x^2-2x+1=0$. The coefficients in which, read either way, are 1, -2, $\left(2-\frac{25}{144}\right)$, -2, $+1$.

$$\therefore 2z-\frac{3}{2}=\sqrt{\left(\frac{406}{25}+\frac{9}{4}\right)}=\sqrt{\frac{1849}{100}}=\pm\frac{43}{10}$$

$$\therefore z=\frac{29}{10}, \text{ or } -\frac{7}{5}. \therefore \frac{x}{y}+\frac{y}{x}=\frac{29}{10}, \text{ or } -\frac{7}{5}.$$

Hence, the first of the given equations is either

$$\frac{x^2}{y^2}+\frac{y^2}{x^2}=\frac{87}{20}+\frac{103}{50}=\frac{641}{100} \quad \ldots \text{ [A]},$$

$$\text{or } \frac{x^2}{y^2}+\frac{y^2}{x^2}=\frac{103}{50}-\frac{21}{10}=-\frac{6}{100} \quad \ldots \text{ [B]}.$$

Subtracting 2 from each side of [A],

$$\left(\frac{x}{y}-\frac{y}{x}\right)^2=\frac{441}{100} \therefore \frac{x}{y}-\frac{y}{x}=\pm\frac{21}{10},$$

But $\frac{x}{y}+\frac{y}{x}=\frac{29}{10}$.

$$\therefore \frac{x}{y}=\frac{5}{2}, \text{ or } \frac{2}{5}. \therefore x=\frac{5}{2}y, \text{ or } \frac{2}{5}y.$$

$\therefore$ (Second equa.), $4x-5y=10y-5y=10$

$$\therefore y=2, \text{ or } 4x-5y=\frac{8}{5}y-5y=10$$

$$\therefore y=-\frac{50}{17} \therefore x=\frac{5}{2}y=5, \text{ or } x=\frac{2}{5}y=-\frac{20}{17}.$$

And proceeding in a similar way, another pair of values may be got from the equation [B].

(10) $12(x-1)y=5x$

$$y^2-x^2=1.$$

From the second, $y=\sqrt{(x^2+1)}$.

Substituting in the first,

$$12(x-1)\sqrt{(x^2+1)}=5x.$$

Squaring, $144(x-1)^2(x^2+1)=25x^2$, that is, multiplying out,

$$144(x^4-2x^3+2x^2-2x+1)=25x^2.$$

Dividing by $144x^2$, we have

$$x^2+\frac{1}{x^2}-2\left(x+\frac{1}{x}\right)=\frac{25}{144}-2.$$

Adding 2 to each side,

$$\left(x+\frac{1}{x}\right)^2-2\left(x+\frac{1}{x}\right)=\frac{25}{144}.*$$

Putting z for $x+\frac{1}{x}$, and completing the square,

$$z^2-2z+1=\frac{169}{144}$$

$$\therefore z-1=\pm\frac{13}{12} \therefore z=\frac{25}{12}, \text{ or } -\frac{1}{12}$$

$$\therefore x+\frac{1}{x}=\frac{25}{12}, \text{ or } x+\frac{1}{x}=-\frac{1}{12}$$

$$\therefore x^2-\frac{25}{12}x=-1, \text{ or } x^2+\frac{1}{12}x=-1.$$

Completing squares,

$$x^2-\frac{25}{12}x+\left(\frac{25}{24}\right)^2=\frac{49}{24^2}$$

$$x^2+\frac{1}{12}x+\frac{1}{24^2}=-\frac{575}{24^2}.$$

From the first of these, $x=1\frac{1}{3}$, or $\frac{3}{4}$.

$$\therefore y=\frac{5}{12}\cdot\frac{x}{x-1}=1\tfrac{2}{3}, \text{ or } -1\tfrac{1}{4}.$$

The other values are imaginary.

NOTE.—When dealing with a pair of simultaneous equations, and having obtained a value of one of the unknowns, we substitute it in one of the given equations, and thence deduce values for the other unknown, it will not always happen that each of these values, taken in conjunction with that substituted, will satisfy the other given equation. As an easy exemplification of this, suppose we have $x-y=1$, and $x^2-y^2=3$; then, by division, we get $x+y=3$, from which, by subtraction, we get $y=1$. Suppose, now, we substitute this value of y in the second equation, we then get $x=\pm 2$; but the values $x=-2$, $y=1$, fail to satisfy the first equation.

76. The student must bear in mind that the sign of a radical is ambiguous only when there is no overruling condition to control it. In ex. 10, for instance, worked out above, if we had substituted the values of x, namely, $x=1\frac{1}{3}$, and $x=\frac{3}{4}$, in the *second* of the given equations, we should have got $y=\sqrt{(x^2+1)}=\pm 1\frac{2}{3}$, or $\pm 1\frac{1}{4}$, but the first equation renders it necessary that the minus sign be rejected in the first value of y, and the plus sign in the second value.

It is proper, therefore, at least as regards real values, to verify our results by actual substitution in both equations, when such ambiguities occur.

The last term, 1, is with propriety called a coefficient, since the term may be written $1x^0$, or x^0.

* See foot note, p. 55.

(1) $\begin{cases} x^2+xy=12 \\ xy-2y^2=1. \end{cases}$

(2) $\begin{cases} 2x^2-3xy+y^2=4 \\ x^2-2xy+3y^2=9. \end{cases}$

(3) $\begin{cases} 3x^2-7y^2+1=0 \\ xy=6. \end{cases}$

(4) $\begin{cases} x^2-y^2=\frac{11}{30}xy \\ x^2+xy=66. \end{cases}$

(5) $\begin{cases} x^3+y^3=18xy \\ x+y=12. \end{cases}$

(6) $\begin{cases} \frac{1}{x}+\frac{1}{y}=1 \\ \frac{1}{x^2}+\frac{1}{y^2}=5. \end{cases}$

(7) $\begin{cases} x+y=4 \\ x^4+y^4=82. \end{cases}$

(8) $\begin{cases} x-y=2 \\ x^4+y^4=272. \end{cases}$

(9) $\begin{cases} \frac{x^2}{y}+\frac{y^2}{x}=18 \\ x+y=12. \end{cases}$

(10) $\begin{cases} x-2\sqrt{xy}+y=\sqrt{x}-\sqrt{y} \\ \sqrt{x}+\sqrt{y}=5. \end{cases}$

(11) $\begin{cases} x+\frac{1}{y}=a \\ y+\frac{1}{x}=b. \end{cases}$

(12) $\begin{cases} (x^2+y^2)(x-y)=51 \\ x^2+y^2+x-y=20. \end{cases}$

(13) $\begin{cases} (x+y)^3-27(x-y)=0 \\ (x^2-xy+y^2)(x-y)=x+y. \end{cases}$

(14) $\begin{cases} (x+y)^3-64(x-y)=0 \\ (x^3+y^3)(x+y)=76. \end{cases}$

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77. *Miscellaneous Questions requiring Simultaneous Quadratics.*

(1) Find three numbers, such that their sum may be 33, the sum of their squares 467, and that the difference of the first and second may exceed the difference of the second and third by 6.

(2) Find two numbers, such that their sum, product, and the difference of their squares may all be equal.

(3) What number is that which, if divided by the product of its two digits, the quotient is 2, and, if 27 be added to the number, the digits will be reversed?

(4) When 962 men were drawn up in two square columns, it was found that one column consisted of 18 ranks more than the other. What was the strength of each column?

(5) When 732 men were drawn up in column, the number in front and the number of ranks together made up 73. How many ranks were there?

(6) The fore-wheel of a carriage made 6 revolutions more than the hind-wheel in going 120 yards; but if the circumference of each wheel were to be increased 1 yard it would make only 4 revolutions more than the hind-wheel in going that distance. Required, the circumference of each wheel.

(7) Find two numbers, such that the difference of their squares is equal to their product, and the sum of their squares equal to the difference of their cubes.

(8) A person rows a distance of 20 miles and back in 10 hours, the stream flowing uniformly in the same direction, and he finds that he rows 3 miles with the stream in the same time that he rows 2 miles against it. Required, the time of going and returning, as also the velocity of the stream.

(9) By adding m times the sum of two numbers to the sum of their squares we obtain the square of m, and by adding the product of the numbers to the sum of their squares we obtain the same square m^2: what are the numbers?

(10) The hypotenuse of a right-angled triangle is to be 13 feet, and

the measure of its surface is to be 30 square feet, what must its sides be? [NOTE.—The surface of a triangle is found by taking half the product of its base and height.]

(11) Find the sum of the squares of the roots of the equation $x^2+px+q=0$, without solving it.

(12) Find the sum of the cubes of the same roots.

(13) How many sides must a polygon have in order that it may have n diagonals?

(14) The men in each of two columns of troops A, B, were formed into squares: in the two squares together there were 84 ranks. Upon changing ground A was drawn up with the front that B had, and B with the front that A had, when the number of ranks in the two columns was 91; what was the number of men in each column?

(15) Can two real numbers be found, such that their sum, product, and sum of their squares shall all be equal?

(16) The product of four consecutive positive whole numbers is 3024: find them.

(17) A certain number of two figures is such that the left-hand figure is equal to 3 times the right-hand one, and if 12 be subtracted from the number, the remainder will be equal to the square of the left-hand figure: required the number.

(18) Find two real numbers whose sum is a and product b, and show that such numbers cannot exist if $b > \left(\frac{a}{2}\right)^2$.

(19) Find two numbers, such that their sum, product, and difference of their squares may all be equal.

(20) There are three whole numbers, such that the squares of the first and second together with their product make 13, the squares of the second and third together with their product make 49, and the squares of the first and third together with their product make 31: what are the numbers?

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78. **Proportion.**—When one quantity is divided by another of the same kind, the quotient is also called the *ratio* of the two quantities: the dividend, or numerator of the fraction expressing the ratio, is called the antecedent of that ratio, and the divisor, or denominator of the fraction, the consequent. In expressing the ratio of a to b, which is no other than the quotient $a \div b$, the small line between the dots is suppressed, and the ratio is written $a : b$.

Proportion is an equality of ratios; thus if $\frac{a}{b}=\frac{c}{d}$, or which is the same thing, if $a : b = c : d$, the four quantities a, b, c, d are said to be in proportion; but instead of the sign of equality, it is usual to write four dots, thus, $a : b : : c : d$, which is read *as a is to b, so is c to d.*

From these explanations it is plain that any two equal fractions may be converted into a proportion, and any proportion may be replaced by two equal fractions. The following are the principal theorems respecting proportion.

79. **Theorems in Proportion.**—1. The product of the extremes is equal to that of the means, and conversely if two products are equal, the pair of factors from which one is produced will be the extremes, and the pair of factors of the other product, the means of a proportion.

For $a : b :: c : d$ implies that $\frac{a}{b}=\frac{c}{d}$ $\therefore ad=bc$.

Again, if $ad=bc$, then $\frac{ad}{bd}=\frac{bc}{bd}$ $\therefore \frac{a}{b}=\frac{c}{d}$, that is, $a:b::c:d$.

If the two means are equal, that is, if $a : b :: b : d$, then $ad=b^2$, that is, if three quantities, a, b, d are in *continued* proportion, the product of the extremes is equal to the square of the mean. And since also $\frac{a}{b}=\frac{b}{d}$ $\therefore \left(\text{multiplying by } \frac{b}{d}\right)$, $\frac{a}{d}=\frac{b^2}{d^2}=\frac{a^2}{b^2}$, so that the first is to the third as the square of the first to the square of the second, or as the square of the second to that of the third.

2. If four quantities are in proportion, they are also in proportion when taken inversely, that is, the second is to the first as the fourth to the third.

Let $a : b :: c : d$ $\therefore \frac{a}{b}=\frac{c}{d}$ $\therefore \frac{b}{a}=\frac{d}{c}$ $\therefore b : a :: d : c$.

3. If the quantities be all of the same kind, they are also in proportion when taken alternately, that is, the first is to the third, as the second to the fourth.

Let $a : b :: c : d$ $\therefore \frac{a}{b}=\frac{c}{d}$ $\therefore \left(\text{mult. by } \frac{b}{c}\right)$, $\frac{a}{c}=\frac{b}{d}$ $\therefore a : c :: b : d$.

NOTE.—As there cannot be any ratio between two quantities different in kind, alternation can be applied only to quantities all of the same kind.

4. If $a : b :: c : d$ then also $a+mb : a+nb :: c+md : c+nd$

For since $\frac{a}{b}=\frac{c}{d}$ $\therefore \frac{a}{b}+m=\frac{c}{d}+m$, and $\frac{a}{b}+n=\frac{c}{d}+n$.

By dividing, $\frac{a+mb}{a+nb}=\frac{c+md}{c+nd}$ $\therefore a+mb : a+nb :: c+md : c+nd$.

*** In each of these expressions, as well as in the equation below, the sign $\sim$ may be put for $+$.

NOTE.—The above is only a particular case of a theorem much more general, for if two fractions are equal as $\frac{a}{b}=\frac{c}{d}$, then we may write any combinations of a, b, for a new num. and any other combinations for a new den., provided we only take care that each term shall be of the same dimensions as respects a and b, and if in the new fraction thus got we change the a and b for c and d, we shall get another fraction equal to it:

thus, $\frac{5a^2+7ab+3b^2}{6(a+b)^2}=\frac{5c^2+7cd+3d^2}{6(c+d)^2}$.

For let m be a quantity such that $c=ma$, then necessarily $d=mb$. Now, if in the second fraction these substitutions be made, and then num. and den. divided by m^2, the first fraction will be the result.

5. If any number of quantities of the same kind are proportionals, then as one antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Let $a : b :: c : d :: e : f$, &c. Put $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, &c. $=m$ $\therefore a=mb$, $c=md$, $e=mf$, &c.

$\therefore a+c+e+$ &c. $=m(b+d+f+$ &c.$)$ $\therefore \frac{a+c+e+\text{ &c.}}{b+d+f+\text{ &c.}}=m=\frac{a}{b}=\frac{c}{d}$ &c.

$\therefore a : b :: a+c+e+$ &c. $: b+d+f+$ &c.

6. If several sets of four be proportionals, then the products of the corresponding antecedents and consequents will be proportional.

Thus, if $\frac{a}{b}=\frac{c}{d}$, $\frac{e}{f}=\frac{g}{h}$, $\frac{k}{l}=\frac{m}{n}$, &c., then $\frac{aek\ \&c.}{bfl\ \&c.}=\frac{cgm\ \&c.}{dhn\ \&c.}$

7. If four quantities be proportional according to the algebraic definition, they are proportional according to Euclid's definition.

In order that a, b, c, d may be proportional according to the algebraic definition, it is necessary and sufficient that $\frac{a}{b}=\frac{c}{d}$. Multiply each by $\frac{m}{n}$, then $\frac{ma}{nb}=\frac{mc}{nd}$.

Now, whatever be m and n, if $ma>nb$, then necessarily $mc>nd$
$ma=nb$, ,, ,, $mc=nd$
$ma<nb$, ,, ,, $mc<nd$.

And if a, b, c, d, satisfy these conditions, they are proportional according to Euclid.

80. **Variation.**—This is a term often used instead of *proportion*, to indicate the same thing, when the quantities compared are *variable*. We say, for instance, that the price of a commodity varies as its weight, that the work done varies as the time expended on it, the wages paid as the number of men employed, and so on. In all such forms of expression, proportion is of course implied; but, by aid of the new mark, $\propto$, to stand for *varies as*, it is more concisely indicated thus: $A \propto B$ implies that A varies as B, and *this* means that if A and B be any two corresponding values, and A', B' any other two corresponding values, then $A : A' :: B : B'$, the *ratios* $\frac{A}{A'}, \frac{B}{B'}$ remaining equal throughout all the corresponding changes of A, B.

We know (Arithmetic) that every proportion among magnitudes of any kind may be converted into a proportion among *abstract numbers*. We have only to divide the first two terms by their common unit of measure, and the other two by *their* common unit of measure. This being done, we have from the equal ratios $\frac{A}{A'}=\frac{B}{B'}$, $A=\frac{A'}{B'}B$, which may be written $A=mB$, where m is a *constant* number, so that $A \propto B$ may always be changed for $A=mB$, A and B being the numerical representatives of the magnitudes or quantities themselves when they are different in kind.

If $A \propto B$, A is said to vary *directly* as B
If $A \propto \frac{1}{B}$,, ,, inversely ,,
If $A \propto \frac{B}{C}$,, ,, directly as B and inversely as C
If $A \propto BC$,, ,, conjointly as B, C.

We may illustrate all this by a reference to the plane triangle, the surface of which, as already noticed (p. 58), is equal to half the product of the base and altitude. Calling the area A, the base B, and the altitude C, then—1. If C be invariable, $A \propto B$. 2. If A be invariable, $C \propto \frac{1}{B}$.

3. If neither be invariable, $B \propto \frac{A}{C}$. 4. And $A \propto BC$. For 1. $A=mB$, where m is the constant, $\frac{1}{2}C$. 2. $C=\frac{m}{B}$, or $BC=m$, where m is the constant, $2A$. 3. $B=m\frac{A}{C}$, or $BC=mA$, where m is the constant 2. 4. $A=mBC$, where m is the constant $\frac{1}{2}$.

81. Since a variation may always be converted into a proportion, and a proportion into an equation, it follows that problems involving either are virtually problems involving equations only; but, in certain inquiries, the idea and notation of variation may be very conveniently introduced; as, for instance, in the class of questions which occur in arithmetic under the head of the Double Rule of Three. Take the following for example:—

If 8 men can mow 112 acres of grass in 14 days, how many men will be required to mow 2000 acres in 10 days?

As the number of men varies *directly* as the number of acres, and *inversely* as the number of days, we have—

$$\text{No. of men} \propto \frac{\text{no. of acres}}{\text{no. of days}} \therefore \frac{112}{14} : \frac{2000}{10} :: \overset{\text{men}}{8} : \text{no. of men required} = 200.$$

The proportion may, of course, be dispensed with; for since

$$\text{No. of men} = m\frac{\text{no. of acres}}{\text{no. of days}} \therefore \text{the constant } m = 8 \div \frac{112}{14} = \frac{8\times 14}{112}$$

$$\therefore \text{No. of men} = \frac{8\times 14}{112} \cdot \frac{2000}{10} = 200.$$

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82. **Arithmetical Progression.**—When a set of quantities increase or decrease, each by a constant difference, they are said to be in arithmetical progression. If a represent the first term of the series, d the constant difference—whether positive or negative—and n the number of terms, then the general form of the arithmetical progression is

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots\ldots,a+(n-1)d.$$

The nth or last term of such a series is usually represented by l, and the sum of the series by S, so that

$$l=a+(n-1)d \ldots\ldots \text{[I.]}$$

and $$S=a+(a+d)+(a+2d)+(a+3d)+\ldots\ldots+l$$

or reversing the terms, $$S=l+(l-d)+(l-2d)+(l-3d)+\ldots\ldots+a$$

$$\therefore \text{adding, } 2S=(a+l)+(a+l)+(a+l)+(a+l)+\ldots\ldots+(a+l)=n(a+l)$$

$$\therefore S=\tfrac{1}{2}n(a+l), \text{ or [I]}, S=\tfrac{1}{2}n\{2a+(n-1)d\} \ldots\ldots \text{[II.]}.$$

And these two formulæ for l and S embrace the whole theory of arithmetical progression.

83. One of the most obvious properties of such a progression is that the sum of the two extreme terms is equal to the sum of any two terms equally distant, one from one extreme, and the other from the other extreme. Let a and l be the extremes, d the common difference, t the term which has m terms *before* it, and t' that which has m terms *after* it; then [I.], $t=a+md$, and $l=t'+md$ $\therefore$ $t-l=a-t'$ $\therefore$ $t+t'=a+l$. If t be the *middle* term, then $t=t'$ $\therefore$ $t=\frac{1}{2}(a+l)$ that is, any term is equal to half the sum of two terms equally distant from it, for the series may be considered to commence at *any* term and to end at any term.

(1) Required, the 16th term of the series 1, 3, 5, 7, ... Here $a=1, n-1=15$, and $d=2$, $\therefore l=1+15.2=31$.

(2) Required, the sum of 67 terms of a *decreasing* arithmetical series of which the first term is 199, and the common difference 3. Here $a=199$, $n-1=66$, and $d=-3$, $\therefore l=199-66.3=1$ $\therefore S=\frac{1}{2}67.200=6700$.

(3) It is required to insert four *arithmetical means* between 5 and 11; that is, 5 being the first term, and 11 the sixth, supply the four intermediate terms. Here d is at present unknown, but $a=5$, $n=6$, and $l=11$; and since [1], $11=5+5d$ $\therefore d=1\frac{1}{5}$; hence the four means are $6\frac{4}{5}$, $7\frac{2}{5}$, $8\frac{3}{5}$, $9\frac{4}{5}$, the entire series being 5, $6\frac{1}{5}$, $7\frac{2}{5}$, $8\frac{3}{5}$, $9\frac{4}{5}$, 11.

(4) How many terms of the series 12, $11\frac{1}{2}$, 11, $10\frac{1}{2}$ . . . . must be taken to make up the number 55? Here n is unknown, but $a=12$, $d=-\frac{1}{2}$, and $S=55$ $\therefore$ [II.], $55=\frac{1}{2}n\{24-\frac{1}{2}(n-1)\}$ $\therefore 220=n\ (49-n)$ $\therefore n^2-49n=-220$ $\therefore n=\dfrac{49\pm39}{2}=5$, or 44. It thus appears that the sum of the terms will make 55 whether the number of them be 5 or 44. As the terms continually decrease by $\frac{1}{2}$, it is plain that the 25th term must be 0; the series then proceeds with *negative* terms, the first nineteen of which, balancing the nineteen positive terms before the zero, leaves for the aggregate of the whole 44 terms, only the five leading ones.

Examples for Exercise.

(1) Find the eleventh term of the series 1, 4, 7, 10, &c.

(2) Find the sixth term of the series $\frac{1}{2}$, $\frac{9}{20}$, $\frac{2}{5}$, &c.

(3) Find the sum of 7 terms of the series $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+\ldots$

(4) The first term of an arith. prog. is 14, and the sum of 8 terms, 28. Required, the common difference.

(5) The sum of the series 1, 5, 9, &c., is 190. How many terms are there?

(6) Insert five arith. means between $\frac{1}{3}$ and $\frac{5}{6}$.

(7) What must the first term of an arith. prog. be whose sum is -28, and the com. diff. of the terms $\frac{1}{5}$?

(8) The first term of an arith. prog. is $3\frac{1}{3}$, the com. diff. $1\frac{4}{9}$, and the sum 22. Find the number of terms.

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84. **Geometrical Progression.**—A set of quantities are said to be in geometrical progression when each is equal to that which immediately precedes, multiplied by a constant quantity, called the *common ratio* of the progression. According as this common ratio is greater or less than unity is the series increasing or decreasing.

Let a represent the first term, r the common ratio, l the last term, and n the number of terms, then the general form of the geom. prog. is

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ \ldots,\ ar^{n-1}=l \ldots [\text{I.}].$$

As the exponents of r form the arithmetical series, 0, 1, 2, 3, 4, 5, 6, 7, 8, &c., and since whatever two of these be taken for extremes, the sum of them will always be equal to the sum of any other two equally distant from them (83), it follows that the *product* of any two terms of [I.] is equal to the product of any other two terms equally distant from

them, as also to the square of the middle term when the number of terms is odd. Representing the sum of the n terms of the geom. prog. by S, we have

$$S=a+ar+ar^2+ar^3+ar^4+\ldots ar^{n-2}+ar^{n-1}$$
$$\therefore Sr= \quad ar+ar^2+ar^3+ar^4+\ldots ar^{n-2}+ar^{n-1}+ar^n$$
$$\therefore Sr-S=ar^n-a \therefore S=\frac{ar^n-a}{r-1}=a\frac{r^n-1}{r-1}=\frac{rl-a}{r-1} \ldots [\text{II.}].$$

85. The formula [I.] for the last term, and these expressions for the sum, comprehend the entire theory of geometrical progression. When $r>1$, the terms continually increase in magnitude. If the number of them be indefinitely extended, the sum, therefore, must at length exceed in magnitude all finite limits; but if $r<1$, and therefore the series be decreasing, the sum always has a finite value, even though the number n of terms be regarded as infinite. For it is plain, when $r<1$, that r^n, by taking n greater and greater, will at length become less than any number we may assign, however small that number may be. If n be infinitely great, r^n must be regarded as absolutely 0, for whatever value we assign to it short of 0 must evidently be too great. In the case, therefore, of an infinite decreasing geom. series, the symbol l, above, must be 0; and, putting Σ for the sum, in this case, instead of S, we have the following expression for the sum of an infinite decreasing geom. prog.:—

$$\Sigma=\frac{a}{1-r} \ldots [\text{III.}]$$

which, upon comparing with [II.], justifies the conclusion that the finding the sum of the terms continued to infinity is really an easier matter than the finding the sum of only a few of the leading terms.

(1) Required the sum of five terms of $1-2+4-\ldots$. Here $a=1$, $r=-2$, $n=5$

$$\therefore l=(-2)^4=16 \therefore S=\frac{-2.16-1}{-3}=11.$$

(2) Sum six terms of $\frac{1}{3}+\frac{1}{6}+\frac{1}{12}\ldots$. Here $a=\frac{1}{3}$, $r=\frac{1}{2}$, $n=6 \therefore l=\frac{1}{3}\left(\frac{1}{2}\right)^5=\frac{1}{96}$

$$\therefore S=\frac{\frac{1}{2}\cdot\frac{1}{96}-\frac{1}{3}}{-\frac{1}{2}}=\frac{\frac{1}{96}-\frac{2}{3}}{-3}=\frac{21}{96}=\frac{7}{32}.$$

(3) Required the sum of the infinite series $1+\frac{1}{3}+\frac{1}{9}+\ldots$. Here $a=1$, $r=\frac{1}{3}$, $l=0$

$$\therefore \Sigma=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}.$$

(4) Find a geometrical mean between $\frac{1}{3}$ and $\frac{1}{12}$. Here $\sqrt{\left(\frac{1}{3}\times\frac{1}{12}\right)}=\frac{1}{6}$, the mean.

(5) Insert three geom. means between $\frac{1}{2}$ and $\frac{2}{9}$. Here $a=\frac{1}{2}$, $n=5$, $l=\frac{2}{9}=\frac{1}{2}r^4$.

$$\therefore r^4=\frac{4}{9} \therefore r^2=\frac{2}{3} \therefore r=\sqrt{\frac{2}{3}}=\frac{1}{3}\sqrt{6}: \text{hence the means are}$$
$$\frac{1}{2}\sqrt{\frac{2}{3}}, \frac{1}{3}, \frac{1}{3}\sqrt{\frac{2}{3}}, \text{ or } \frac{1}{6}\sqrt{6}, \frac{1}{3}, \frac{1}{9}\sqrt{6}.$$

(6) To find the value of the decimal $\cdot NPPP\ldots$, in which $PPP\ldots$ represents an infinite series of recurring figures, and N any number of figures preceding the recurring

decimals. Let N consist of n figures, and P of p figures, then the value of the proposed decimal will be

$$\frac{N}{10^n}+\left(\frac{P}{10^{n+p}}+\frac{P}{10^{n+2p}}+\frac{P}{10^{n+3p}}+\ldots.\right).$$

Where it is plain that the terms within the brackets form an infinite geometric series, of which the common ratio is $\frac{1}{10^p}$.

The sum of this series is therefore $\Sigma=\frac{P}{10^{n+p}}\div\left(1-\frac{1}{10^p}\right)=\frac{P10^p}{10\quad(10^p-1)}$

$$\therefore \cdot NPPP\ldots=\frac{N(10^p-1)+P}{10^n(10^p-1)},$$ where 10^p-1 is p *nines*.

Hence the fraction, equivalent to the proposed decimal, has for num. N times the number formed by p *nines*, plus P; and for den., that same number of nines followed by n ciphers.

In multiplying by nines, we of course merely annex so many zeros to the number multiplied, and then subtract that number.

Ex. Required the fraction equal to ·54123123 ....

Here $N=54$, $n=2$, $P=123$, $p=3$. Also $54\times999=54000-54=53946$.

$$\therefore \cdot54123123\ldots=\frac{53946+123}{99900}=\frac{54069}{99900}=\frac{18023}{33300}.$$

[This process may be conveniently described in words, as in *Weale's* RUDIMENTARY ARITHMETIC, p. 138.]

EXAMPLES FOR EXERCISE.

(1) Find the tenth term of 7, 10, 13, &c.
(2) Find the sum of six terms of 5, 15, 45, &c.
(3) The first term is 7, and the common ratio $\frac{2}{3}$, find Σ.
(4) Find the sum of $2\frac{1}{2}-\frac{1}{2}+\frac{1}{10}-\ldots.$ to infinity.
(5) Find the sum of the infinite series $1+\frac{1}{x}+\frac{1}{x^2}+\ldots.$
(6) Insert three geom. means between 2 and $10\frac{1}{2}$.
(7) Insert five geom. means between 2 and 1458.
(8) Sum the infinite series $2-\frac{4}{3}+\frac{8}{9}-\ldots.$

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86. **Harmonic Progression.**—Quantities are said to be in harmonic progression when the reciprocals of them are in arithmetic progression. The designation arises from the fact that musical strings of equal thickness and tension, in order to produce harmony, when sounded together, must have their lengths as the reciprocals of the arithmetical series 1, 2, 3, &c., that is, as 1, $\frac{1}{2}$, $\frac{1}{3}$, &c.

If three quantities are in harm. prog. the first is to the third as the difference of the first and second to the difference of the second and third. For if a, b, c, be in harm. prog., then $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, are in arith. prog.

$\therefore \frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}$ $\therefore$ mult. by abc, $ac-bc=ab-ac$ $\therefore a:c::a-b:b-c$.

And if a similar property hold with respect to four quantities a, b, c, d, that is, if $a:d::a-b:c-d$, the four are said to be in harmonic proportion.

From the first property, $b=\frac{2ac}{a+c}$ is the har. mean between a and c, and $c=\frac{ab}{2a-b}$ is a third har. proportional to a and b.

There is no general formula for the sum of any number of terms of an harmonic progression. But as, by taking the reciprocals of the terms, we change the harmonic progression into an arith. prog. many particulars respecting such series may be easily deduced: thus,—

(1) Add two more terms to the harmonic series 1, $\frac{6}{5}$, $\frac{3}{2}$. The reciprocals are 1, $\frac{5}{6}$, $\frac{2}{3}$, an arith. prog., in which the com. diff. is $-\frac{1}{6}$ ∴ two additional terms are $\frac{3}{4}$, $\frac{2}{3}$, that is, $\frac{1}{2}$, $\frac{1}{3}$: hence the required harmonicals are 2 and 3.

(2) Insert three har. means between 2 and 3. Here we have to insert three arith. means between $\frac{1}{2}$ and $\frac{1}{3}$. Since $a=\frac{1}{2}$, $n=5$, and $l=\frac{1}{3}$ ∴ $\frac{1}{3}=\frac{1}{2}+4d$ ∴ $d=-\frac{1}{24}$ ∴ the arith. means are $\frac{11}{24}$, $\frac{10}{24}$, $\frac{9}{24}$, and ∴ the har. means are $\frac{24}{11}$, $\frac{24}{10}$, $\frac{24}{9}$, or $2\frac{2}{11}$, $2\frac{2}{5}$, $2\frac{2}{3}$.

Examples for Exercise.

(1) Insert two har. means between 3 and 12.

(2) Insert three har. means between 4 and $1\frac{7}{9}$.

(3) The arith. mean between two numbers is 2, the har. mean $1\frac{1}{2}$: find the numbers.

(4) The sum of three numbers in har. prog. is 13, and the sum of their squares 61: what are the numbers?

---

87. Miscellaneous Questions in Proportion and Progressions.

(1) British wine at 5*s.* per gal. is mixed with spirits at 11*s.* per gal. in such proportion that the mixture at 9*s.* per gal. may produce a profit of 35 per cent., what is the proportion?

Let the prop. of the wine to the spirits be $x : y$, then $5x+11y=$ cost of $x+y$ gal.
and $9x+9y=$ selling price,
∴ $4x-2y=$ profit.

Hence by the question,
$5x+11y : 4x-2y :: 100 : 35$
or $20 : 7$.

Multiplying extremes and means, and equating the results (79),
∴ $80x-40y=35x+77y$
∴ $45x=117y$ ∴ $5x=13y$.

Consequently (79) $x : y :: 13 : 5$, so that the mixture must be at the rate of 13 gal. of wine to 5 gal. of spirits.

(2) In a mixture of brandy and water, it was found that if there had been 6 gal. more of each there would have been 7 gal. of brandy to every 6 gal. of water; but if there had been 6 gal. less of each, there would have been 6 gal. of brandy to every 5 gal. of water, what was the original proportion?

Let $7x-6$ represent the number of gal. of brandy, and $6x-6$ the number of gal. of water, then the first condition will be at once fulfilled; and for the second we have

$7x-12 : 6x-12 :: 6 : 5$
∴ (p. 59), $x : 6x-12 :: 1 : 5$
∴ $6x-12=5x$ ∴ $x=12$
∴ $7x-6=78$ gal. of brandy
$6x-6=66$ gal. of water.

(3) The product of three positive whole numbers in geom. prog. is 64, and the sum of their cubes is 584, required the numbers.

Let them be x, xy, xy^2, then $x^3y^3=64$, and $x^3+x^3y^3+x^3y^6=584$.

From the 1st, $y^3=\frac{64}{x^3}$ ∴ $y^6=\frac{4096}{x^6}$

∴ by substitution in 2nd,

$$x^3+64+\frac{4096}{x^3}=584$$

∴ $x^6-520x^3=-4096$
∴ $x^3=8$ ∴ $x=2$ ∴ $y=2$
∴ the numbers are 2, 4, and 8.

(4) Given the sum S, of n terms of an arith. prog., $a+b+c+\ldots+l$, in which the common difference is δ, to find the sum S_2, of n terms of the series of squares $a^2+b^2+c^2+\ldots+l^2$.

Since $b=a+\delta$, $c=b+\delta$, $d=c+\delta$, ..., $l=k+\delta$, we have

$$\begin{aligned}
b^3&=a^3+3a^2\delta+3a\delta^2+\delta^3\\
c^3&=b^3+3b^2\delta+3b\delta^2+\delta^3\\
d^3&=c^3+3c^2\delta+3c\delta^2+\delta^3\\
&\vdots\qquad\qquad\vdots\\
l^3&=k^3+3k^2\delta+3k\delta^2+\delta^3\\
(l+\delta)^3&=l^3+3l^2\delta+3l\delta^2+\delta^3
\end{aligned}$$

Adding all these n equations together, omitting the quantities common to both sides, we have

$$\begin{aligned}
(l+\delta)^3&=a^3+3\delta(a^2+b^2+c^2+\ldots+l^2)+3\delta^2(a+b+c+\ldots+l)+n\delta^3\\
&=a^3+3\delta S_2+3\delta^2 S_1+n\delta^3 \therefore S_2=\frac{(l+\delta)^3-a^3}{3\delta}-S_1\frac{\delta}{3}-\frac{n}{3}\delta^2.
\end{aligned}$$

Suppose the arith. prog. to be $1+2+3+\ldots+l$, where $a=1$, and $\delta=1$, also $l=n$, then $S_1=\frac{1}{2}n(n+1)$, and by substitution in S_2, we have $S_2=\frac{(n+1)^3-1}{3}-\frac{n(n+1)}{2}-\frac{n}{3}$, that is,

$$S_2=\frac{n^3+3n^2+3n}{3}-\frac{n(n+1)}{2}-\frac{n}{3}=\frac{n^3+3n^2+2n}{3}-\frac{n(n+1)}{2}.$$

But the numerator, $n(n^2+3n+2)=n(n+1)(n+2)$. [See art. 68.] So that

$$S_2=n(n+1)\left\{\frac{n+2}{3}-\frac{1}{2}\right\}, \text{ and since } \frac{n+2}{3}-\frac{1}{2}=\frac{2n-1}{6},$$

$$\therefore S_2, \text{ or } 1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

This formula will be found of useful application hereafter (p. 68).

If, instead of the cubes, we take the fourth powers of b, c, d, &c., we may, in like manner, find S_3 in terms of S_2, S_1; and similarly, we may find S_4, S_5, &c., each in terms of the sums of the inferior powers.

---

(1) A starts from a certain place, and travels so that his second day's journey shall be twice the first, the third three times, and so on. B starts 4 days after to overtake him, and travels uniformly per day 9 times the distance A went the first day: when will B come up with A?

(2) A hundred stones are placed in a straight line, at intervals of 1 yard: how far must a person walk who shall bring the stones, one by one, to a basket, at the place of starting, 10 yards from the first stone?

(3) If the first term of an arith. prog. be n^2-n+1, and the common difference 2, prove that the sum of n terms will be n^3.

(4) Show that the series of cubes 1^3, 2^3, 3^3, 4^3, &c., are severally obtained by summing the following portions of the arithmetical series of odd numbers, namely 1 | 3, 5 | 7, 9, 11 | 13, 15, 17, 19 | &c.

(5) A brigade of sappers carried on 15 yards of sap the first night, 13 the second, and so on, decreasing 2 yards each night, till 3 yards only were left for the last night. How many nights were they employed, and what was the whole length of the sap?

(6) A servant agrees to serve his master for 12 years on condition that he is paid a farthing for the first year, a penny for the second, fourpence for the third, and so on: what would be due to him at the end of the 12 years?

(7) What fraction is equivalent to the recurring decimal ·27543543....?

(8) Two vessels A and B each contain a mixture of wine and water. In A the wine : water : : 1 : 3, but in B as 3 : 5 : how much must be taken from each vessel to make 14 gal. of mixture containing 5 gal. of wine and 9 gal. of water?

(9) If $a+b \propto a-b$, prove that $a^2+b^2 \propto ab$.

10 If the sum of three numbers in arith. prog. be 9, and the sum of their cubes 153; what are the numbers?

(11) Find three numbers in geom. prog. such that their sum may be 7, and the sum of their squares 21.

(12) Find four numbers in geom. prog. such that the sum of the means may be 36, and the sum of the extremes 84.

(13) What two numbers are those whose difference, sum, and product are as the numbers 2, 3, and 5 respectively?

(14) The three digits of a certain number are in arith. prog.: if the number be divided by the sum of the digits, the quotient will be 26; but if 198 be added to the number, the digits will occur in reverse order: what is the number?

(15) There are three infinite series, namely:—

$$S_1=1+\frac{1}{r}+\frac{1}{r^2}+\ldots., \quad S_2=1-\frac{1}{r}+\frac{1}{r^2}-\ldots., \quad S_3=1+\frac{1}{r^2}+\frac{1}{r^4}+\ldots.$$

Prove that $S_1 \times S_2 = S_3$.

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88. **Piling of Balls and Shells.**—Cannon shot and shells are deposited in arsenals in piles of three different forms, called respectively triangular, square, and rectangular or oblong piles, according to the figure of the base of the pile. Each side of a pile presents an inclined face of rows one above another, each row diminishing by a single shot. If the pile be triangular, or square, it terminates in a single shot at top; if it be rectangular, it terminates in a ridge, or row of shot. The three different forms of a pile are here represented.



Triangular Pile.



Square Pile.



Rectangular Pile.

Every horizontal layer of shot is called a *course*.

89. **Triangular Pile.**—In this pile each layer or course of shot, till the top shot is reached, is an equilateral triangle: the side of the first of these triangles, proceeding from the top, is 2 ∴ the number of shot in that course is 2+1, or 3; the number in the next course, the side of which is 3, is 3+2+1, or 6; the number in the next, the side of which is 4, is 4+3+2+1, or 10; and so on ∴ the number of shot in the nth course, counting from the top is $1+2+3+\ldots.+n=\frac{1}{2}n(n+1)$ (art. 82), so that if the nth be the bottom course, and we represent the sum of all the courses by S, we shall have—

$$S=1+3+6+10+\ldots. +\frac{n(n+1)}{2}.$$

$$\text{or, } S= \quad 1+3+6+10+\ldots. +\frac{(n-1)n}{2}+\frac{n(n+1)}{2}$$

$$\therefore \text{ adding, } 2S=1+2^2+3^2+4^2+\ldots\ldots+n^2+\frac{n(n+1)}{2}$$

$$\text{that is (p. 66), } 2S=\frac{n(n+1)(2n+1)}{6}+\frac{3n(n+1)}{6}=\frac{n(n+1)}{6}\{(2n+1)+3\}$$

$$=\frac{n(n+1)(2n+4)}{6} \therefore S=\frac{n(n+1)(n+2)}{6} \ldots\ldots [\text{I.}].$$

If the pile be reduced by the removal of any of the top courses, the number of shot in the remaining incomplete pile is found by subtracting the pile removed from the original pile.

90. **Square Pile.**—As the courses here form the series of squares $1+2^2+3^2+\ldots+n^2$, the number S of shot is $S=\frac{n(n+1)(2n+1)}{6}$ [II.], and the number in an incomplete pile is found from subtracting the pile wanting from the complete pile.

Rectangular Pile.—It is convenient to conceive a rectangular pile to be formed thus. To one of the slant faces of a square pile, each side of whose base is equal to the *shorter* side of the intended rectangular pile, imagine an equal face of shot to be applied (see third *fig.* at p. 67): we shall thus have a rectangular pile terminating in a ridge of two shot, and the base of the rectangle will have one shot more in its length than in its breadth; and continuing to add faces of shot in this way, when we have added m faces, there must be $m+1$ shot in the top ridge, and m shot more in the length of the base than in the breadth. Calling, therefore, the number of shot in the shorter side of the base n, the entire number in the rectangular pile will be—

No. in square pile of base n^2, +No. in the m faces added to it.

The number of shot in the m faces will, of course, be $m\ (1+2+3+\ldots+n)=m\frac{n(n+1)}{2}$, and the number in the square pile is $\frac{n(n+1)(2n+1)}{6}$,

$\therefore S=\frac{n(n+1)}{6}\left\{3m+2n+1\right\}$ ...[III.], where n is the number of shot in the shorter side of the base, and m is the difference of the numbers in the longer and shorter sides, or it is the number of shot in the top ridge *minus* 1. If p represent the number of shot in the longer side of the base, then $m=p-n$; and making this substitution, S becomes

$$S=\frac{n(n+1)(3p-n+1)}{6}\ldots[\text{IV.}].$$

As the number of shot in the top row is $m+1=p-n+1$, if we call this number t, we shall have $S=\frac{n(n+1)(3t+2.n-1)}{6}\ldots[\text{V.}]$.

91. Now, it is worthy of notice that the last factor in the num. of this expression being the same as $t+2(t+n-1)$, and that $p=t+n-1$, the factor referred to is $t+2p$, where t is the top row, and $2p$ the two equal base rows parallel to it. Moreover, as $\frac{n(n+1)}{2}$ expresses the number of shot in the triangular or end face of the pile, if we call this number Δ, we may express S thus: $S=\Delta\frac{t+2p}{3}$, which, being independent of n,

equally applies to a square pile, in which $t=1$. With slight modification, it also applies to a triangular pile, for in this, the last factor in the num. of the expression for S is the same as $1+n+1$, which we may regard as expressing the top shot + the two opposite base rows, one of these rows being n, and the opposite one merely a single shot. Consequently, allowing this latitude to the word *row*, and representing the two opposite rows of the base, which are parallel to the top row, by p, p', we have in general, for each pile, $S=\Delta\frac{t+p+p'}{3}$...[VI.], a formula which may be expressed in words as follows:—

92. **General Rule.**—Multiply the number of shot in a triangular face of the pile by one-third of the number of shot in the three parallel edges of the pile: the product will be the number in the pile.

In all the cases an incomplete pile is regarded as the difference of two complete piles.

(1) How many shot are there in an incomplete triangular pile of 15 courses, 6 shot being in a side of the upper course? As 6 shot are in the side of the upper course, 5 courses have been removed, so that the complete pile had 20 courses: hence in the formula [I.] putting first 20 for n, and then 5, and subtracting, we have $S=\frac{20.21.22-5.6.7}{6}=7(10.22-5)=5.7(2.22-1)=1505$. The operation would be just the same by using the formula [VI.], or the above rule; for $t+p'$ is here=2, so that the number of shot in the three parallel edges is 22 for the complete pile, and 7 for the pile removed; also Δ is $\frac{20.21}{2}$ for the former, and $\frac{5.6}{2}$ for the latter.

(2) There are 18 courses in a rectangular pile, and 45 shot in the ridge: how many shot are there in the pile?

In the formula [III.], $n=18$, and $m=45-1=44$

$$\therefore S=\frac{18.19.\{3.44+2.18+1\}}{6}=3.19.169=9633:$$

By the formula, or rule above, $S=\frac{18.19}{2}.\frac{45+2.62}{3}=3.19.169=9633:$ for $p=p'=t+n-1=62$.

Examples for Exercise.

(1) Find the number of shot in a triangular pile, and also in a square pile, each composed of 12 courses.

(2) How many shells are there in a rectangular pile of which one side of the base contains 16 shells and the other 7?

(3) How many shot are there in a rectangular pile of 15 courses, 21 shot being in the top row?

(4) If the number of courses be 30, and the top row contain 31 shot, how many will there be in the pile?

(5) How many shot are there in an incomplete rectangular pile of 12 courses, the longer and shorter sides of the base containing 40 and 20 shot respectively?

93. **The Binomial Theorem.**—This theorem, which is one of the most important theorems in algebra, was first given in a general form by NEWTON: its object is to express any power or root of a binomial without actually performing the involution or extraction: in other words, its object is to express the *development*, as it is called, of $(a+x)^n$, by a formula applicable to every value of n. To arrive at this general formula, we must first establish the following preliminary theorem, namely:

94. **Preliminary Theorem.**—If the series $A+Bx+Cx^2+Dx^3+\ldots$, whether the number of terms be finite or infinite, be equal, as a whole, to the series $A'+B'x+C'x^2+D'x^3+\ldots$, (the coefficients being all finite) whatever value be given to x, then the coefficients of the like powers of x must be equal: that is, $A=A'$, $B=B'$, $C=C'$, $D=D'$, &c.

For the condition being that the series—each taken as a whole—are equal, whatever value be given to x, put for x the extreme value $x=0$: then all the terms in each, after the first, must vanish, since the coefficients are finite; so that the two equal series are reduced to $A=A'$.

Having thus ascertained that the first term in one series is equal to the first in the other, we have, by suppressing these terms, $Bx+Cx^2+Dx^3+\ldots=B'x+C'x^2+D'x^3+\ldots$, and therefore, dividing by x, $B+Cx+Dx^2+\ldots=B'+C'x+D'x^2+\ldots$, which, when $x=0$, becomes simply $B=B'$. Thus the first two coefficients in the one series are equal, each to each, to the first two in the other. And proceeding uniformly in this way, we find that $A=A'$, $B=B'$, $C=C'$, $D=D'$, &c.

It follows from this, that if $A+Bx+Cx^2+\ldots=0$, or, which is the same thing, that if $A+Bx+Cx^2+\ldots=0+0x+0x^2+\ldots$, for all values of x, the coefficients A, B, C, &c., must each be 0.*

95. The binomial theorem, the proof of which the above will assist us in establishing, is this, namely:

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{2.3}a^{n-3}x^3+\text{ &c.}$$

or, dividing each side by a^n, and for brevity putting z for $\frac{x}{a}$, it is

$$(1+z)^n=1+nz+\frac{n(n-1)}{2}z^2+\frac{n(n-1)(n-2)}{2.3}z^3+\text{ &c.}$$

It will be sufficient to prove it in this more simple form; for if this be true, the former development must be true, inasmuch as this is converted into that by simply putting $\frac{x}{a}$ for z (which latter symbol may be anything) and then multiplying each side by a^n.

I. NO FRACTIONAL OR NEGATIVE EXPONENT CAN ENTER THE DEVELOPMENT OF $(1+z)^n$ WHATEVER BE n. Or, to be more explicit, the development must always be of the form

$$(1+z)^n=1+Pz+Qz^2+Rz^3+Sz^4+\text{ &c...[1.]}$$

1. *When n is a positive Integer.* In this case, since the development of $(1+z)^n$ may be obtained by repeated multiplications of $1+z$ by itself, it is plain that the first term of this development will be 1, and that all the powers of z entering it must be positive integers. In this case, therefore, [1] is obviously the form of the development.

* It may be proved, too, that if $A+Bx+Cx^2+\ldots+Lx^n=0$ be satisfied for $n+1$ values of x, the coefficients must each be 0.

2. *When n is a positive fraction.* Let $\frac{p}{q}$ be the fraction: we know that $(1+z)^{\frac{p}{q}}$ indicates that after the power p of $1+z$ has been found, the qth root of that power is to be extracted. But the power p of $1+z$ has already been proved to be of the form [1], and if the qth root of [1] be extracted, this root must be such as, by repeated multiplications by itself, to reproduce [1]: its leading term must, therefore, be 1, and no fractional exponent of z can occur in the root, for this reason, namely, that whatever fractional exponent enters an expression whose leading term is 1, must *also* enter every integral power of that expression: the multiplier 1 obviously necessitates this. Hence, a fractional exponent of z cannot enter the development of $(1+z)^{\frac{p}{q}}$. Neither can negative exponents enter; for if this be supposed possible, let $-k$ be the *greatest* negative exponent of z in the development; then, if every term containing z were to be divided by z^{-k}, or, which is the same thing, if each exponent of z were increased by k, *all* the resulting exponents would be positive, and one of these exponents, namely, that which was originally $-k$, would be 0: hence, calling these increased exponents, when arranged in order of magnitude, 0, α, β, γ, &c., the form of the development of $(1+z)^{\frac{p}{q}}$ would be $1+z^{-k}(a+bz^{\alpha}+cz^{\beta}+dz^{\gamma}+$ &c.), a being the coef. of z^{-k}, so that multiplying by z^k, we should have $(1+z)^{\frac{p}{q}}z^k=z^k+a+bz^{\alpha}+cz^{\beta}+dz^{\gamma}+$ &c., whatever be z, since the two expressions are identical. But if $z=0$, the equation becomes $0=a$. Hence, the coef. a, of the supposed negative power, is 0; that is to say, such negative power cannot enter the development of $(1+z)^{\frac{p}{q}}$; and since the entrance of fractional powers has been shown to be also impossible, the development in this case too is of the form [1].

3. *When n, either integral or fractional, is negative.*—Here we have to seek the form of the development of $(1+z)^{-n}$, or of its equal $\frac{1}{(1+z)^n}$, that is, we have to determine the form of the quotient arising from dividing 1 by an expression of the form [1]. The first term of this quotient must, of course, be 1; and since the first remainder cannot contain any fractional or negative power of z, neither can the next term of the quotient, neither, therefore, can the second remainder, nor, in consequence, the next term of the quotient, and so on. Hence, generally, whether n be integral or fractional, positive or negative, the development of $(1+z)^n$ will always be of the form [1].

II. DETERMINATION OF THE COEFFICIENT P IN THE GENERAL FORM [1].

1. *When n is a positive integer.*—Suppose the first two terms of *any* positive integer power of $1+z$ to be $1+mz$, then the first two terms of the power a unit higher, got by multiplying again by $1+z$, will be $1+(m+1)z$; so that by increasing the exponent of the power by a unit, the coefficient of z becomes increased by a unit. In other words, the difference between the exponent and the resulting coef. of z is always the same; but if the exp. be 2, we know that this difference is 0, for $(1+z)^2=1+2z+z^2$; hence the difference is always 0. Consequently, when n is

a positive integer, $(1+z)^n=1+nz+Qz^2+Rz^3+Sz^4+$ .....[2], in which, however, nothing at present is known about the coefficients Q, R, S, &c.

2. *When n is a negative integer.*—In this case $(1+z)^{-n}=1\div(1+z)^n=1\div(1+nz+$ ...), and actually performing the division, as far as the second term of the quotient, we have $1+nz+$ ...)1 $(1-nz+$ .... Hence, whether the integer n be positive or negative, the form of the development is [2].

3. *When n is a fraction, either positive or negative.*—Let $n=\frac{p}{q}$, p being either positive or negative, and, for brevity, put z' for $Pz+Qz^2+$ &c. in [1], then $(1+z)^{\frac{p}{q}}=1+z'$ $\therefore (1+z)^p=(1+z')^q$ $\therefore 1+pz+$ &c., $=1+qz'+$ &c., by [1] and [2] above, that is, replacing the value of z', $1+pz+$ &c. $=1+q(Pz+Qz^2+$ &c.) + &c. Now, this being an identical equation, is true, whatever be the value of z, therefore (94), $p=qP$ $\therefore P=\frac{p}{q}=$ the proposed exponent. Hence, whatever be n, that n is always the coefficient of z in the development of $(1+z)^n$: in other words, the development is always of the form [2].

If now we replace z by $\frac{x}{a}$, and multiply by a^n, we have [2],

$$(a+x)^n=a^n+na^{n-1}x+Qa^{n-2}x^2+Ra^{n-3}x^3+Sa^{n-4}x^4+ \text{ \&c. ...[3]}$$

whatever be the value of n.

III. DETERMINATION OF THE COEFFICIENTS Q, R, S, &c., IN THE GENERAL FORMS [2], [3].—In the identity [2], as z may be anything, put $b(1+x)$ for z; we shall then have $[1+b(1+x)]^n=1+nb(1+x)+Qb^2(1+x)^2+Rb^3(1+x)^3+Sb^4(1+x)^4+$ &c. Now we know from [2] that the first two terms of the development of $(1+x)^m$ are $1+mx$; so that we know every first term and every second term furnished by $(1+x)^1$, $(1+x)^2$, $(1+x)^3$, &c., in the right hand member of this last equation; and collecting all the first terms together, and then all the second terms, we have

$$[1+b(1+x)]^n=(1+nb+Qb^2+Rb^3+Sb^4+ \text{ \&c.})+(nb+2Qb^2+3Rb^3+4Sb^4+ \text{ \&c.})x+ \text{ \&c.}$$

Again, since $1+b(1+x)$ is the same as $(1+b)+bx$, and that by [3] $[(1+b)+bx]^n=(1+b)^n+n(1+b)^{n-1}bx+$ &c., the two series on the right, this and the one above, must be equal, whatever be the value of x; hence

$$n(1+b)^{n-1}b=nb+2Qb^2+3Rb^3+4Sb^4+ \text{ \&c.}$$
$$\therefore n(1+b)^{n-1}=n+2Qb+3Rb^2+4Sb^3+ \text{ \&c.}$$

Now multiply each side of this by $1+b$, and there results

$$n(1+b)^n=n+(2Q+n)b+(3R+2Q)b^2+(4S+3R)b^3+ \text{ \&c.}$$

But [2], $n(1+b)^n=n+n^2b+nQb^2+nRb^3+$ &c.

The two series on the right are, therefore, equal, whatever be the value of b, consequently (94)

$$2Q+n=n^2 \therefore Q=\frac{n(n-1)}{2}$$

$$3R+2Q=nQ \therefore R=\frac{n(n-1)(n-2)}{2.3}$$

$$4S+3R=nR \therefore S=\frac{n(n-1)(n-2)(n-3)}{2.3.4}$$

&c., &c.,

where the law of the coefficients, Q, R, S, &c., is abundantly obvious. Hence the development [3] is

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{2.3}a^{n-3}x^3+ \text{ \&c. . . . [A.]}$$

which development is the BINOMIAL THEOREM.

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96. In the preceding development [*A*], the student will do well to attend not only to the law of the coefficients, but also to that of the exponents; he will observe that while the exponents of the first term of the binomial, namely a, commencing with n, regularly descend by unity, those of the second term x as regularly ascend by unity, so that the sum of the exponents is always n; it thus appears that, omitting the coefficients, each term is equal to that which immediately precedes it, multiplied by $\frac{x}{a}$. Knowing the first term, therefore, we can always write down the subsequent terms (omitting coefficients) without any trouble. And leaving spaces for the coefficients, these may afterwards be supplied from observing that the coefficient of the second term is always the given exponent n, or $\frac{n}{1}$, and that every subsequent coefficient is found by introducing an additional factor into both num. and den., the factors of the num. uniformly decreasing by 1, and those of the den. as uniformly increasing by 1.

The additional factor in the num. is no other than the exponent of a in the preceding term, and that in the den. is no other than the exponent of x in that term increased by 1. Suppose, for example, we had to develope $(a+x)^5$; then, knowing that the first term is a^5, by leaving spaces for the coefficients, we might very rapidly first write

$$a^5+ \quad a^4x+ \quad a^3x^2+ \quad a^2x^3+ \quad ax^4+ \quad x^5.$$

Then, knowing the coefficient $\frac{5}{1}$ of the second term, and attending to the simple law just described, we have the following succession of coefficients, namely, 5, 10, 10, 5, 1. Hence, filling in the coefficients, we have

$$(a+x)^5=a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5.$$

97. Whenever the exponent is a positive whole number, the coefficients, as in this instance, form the same series of numbers, whether taken in order from left to right, or from right to left. It is obvious that such must always be the case; for, as the coefficients must remain the same, whatever particular *letters* form the binomial, the development of $(x+a)^n$ must have its coefficients, taken in order, the same as those of the development of $(a+x)^n$; but, omitting the coefficients, the latter development is

$$a^n+ \quad a^{n-1}x+ \quad a^{n-2}x^2+\ldots\ldots\ldots+a^2x^{n-2}+ \quad ax^{n-1}+ \quad x^n$$

and the former,

$$x^n+ \quad x^{n-1}a+ \quad x^{n-2}a^2+\ldots\ldots\ldots+x^2a^{n-2}+ \quad xa^{n-1}+ \quad a^n$$

But either of these, written in reverse order, becomes the other, so

that the coefficients must be the same series of numbers, whether they be taken in order from left to right, or from right to left.

98. In the foregoing general development x is preceded by the plus sign; but as x is any value whatever, we may replace it by $-y$, and thus get the development of $(a-y)^n$, or, putting the symbol x for y, of $(a-x)^n$. But the changing of x into $-x$ can affect only the signs of the *odd* powers of x; so that, in this case, the development [A] will have its signs alternately positive and negative when n is a positive integer; but if n be a negative integer, then, on account of the coefficients—commencing with that of the second term—being alternately negative and positive, the signs will be all plus.

99. As to the number of terms, when n is a positive integer, we may readily determine it by observing that in the first term the power of x is x^0, and that in the last it is x^n; and as the exponents of x regularly increase by unity, the entire number of terms must be $n+1$. We say that the term involving x^n is the *last* term, because, when we arrive at this term, the coefficient becomes 1; and, by extending the terms beyond this limit, each coefficient becomes 0.

But when the exponent is negative or fractional, then there can be no last term. The coefficients never disappear, and the series may be continued indefinitely; for the *subtraction* of a positive number from $-n$ increases it, and the subtraction of a whole number from a fraction can never destroy that fraction.

100. Confining our attention, then, for the present, to the *finite* development, that is, to the case in which n is a positive integer, and for which the terms are limited to the number $n+1$, we see, from the principle (97), that when n is even, we need compute only the first $\frac{n}{2}+1$ coefficients of the development of $(a+x)^n$; and that when n is odd, only the first $\frac{n+1}{2}$; for the remaining coefficients will be these written in reverse order, observing that, in the latter case, the $\frac{n+1}{2}$th coefficient is to be repeated. In the former case we compute up to the middle coefficient inclusive; in the latter, only up to the first of the *two* middle and equal coefficients. These coefficients are necessarily all *integers*, since, from the nature of multiplication, $(a+x)^n$, when developed, cannot involve fractions. The same is true when n, instead of a positive, is a negative integer; for the development of $(a+x)^{-n}$, that is, the quotient of $1 \div (a^n+na^{n-1}x+\ldots)$, seeing that the coefficients of the divisor are all integral, and the *leading* one *unit*, cannot possibly involve a fractional coefficient.

We may infer, from what is here shown, that the product of n consecutive whole numbers is always divisible by 1. 2. 3. 4...n.

101. Another interesting deduction is, that so long as n is a positive integer, the sum of the coefficients, when they are all positive, is equal to 2^n; and when they are alternately positive and negative, the sum is 0; that is, the sum of the positive coefficients is equal to the sum of the negative coefficients. For, making a and x each $=1$, we have

$$(1+1)^n=2^n=1+n+\frac{n(n-1)}{2}+\frac{n(n-1)(n-2)}{2.3}+\ldots+1$$

$$\text{and } (1-1)^n=0=1-n+\frac{n(n-1)}{2}-\frac{n(n-1)(n-2)}{2.3}+\ldots(-1)^n.$$

102. I.—*To develope* $(a+x)^n$ *when* n *is either a positive or negative integer.*

RULE. 1. Write down a^n for the first term of the development; then every subsequent term is found thus:—

2. Multiply the coefficient in the *term last found* by the exponent of a in that term, then divide by the exp. of x in the same term, increased by 1, or, which is the same thing, by the number which marks the place of that term, and the coefficient of the next term will be obtained. As to the *letters*, the powers of a diminish, and those of x increase by 1, successively.

NOTE.—When n is positive, the coefficients need be computed only up to the middle of the development; they are then to be repeated in reverse order. If one of the two terms of the binomial is negative, all the odd powers of that term will be negative.

(1) Required, the development of $(a-x)^6$. By the rule, we have

1st term.	2nd term.	3rd term.	4th term.
a^6	$6a^5x$	$\frac{6.5}{2}a^4x^2=15a^4x^2$	$\frac{15.4}{3}a^3x^3=20a^3x^3$.

We need not proceed further, for we have now arrived at the $\left(\frac{n}{2}+1\right)$th, or middle term. Hence the complete development is

$$(a-x)^6=a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6.$$

(2) Required, the development of $(a+x)^7$. Here we compute $\frac{n+1}{2}=$ 4 terms.

1st.	2nd.	3rd.	4th.
a^7	$7a^6x$	$\frac{7.6}{2}a^5x^2=21a^5x^2$	$\frac{21.5}{3}a^4x^3=35a^4x^3$.

$$\therefore (a-x)^7=a^7+7a^6x+21a^5x^2+35a^4x^3+35a^3x^4+21a^2x^5+7ax^6+x^7.$$

(3) Required, the development of $(2a^2-3bx)^5$. Here, putting a' for $2a^2$, and x' for $3bx$, we might proceed to develope $(a'-x')^5$ as above, and then, in the result, replace the values of a' and x'. But the operation will be easier by considering the general development [A] under the form

$$(a+x)^n=a^n\left\{1+n\frac{x}{a}+\frac{n(n-1)}{2}\left(\frac{x}{a}\right)^2+\frac{n(n-1)(n-2}{2.3}\left(\frac{x}{a}\right)^3+\ldots+\left(\frac{x}{a}\right)^n\right\}^*\ldots \text{[B]}$$

where $\frac{x}{a}$ is the second term of the proposed binomial divided by the first; and since, in the present example, this is $-\frac{3bx}{2a^2}$, we have

$$(2a^2-3bx)^5=(2a^2)^5\left\{1-5\frac{3bx}{2a^2}+10\left(\frac{3bx}{2a^2}\right)^2-10\left(\frac{3bx}{2a^2}\right)^3+5\left(\frac{3bx}{2a^2}\right)^4-\left(\frac{3bx}{2a^2}\right)^5\right\}.$$

(4) Required the development of $\frac{1}{(a+x)^3}$, or $(a+x)^{-3}$. Here the coefficients will be

$$1,\ -3,\ \frac{-3\times-4}{2}=6,\ \frac{6\times-5}{3}=-10,\ \frac{-10\times-6}{4}=15,\ \&c.$$

and by [B], which in this case is an interminable series,

$$(a+x)^{-3}=\frac{1}{a^3}\left(1-\frac{3}{a}x+\frac{6}{a^2}x^2-\frac{10}{a^3}x^3+\frac{15}{a^4}x^4-\frac{21}{a^5}x^5+\&c.\right)$$

---

* When n is negative, or fractional, there is, of course, no final term here, the series being then interminable.

EXAMPLES FOR EXERCISE.

(1) $(a-x)^7=$.
(2) $(x+y)^6=$.
(3) $(a+x)^8=$.
(4) $(x-y)^9=$.
(5) $(2a^2+3x^3)^4=$.
(6) $(a+b\sqrt{-1})^6=$.
(7) $(2ax-3y)^5=$.
(8) $(a+x)^{-2}=$.
(9) $(1+x^2)^{-3}=$.
(10) $(a+2x)^{-3}=$.

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103. II.—*To develope* $(a+x)^n$ *when n is a positive or negative fraction.*—The rule is the same as in the former case, the fractional exponents merely replacing the integral ones.

(1) Required the development of $\sqrt{(a+x)}$, or $(a+x)^{\frac{1}{2}}$.

The coefficients are 1, $\frac{1}{2}$, $\frac{\frac{1}{2}(-\frac{1}{2})}{2}=-\frac{1}{8}$, $\frac{-\frac{1}{8}(-\frac{3}{2})}{3}=\frac{1}{16}$, $\frac{\frac{1}{16}(-\frac{5}{2})}{4}=-\frac{5}{128}$, &c.

$$\therefore \sqrt{(a+x)}=a^{\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}x-\frac{1}{8}a^{-\frac{3}{2}}x^2+\frac{1}{16}a^{-\frac{5}{2}}x^3-\frac{5}{128}a^{-\frac{7}{2}}x^4+\text{ &c.}$$

$$=\sqrt{a}.\left(1+\frac{x}{2a}-\frac{x^2}{8a^2}+\frac{x^3}{16a^3}-\frac{5x^4}{128a^4}+\text{ &c.}\right)$$

which last is the form at once obtained from [B].

(2) Required the development of $\frac{1}{(b^2+x)^{\frac{1}{2}}}$, or $(b^2+x)^{-\frac{1}{2}}$.

The coefficients are 1, $-\frac{1}{2}$, $\frac{-\frac{1}{2}(-\frac{3}{2})}{2}=\frac{3}{2.4}$, $\frac{\frac{3}{2.4}(-\frac{5}{2})}{3}=-\frac{3.5}{2.4.6}$, &c.

$$\therefore \text{[B]}, \frac{1}{(b^2+x)^{\frac{1}{2}}}=\frac{1}{b}\left(1-\frac{x}{2b^2}+\frac{3x^2}{2.4b^4}-\frac{3.5x^3}{2.4.6b^6}+\frac{3.5.7x^4}{2.4.6.8b^8}-\text{ &c.}\right)$$

There is an advantage in thus putting the numerical factors in evidence, since, after a few terms have been computed, they may be continued at pleasure.

EXAMPLES FOR EXERCISE.

(1) $\sqrt{(b^2+x)}=$.
(2) $\sqrt[3]{(b^3+x)}=$.
(3) $\frac{1}{(a-x)^{\frac{1}{2}}}=$.
(4) $(a+x)^{\frac{3}{2}}=$.
(5) $(a^2-x)^{\frac{3}{2}}=$.
(6) $(a+x)^{-\frac{2}{3}}=$.
(7) $\sqrt[3]{9}=\sqrt[3]{(8+1)}=$.
(8) $\sqrt[3]{6}=\sqrt[3]{(8-2)}=$.
(9) $\frac{a^2}{(a^2-x)^{\frac{1}{2}}}=$.
(10) $\frac{1}{(3-7x)^{\frac{1}{3}}}=$.

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104. Instead of the entire development of $(a+x)^n$, it is sometimes required to express only the single term involving a specified power of x, as the power x^r. From the general form, we know that the den. of the coef. belonging to x^r is 2. 3. 4...r, and that the last of the r factors in the num. is $n-(r-1)$; the required term is therefore readily found. For instance, let the term involving x^r in the development of $(a+x)^{-\frac{5}{2}}$ be required. The coefficient would be

$$\frac{\frac{5}{2}(\frac{3}{2})(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\ \ldots\ (\frac{5}{2}-r+1)}{2.3.4\ \ldots\ .\ r}=\frac{5.3.1.1.3\ \ldots\ .\ (7-2r)}{2.3.4\ \ldots\ .\ r.\ 2^r}$$

And the term itself would be $\frac{5.3.1.1.3\ \ldots\ (7-2r)}{2.3.4\ \ldots\ r.\ 2^r}a^{n-r}x^r$, which is called the *general term* of the series.

It is sometimes, too, required to find, for given numerical values of a and x, which is the *greatest* term of the series. Now, from $[B]$ we see that each term within the brackets is equal to the constant factor $\frac{x}{a}$, multiplied by the preceding term, and also by $\frac{n-(r-1)}{r}$, where the number r marks the place of that preceding r term; that is, this term is the rth of the series, or that in which the exponent of x is $r-1$. So long, then, as the multiplier $\frac{n-(r-1)}{r}.\frac{x}{a}$ exceeds unit, will the successive terms continue to increase in magnitude, but when at length r becomes so great that the multiplier becomes less than unit, we may be sure that the numerically greatest term is about to be passed; that is, that the term, to which this multiplier must be applied to obtain the next, is the greatest term of the series; for, as r increases, the factor $\frac{n-(r-1)}{r}$ decreases, so that when once the terms begin to diminish, they must continue to diminish. This is readily proved thus: take the next term of the series, that is, change r into $r+1$; we have to prove that

$\frac{n+1-r}{r} > \frac{n-r}{r+1}$, or that $\frac{n+1}{r}-1 > \frac{n+1}{r+1}-1$,* or that $\frac{n+1}{r} > \frac{n+1}{r+1}$, which is necessarily the case, because $\frac{1}{r} > \frac{1}{r+1}$. For example: What is the greatest term in the development of $\left(4+\frac{5}{2}\right)^{\frac{3}{2}}$? Here $\frac{x}{a}=\frac{5}{8}$, and $n-(r-1)=\frac{5}{2}-r$ $\therefore$ $\frac{5}{8}\left(\frac{5}{2}-r\right)<r$, or $\frac{25}{16}-\frac{5}{8}r<r$ $\therefore \frac{25}{2}<13r$ $\therefore r=1$. Hence the *first* is the greatest term, and it is equal to $4^{\frac{3}{2}}=8$.

Whenever it happens that $\frac{n-(r-1)}{r}.\frac{x}{a}=1$, or $(n+1-r)\frac{x}{a}=r$, then the rth term and the $(r+1)$th will be *equal*, and each greater than any other term. Again: suppose we had $\left(1+\frac{5}{2}\right)^{\frac{3}{2}}$; then $\frac{x}{a}=\frac{5}{2}$, and $n-(r-1)=\frac{5}{2}-r$ $\therefore$ $\frac{5}{2}\left(\frac{5}{2}-r\right)<r$, or $\frac{25}{4}-\frac{5}{2}r<r$ $\therefore \frac{25}{2}<7r$ $\therefore r=2$. Hence the *second* is the greatest term, and it is equal to $\frac{3}{2}.\frac{5}{2}=3\frac{3}{4}$.

Lastly: suppose we had $\left(1+\frac{5}{2}\right)^{-\frac{3}{2}}$; then $n-(r-1)=-\frac{1}{2}-r$ $\therefore$ $-\frac{5}{2}(\frac{1}{2}+r)<r$, or disregarding sign, as we are concerned only with magnitude, $\frac{5}{2}(\frac{1}{2}+r)<r$, which is manifestly impossible; consequently it is impossible that there can be a *greatest* term; the terms must go on increasing to infinity.

105. The existence of such series, as results of the application of the binomial theorem, must awaken the student to the fact that there is not, in *all* cases, an equivalence between $(a+x)^n$ and what is called its development. The right-hand member of $[A]$, when the series is interminable, may consist of that infinite series, and something else, concealed under the &c. If, for particular values of a and x, this supple-

* $n-r=n+1-(r+1)$.

mental correction of the series, whatever it be, do not vanish, we shall commit error in regarding the series alone—though viewed in all its infinitude of terms—as the equivalent of $(a+x)^n$. In fact, the demonstration of the theorem [A], before given, and every other demonstration of it, tacitly *assumes* that there is no supplemental correction under the &c., but that the series *alone* completes the items which make up $(a+x)^n$. When such is *not* the case, the terms, as in the above example, instead of continually diminishing in numerical magnitude, and tending to evanescence, must, after a certain stage, increase and tend *from* zero. All such infinite series are useless for the purposes of calculation, having no sum.

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106. **The Exponential Theorem.**—The object of this theorem is to develope a^x in a series of terms according to the ascending powers of x. In order to bring this exponential quantity, as it is called, under the dominion of the binomial theorem, let $1+b$ be written for a, then we shall have, by the theorem alluded to, $a^x=(1+b)^x=1+xb+\frac{x(x-1)}{2}b^2+\frac{x(x-1)(x-2)}{2.3}b^3+\frac{x(x-1)(x-2)(x-3)}{2.3.4}b^4+$ &c., from which it is evident that if the factors in the several numerators were actually multiplied together, we should get a series of the form $1+Ax+Bx^2+Cx^3+$ &c., so that the form of development proposed is really possible.

It is plain that $A=b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+$ &c. Put then

$a^x=1+Ax+Bx^2+Cx^3+Dx^4+$ &c., and since x is anything,

$\therefore a^y=1+Ay+By^2+Cy^3+Dy^4+$ &c.

and $a^{x+y}=1+A(x+y)+B(x+y)^2+C(x+y)^3+D(x+y)^4+$ &c.

But $a^{x+y}=a^x\times a^y$, so that the last development must be equal to the product of the two preceding developments. Now by the binomial theorem, the coefficient of y in the last development is

$$A+2Bx+3Cx^2+4Dx^3+\ldots\ldots[1]$$

and the coefficient of y in the product of the two preceding developments, which coefficient is simply the product of the first development by A, is

$$A+A^2x+ABx^2+ACx^3+\ldots\ldots[2]$$

Consequently, since the coefficients of the like powers of y, in the two identical developments are equal (94), we must have [1]=[2], whatever be a; hence, by the same principle (94), we must have

$$2B=A^2,\ 3C=AB,\ 4D=AC,\ \&c.$$

$$\therefore B=\frac{A^2}{2},\ C=\frac{A^3}{2.3},\ D=\frac{A^4}{2.3.4},\ \&c.$$

$$\therefore a^x=1+Ax+\frac{A^2}{2}x^2+\frac{A^3}{2.3}x^3+\frac{A^4}{2.3.4}x^4+\ \&c.\ \ldots\ [3]$$

In which it will be remembered that $A=b-\frac{1}{2}b^2+\frac{1}{3}b^3-$ &c.; or, since $b=a-1$,

$$A=(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+\ \&c.\ldots\ldots[4].$$

The development [3] is the EXPONENTIAL THEOREM.

107. This development, unlike that of the binomial theorem, is always

interminable, whether x be positive or negative, integral or fractional. Not only so, but even the constant A, which enters each term, is exhibited, at present, only under the form of an interminable series, namely, the series [4]; but A will be computed independently of this series hereafter.

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108. **Theory of Logarithms.**—If we take any positive number a other than unity, then n representing any positive number whatever, it is always possible to determine x so that a^x may be equal to that positive number; that is, it is always possible to solve the *exponential equation* $a^x=n$, as we shall presently show. For the same proposed number n, x will of course be different for different values of the assumed *base* a; but this base being fixed once for all, the value of x, which satisfies the equation $a^x=n$, is called the *logarithm* of the number n, *to the base* a: we express this by the notation $x=\log n$. Making $n=1, 2, 3, 4$, &c., up to any proposed limit, and determining the corresponding values of x, as will be hereafter shown, we shall obtain the logarithms of all the natural numbers in succession up to the prescribed limit: these, arranged in a table, constitute a *Table of Logarithms*. Such a table is therefore simply a table of exponents. The assumed base a, to which these exponents all belong, is, in modern tables, the number 10; any number, therefore, being proposed—the number 256 for instance—the table informs us, by inspection, what exponent (x) must be put over the 10, so as that 10^x may be equal to 256; we should find that $x=2{\cdot}40824$, in other words, that $\log 256=2{\cdot}40824$.

Reserving the methods of constructing logarithms for the next section, we shall here enumerate their fundamental properties.

1. The sum of the logarithms of two or more numbers is the logarithm of the product of those numbers.

Let $a^x=n$, and $a^{x'}=n'$ $\therefore a^x\times a^{x'}=a^{x+x'}=nn'$. But by definition,

$x=\log n$, $x'=\log n'$, and $x+x=\log nn'$ $\therefore \log n+\log n'=\log nn'$.

Similarly, putting N for nn', $\log N+\log n''=\log Nn''$ $\therefore \log n'+\log n'+\log n''=\log nn'n''$, and so on.

2. The difference of the logarithms of two numbers is the logarithm of the quotient of those numbers.

Let $a^x=n$, and $a^{x'}=n'$ $\therefore a^x\div a^{x'}=a^{x-x'}=\frac{n}{n'}$.

But $x=\log n$, $x'=\log n'$, and $x-x'=\log\frac{n}{n'}$ $\therefore \log n-\log n'=\log\frac{n}{n'}$.

3. The logarithm of any power of a number, whether the exponent be integral or fractional, positive or negative, is the logarithm of the number itself multiplied by that exponent. Let the number be n, and let m be any exponent over it, and as before put

$a^x=n$ $\therefore a^{mx}=n^m$ $\therefore mx=\log n^m$, that is, $m\log n=\log n^m$.

109. These general principles sufficiently show the use that may be made of a table of logarithms in arithmetical operations that would be laborious without such aid. If we wish to find the product of several large numbers, we have only to take the log of each out of the table, and to *add* them: the sum will be the log of the product; and against this log in the table the product will be found inserted. So if we have a power of a number to raise, or a root of it to extract, all we have to do is

to take the log of the number out of the table and to multiply that log by the given exponent, the product is a log against which in the table stands the power or root sought.

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110. **Construction of Logarithms.**—The base of the proposed system of logs being a, and N any positive number, whole or fractional, we have to find x, so as to satisfy the equation $a^x=N$. By [3], since $a^{xy}=N^y$, the two following series are equal; namely:—

$$1+Axy+\frac{A^2}{2}x^2y^2+\text{ \&c.} = 1+A'y+\frac{A'^2}{2}y^2+\text{ \&c.}$$

$$\therefore \text{ by (94), } Ax=A' \therefore x=\frac{A'}{A}=\frac{(N-1)-\frac{1}{2}(N-1)^2+\frac{1}{3}(N-1)^3-\text{ \&c.}}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\text{ \&c.}}, \text{ by [4].}$$

The denom. (A) of this fraction is the same for all values of N, since the base a is constant, but the num. differs for different values of N; it is unaffected, however, by any change in the number a. Substitute $1+n$ for N, and calling unity divided by the den., that is $\frac{1}{A}$, M, we have—

$$\log(1+n)=M(n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\text{ \&c.})$$

where, as already observed, M or $\frac{1}{A}$ is constant: it is called the *modulus* of the particular system of logs adopted; as before remarked, the system in general use is the system whose base (a) is 10, so that for *this* system

$$M=1\div(9-\tfrac{1}{2}9^2+\tfrac{1}{3}9^3-\tfrac{1}{4}9^4+\text{ \&c.})$$

But Napier, the original inventor of logarithms, constructed *his* system on the hypothesis that a was such a number as would render $M=1$, or, which is the same thing, $A=1$. Let Napier's base be distinguished from all other bases by being called e, instead of a; then, since $a=e$ corresponds to $A=1$, we have, by the exponential theorem—

$$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\text{ . . . . [1]}$$

so that, putting $x=1$, $e=1+1+\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}+\text{ . . . [2]}$

If in [1] we put $-x$ for x, we have

$$e^{-x}=1-x+\frac{x^2}{2}-\frac{x^3}{2.3}+\frac{x^4}{2.3.4}-\text{ . . . . [3]}$$

so that each of the series [1], [3], is the reciprocal of the other, whatever be the value of x.

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111. On account of the rapidity with which the series [2] *converges*—that is, the rapidity with which its terms diminish—the value of e may be found to several decimal places by summing only a few of the leading terms: thus, taking only the first nine fractions which follow $1+1$, or 2, and then adding this 2, we find $e=2{\cdot}7182818$, as in the margin; and taking a few more terms, we have $e=2{\cdot}718281828459$. This value of the Napierian base e, would be equally got by putting

2	1
3	·5
4	·1666667
5	416667
6	83333
7	13889
8	1984
9	248
10	27
	3
	2·7182818

$\frac{1}{A}$ for x in [3], page 78; hence $a^{\frac{1}{A}}=e \therefore a=e^A$; and, taking the Napierian log of each side, writing that log thus, $\log_e$, we have

$$\frac{1}{A}\log_e a=1, \text{ that is } M\log_e a=1 \therefore M=\frac{1}{\log_e a},$$

so that the modulus of the common system of logarithms, or that whose base is 10, is $\frac{1}{A}=M=\frac{1}{\log_e 10}$, or 1 divided by the Napierian log of 10.

112. If, in the general series for $\log(1+n)$, p. 80, we make n negative, we have

$$\log(1-n)=M(-n-\tfrac{1}{2}n^2-\tfrac{1}{3}n^3-\tfrac{1}{4}n^4- \&c.)$$

$\therefore$ Subtracting, $\log(1+n)-\log(1-n)$, or $\log\frac{1+n}{1-n}=2M(n+\frac{1}{3}n^3+\frac{1}{5}n^5+ \&c.)\ldots$ [1].

But if in the same series we put $\frac{1}{n}$ for n, we shall have

$$\log(1+n)-\log n=M\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+ \&c.\right) \ldots\ldots [2]$$

$\therefore$ Subtracting, $\log n=M\{(n-n^{-1})-\frac{1}{2}(n^2-n^{-2})+\frac{1}{3}(n^3-n^{-3})- \&c.\} \ldots\ldots$ [3]

113. This last series for $\log n$ will find its application hereafter; but for the actual computation of logarithms none of the foregoing series are directly available, on account of their slow convergency, except in particular cases. But as n is quite arbitrary we may, by judicious substitutions for this symbol, obtain a variety of series of suitable convergency. Put $\frac{1+n}{1-n}=\frac{p+1}{p}$, from which we get $n=\frac{1}{2p+1}$: this converts [1] into the following, when $\log p$ is transposed; namely,

$$\log(p+1)=2M\{\frac{1}{2p+1}+\frac{1}{3(2p+1)^3}+\frac{1}{5(2p+1)^5}+ \&c.\}+\log p \ldots\ldots [4]$$

by means of which we may compute the logs of all the natural numbers in succession, from 1 up to any prescribed limit; for $\log 1=0$, since $a^0=1$, whatever be a (art. 42), so that putting $p=1$, the series gives log2, thence log3, log4, and so on; supposing, that is, that M have been previously computed. It has been seen that M becomes known so soon as the Napierian log of 10 becomes known; we shall first, therefore, show how Napier's logs may be found in succession, as far, at least, as $\log_e 10$. Making, then, $p=1$, 2, &c., we have from [4], remembering that in Napier's system $M=1$,

$$\log_e 2=2\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\ldots\ldots\right)\ldots\ldots\ldots=\cdot 6931472,$$

$$\log_e 3=2\left(\frac{1}{5}+\frac{1}{3.5^3}+\frac{1}{5.5^5}+\ldots\ldots\right)+\log_e 2=1\cdot 0986123,$$

$$\log_e 4=2\log 2 \text{ [art. 108]} \ldots\ldots\ldots\ldots\ldots=1\cdot 3862944,$$

$$\log_e 5=2\left(\frac{1}{9}+\frac{1}{3.9^3}+\frac{1}{5.9^5}+\ldots\ldots\right)+\log_e 4=1\cdot 6094379,$$

$$\vdots$$

$$\log_e 10=\log_e 2+\log_e 5 \text{ [art. 108]} \ldots\ldots\ldots\ldots=2\cdot 3025851,$$

Hence for the modulus M, in the system whose base is 10, we have

[art. 111] $M=1\div 2{\cdot}3025851\ldots\ldots=\cdot 43429448\ldots\ldots$, and for the logs to this base,

$$\log 2=\cdot 86858896\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\ldots\ldots\right)\ldots\ldots\ldots\ldots\ \cdot 3010300,$$

$$\log 3=\cdot 86858896\left(\frac{1}{5}+\frac{1}{3.5^3}+\frac{1}{5.5^5}+\ldots\ldots\right)+\ \log 2=\ \cdot 4771213,$$

$$\log 4=2\log 2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ =\ \cdot 6020600,$$

$$\log 5=\cdot 86858896\left(\frac{1}{9}+\frac{1}{3.9^3}+\frac{1}{5.9^5}+\ldots\ldots\right)+\ \log 4=\ \cdot 6989700,$$

$$\vdots$$

$$\log 10=\log 5+\log 2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ =1{\cdot}0000000,$$

and so on, to any extent. These series, it is seen, increase in rapidity of convergence as the number increases; but, for all numbers above 5, series of much greater convergence will be obtained by putting $\frac{2}{p^3-3p}$ for n in the general series [1]. This substitution would lead to

$$\log(p+2)=\cdot 86858896\left\{\frac{2}{p^3-3p}+\frac{1}{3}\left(\frac{2}{p^3-3p}\right)^3+\frac{1}{5}\left(\frac{2}{p^3-3p}\right)^5+\ldots\ldots\right\}$$
$$+\log(p-2)+2\log(p+1)-2\log(p-1),$$

which is of superior convergence to [4] for the logs of all numbers above 5, for *this* value the two series are the same.

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114. **Advantages of Briggs's Logarithms.**—The logarithms computed to the base $a=10$ are called Briggs's logarithms, because Briggs was the first who changed the base from $2{\cdot}718281828$ to 10, or, which amounts to the same thing, who changed the modulus from 1 to $\cdot 43429448$. This change, although increasing the labour of computing individual logarithms, as the preceding specimens serve to show, introduces in reality very important facilities; for

1. A table computed to the base 10 will be complete, though the logs of the *whole numbers* only, 1, 2, 3, &c., be inserted: the logs of numbers involving decimals would be unnecessary.

2. And even with respect to the logs of the whole numbers, the insertion of the *integral part* of any such log is unnecessary, the decimal part of the log being sufficient.

These advantages are peculiar to the system of logs computed to the base 10, and that they really belong to this system will be seen from the following illustration. Take any number within the limits of the table, the number 6524 for instance. The table gives

$$\log\ 6524\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ =3{\cdot}814514,$$
$$\therefore\log\ 652{\cdot}4=\log(6524\div 10)=\log\ 6524-1=2{\cdot}814514,$$
$$\log\ 65{\cdot}24=\log(652{\cdot}4\div 10)=\log 652{\cdot}4-1=1{\cdot}814514,$$
$$\log\ 6{\cdot}524=\log(65{\cdot}24\div 10)=\log 65{\cdot}24-1=\ \cdot 814514,$$
$$\log\ \cdot 6524=\log(6{\cdot}524\div 10)=\log 6{\cdot}524-1=\bar{1}{\cdot}814514,$$
$$\log\ \cdot 06524=\log(\cdot 6524\div 10)=\log\ \cdot 6524-1=\bar{2}{\cdot}814514,$$

and so on; observing that the minus sign *over* a figure implies that that figure is *subtractive*, the decimals that follow being additive: thus, $\bar{1}{\cdot}814514=-1+\cdot 814514$.

In like manner, $\log 6524000 = \log(6524 \times 1000) = \log 6524 + 3 = 6{\cdot}814514$.

115. It thus appears that the common log of a number always consists of the *same decimals*, however the figures of that number may be separated by the decimal point, or however many zeros may follow or precede it. It is observable, too, that the integer before the decimals in the log, always expresses the number of places which the leading significant figure in the number itself is from the units' place: thus, in the number 6524, the leading figure is *three* places from the units' place; in 652·4, it is *two* places, in 65·24, *one* place, in 6·524 *no* places distant, in ·6524, it is distant *one* place in the opposite direction, in ·06524, *two* places in the opposite direction; and so on. Hence, without any reference to a table, the integral part of the log may be written down at once. This integral part is called the *characteristic* of the log: the decimal part is called its *mantissa*. And not only is this mantissa the same for all numbers consisting of the same significant figures—without regard either to the decimal point or to zeros prefixed or appended to the number—but it remains the same for the reciprocal of a number as for the number itself.

This is plain, for $\log\frac{1}{n} = \log 1 - \log n = 0 - \log n = -\log n$; that is, *minus* the log of a number is the log of its reciprocal: thus,

$$\log(1 \div {\cdot}6524) = -\bar{1}{\cdot}814514 = 1 - {\cdot}814514, \log(1 \div {\cdot}06524) = -\bar{2}{\cdot}814514 = 2 - {\cdot}814514, \text{ \&c.}$$

116. As the characteristic, or integral part of the log of any number, is at once suggested by the number itself, the mantissa, or decimal part, is all that is inserted in the common, or Briggs's, table of logarithms. And we see that the logs of the whole numbers alone suffice for every purpose. If the base of the system were other than 10, that is, if log 10 were other than 1, these advantages, it is evident, would not be secured.

Note. The student will find it instructive to return to the general investigations at (106), (111), and observe how the value of A in [4], at page 78, has been obtained for the particular case $a=10$. To have arrived at it from the series [4] itself would have been impracticable, as that series has connected with it, under the "&c.," a value of which we know nothing; the series *alone* has no sum:—its terms tend to infinity. Abandoning, then, this uninterpretable expression for A, as practically useless, we found, from other considerations, that $A = \log_e a$, or in the particular case required, $A = \log_e 10$; and this is the value to which the series [4], even when extended to infinity, would be accurately equal, provided the supplemental addition to it, concealed under the "&c.," were to be introduced. Throughout the present work, the "&c." is always appended to every *general* development, in an infinite series, of an algebraic expression; while for those particular cases in which the series converges, the "&c." is suppressed, and a row of dots supplies its place, signifying that the infinite series itself is the complete equivalent of the undeveloped expression:—what in other cases would be a supplemental correction of it, here vanishing. Such series alone are of any practical use in mathematical researches; and it will be found that even in the most general investigations concerning algebraic developments, *converging* series only are provided for by the conditions introduced. As a *general* proposition, it is

not true that $(a+x)^n=a^n+na^{n-1}x+n\frac{(n-1)}{2}a^{n-2}x^2+\ldots\ldots$, as we have already observed (105): nor is such generality provided for in the *investigation* of the binomial theorem: provision is made only for the cases in which it *is* true. It is sufficient to advert to the preliminary theorem in support of this. There is nothing wrong in putting, as the equivalent of a complicated algebraic expression, written down at random, any series we please—even with the stipulation that the number of terms, all following the assumed uniform law, shall be inexhaustible—*provided*, we append to this arbitrary series an "&c." But such an expression would be practically useless in *all* cases. In what has preceded, every infinite series is practically useless only so long as the "&c." is indispensable; but in these cases the series is never wanted; it is of practical value only when the "&c." disappears.

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117. **Exponential Equations.**—An exponential equation is one in which the unknown enters as an exponent. The most simple equation of this kind is $a^x=b$, which is solved thus:

Taking the log of each side, $x\log a=\log b\ \therefore x=\frac{\log b}{\log a}$. The following are more complicated:—

(1) $Aa^{2x}+Ba^x+C=0$. Putting y for a^x, $Ay^2+By+C=0$: and solving this quadratic, we get the value of y, as $y=b$: it remains then only to solve $a^x=b$, as above.

(2) $5^{x-1}=2+3.5^{2-x}$. Multiply by $5^{x+1}\ \therefore\ 5^{2x}=2.5^{x+1}+3.5^3\ \therefore\ 5^{2x}-10.5^x=375$

$$\therefore 5^x=5\pm 20\ \therefore x\log 5=\log 25\ \therefore x=\frac{2\log 5}{\log 5}=2.$$

(3) $x^2+y^2=a$ } The second equa. is $\log xy=b\ \therefore xy=10^b$, hence the equations
$\log x+\log y=b$ } are the same as $x^2+y^2=a$, $xy=10^b$.

(4) $a^{b^x}=c\ \therefore\ b^x\log a=\log c\ \therefore\ b^x=\frac{\log c}{\log a}\ \therefore x\log b=\log\frac{\log c}{\log a}$

$$\therefore x=\log\frac{\log c}{\log a}\div\log b.$$

If the exponential equation be of the form $x^x=a$, the solution will require the aid of the rule of trial and error, or *double position* as it is called. It may be as well here to show the principle on which that rule is founded. Let x be an unknown number, determinable from the condition $ax+b=c$; let x', x'', be two trial values of x, and c', c'' the corresponding results: then we have the three equations $ax+b=c$, $ax'+b=c'$, $ax''+b=c''$, from which we get for a the three expressions

$$\frac{c'-c''}{x'-x''}=\frac{c-c'}{x-x'}=\frac{c-c''}{x-x''}$$

$$\therefore c'-c'' : x'-x'' :: c-c' : x-x', \text{ or } :: c-c'' : x-x'',$$

which, put into words, is the rule of double position. Let us apply it to the solution of the exponential equation $x^x=100$.

$x\log x=2$, or, putting y for $\log x$, and again taking the logs, $y+\log y=\log 2=\cdot 301030$. We have then to discover what number added to its log will make $\cdot 301030$. The nearest value in the tables, below the true value, is $\cdot 55597$, which, added to its log, $\bar{1}\cdot 745051$ gives $\cdot 301021$; and the nearest value above the truth is $\therefore\cdot 55598$, which, added to its log, $\bar{1}\cdot 745059$ gives $\cdot 301039$; so that here $c'-c''=18$, and $c-c''=9\ \therefore\ 18 : 1 :: 9 : \cdot 5$

$$\therefore y=\cdot 555975=\log x\ \therefore x=3\cdot 59728.$$

118. **Application of Logarithms.—Compound Interest.** Let r=int. of £1 for a year, and put R=£1+r, which is the amount at the end of 1 year; then £1 : R :: R : R^2=amt. at end of 2 years

£1 : R :: R^2 : R^3= „ „ 3 years

and so on; therefore the amt. of £1 at the end of n years is R^n, therefore the amt. A of P pounds at the end of n years is $A=PR^n$ $\therefore \log A=\log P+n\log R$, from which any of the four quantities A, P, R, n, may be found when the other three are given. Instead of years, n may, of course, represent the number of shorter periods at which the interest is due. Thus, if the interest be due m times a year, the interest for the first period is $\frac{1}{m}$ of the yearly int., and $R=1+\frac{r}{m}$; so that the amt. in n years is $\log A=\log P+mn \log\left(1+\frac{r}{m}\right)$. It is scarcely necessary to remark that r is merely the rate per cent. divided by 100.

Ex. What will £600 amount to in 6 years at 4½ per cent., compound interest, supposing the int. to be due half-yearly?

$$\log A=\log 600+12\log\left(1+\frac{2\cdot25}{100}\right)=\log 600+12\log 1\cdot0225=2\cdot894111,$$

the number answering to which in the tables is 783·63 $\therefore A$=£783 12*s.* 7*d.* If we wish to know the time in which a sum of money P will double, or treble, &c., itself, we have only to put $2P$, or $3P$, &c., for A in the above formula, and then to determine n.

119. If m be infinitely great, as it will be if the interest grows as it were continuously, then, dispensing with the logs, we have

$A=P\left(1+\frac{r}{m}\right)^{mn}$, and developing by the binomial theorem,

$$\left(1+\frac{r}{m}\right)^m=1+m\frac{r}{m}+\frac{m(m-1)}{2}\left(\frac{r}{m}\right)^2+\frac{m(m-1)(m-2)}{2.3}\left(\frac{r}{m}\right)^3+\ldots\ldots$$

$$=1+r+\left(1-\frac{1}{m}\right)\frac{r^2}{2}+\left(1-\frac{1}{m}\right)\left(\frac{1}{2}-\frac{1}{m}\right)\frac{r^3}{3}+\ldots\ldots$$

Now the greater the divisor m becomes, the less does the quotient $1\div m$ become; and when m is greater than any finite limit, however large, the quotient must be less than any finite limit, however small. In this case m is *infinite*, and is represented by the symbol ∞; and the quotient is *nothing*, which, of course is represented by the symbol 0; for that which is less than anything, however small, can be no other than 0. In the extreme case supposed therefore, $\frac{1}{m}=0$: hence, expunging $\frac{1}{m}$ from the foregoing development, we have (110)

$$\left(1+\frac{r}{\infty}\right)^\infty=1+r+\frac{r^2}{2}+\frac{r^3}{2.3}+\frac{r^4}{2.3.4}+\ldots\ldots=e^r=(2\cdot718281828\ldots\ldots)^r.$$

Hence, when the interest continuously accumulates, the amount in one year is $A=Pe^r$, and in n years, $A=Pe^{nr}$.

The expressions $\left(1+\frac{r}{\infty}\right)^\infty=e^r$, and $\therefore\left(1-\frac{r}{\infty}\right)^\infty=e^{-r}$, are remarkable, and show that either one is the reciprocal of the other (see art. 110).

120. **Annuities.**—Let a represent an annuity or yearly pension.

If it be left unpaid n years, and then the whole amount A be paid, the first year's annuity will have amounted to aR^{n-1}, seeing that 1 year must have elapsed before a bears any interest: the amount of the second year's annuity will be aR^{n-2}, and so on; the last or nth year's annuity being simply A, as *this* does not remain to accumulate at interest. Hence the whole amount is—

$$A=a(R^{n-1}+R^{n-2}+R^{n-3}+\ldots\ldots+1).$$

The n terms within the brackets form a geom. prog. Hence (84) $A=a\dfrac{R^n-1}{R-1}=a\dfrac{R^n-1}{r}$, from which any three of the quantities A, a, r, n being given, the fourth may be found: for ex., what will an annuity of £30 6*s.* 4*d.* amount to in 12 years, at 5 per cent. compound interest? Here $a=$ £30·31667, $r=$·05, $n=12$ $\therefore$ $A=30\cdot31667.\dfrac{1\cdot05^{12}-1}{\cdot05}$; the computation of which is given in the margin, where it will be observed that the multiplier 30·31667 has been reversed, and contracted multiplication used; but this multiplication might have been avoided by continuing to work by logs: thus, log $A=$ log 30·31667 + log ·7958563. [To every collection of tables is prefixed the necessary explanation of how to use them.]

log 1·05 = ·0211893
12
log $1\cdot05^{12}=$ ·2542716
$\therefore$ $1\cdot05^{12}=1\cdot7958563$
1
·7958563
7661303
23875689
238757
7959
4775
477
55
·05)24·127712
$\therefore$ $A=$£48211*s.*1*d.*=£482·5542

121. **Present Value of an Annuity.**—The present value of an annuity is the principal P, which, accumulating at compound interest during the time the annuity a is forborne, would amount to A, or the same sum that the annuity itself would amount to. In n years—

$$A=PR^n=a\frac{R^n-1}{r} \therefore P=\frac{a}{r}\left(1-\frac{1}{R^n}\right)=\frac{a}{r}(1-R^{-n}).$$

If a be a perpetual annuity, or to continue for ever, then $n=\infty$, and $\therefore$ $1\div R^n=0$ $\therefore$ $P=\dfrac{a}{r}$ is the present value of a perpetuity of a, so that to find the value of the perpetuity we have only to divide the annuity by the interest of £1; that is, $\dfrac{1}{r}$ will express the number of years' purchase.

Ex. (1). What is the present value of an annuity of £170, to continue 20 years at 5 per cent. compound interest? Here $a=170$, $r=$·05, and $n=20$ $\therefore$ $R=1\cdot05$.

$$\log 1\cdot05^{-20}=-20\log 1\cdot05=-\cdot423786 \therefore 1\cdot05^{-20}=\cdot37688948.$$

$$\text{Also } \frac{a}{r}=\frac{170}{\cdot05}=3400 \therefore \frac{a}{r}(1-R^{-n})=3400\times\cdot62311052=2118\cdot575768.$$

Hence the purchase-money P is $P=$£2118 11*s.* 6*d.*

(2) How many years' purchase must be paid for a freehold estate when the interest of money is $3\frac{1}{2}$ per cent.? Here $r=$·035, and $1\div\cdot035=28\frac{4}{7}$, the number of years' purchase.

122. **Annuities in Reversion.**—A reversionary annuity is one of which the payment does not commence till a certain number of years

have elapsed. Let the annuity commence after m years, and then continue n years: the present value will be equal to that of the same annuity for $m+n$ years, diminished by the present value of it for m years, or

$$P=\frac{a}{r}\{1-R^{-(m+n)}-(1-R^{-m})\}=\frac{a}{r}\{R^{-m}-R^{-m-n}\}=\frac{a}{r}R^{-m}\{1-R^{-n}\},$$

that is, the present value of the reversion is found by multiplying the present value of the annuity for n years by the present value of £1 receivable at the end of m years. If a is a perpetuity, then $R^{-n}=0$, in which case $P=\frac{a}{r}R^{-m}$ $\therefore$ $\frac{R^{-m}}{r}$ is the number of years' purchase.

Ex. (1) What present money should be given for the reversion of £30 per annum for 8 years, to be entered upon after the expiration of the next 10 years; compound interest at 5 per cent.? Here $a=30$, $r=\cdot05$,

$$\therefore \frac{a}{r}=600, \text{ also } n=8, \text{ and } m=10,$$

$$\left.\begin{aligned} \therefore R^{-m} &=1\cdot05^{-10}=\cdot613913 \\ R^{-m-n}&=1\cdot05^{-18}=\cdot415521 \end{aligned}\right\} \begin{aligned} &\text{The difference} \times 600=119\cdot0352 \\ &\therefore P=£119\ 18s.\ 3\tfrac{1}{2}d. \end{aligned}$$

(2) How many years' purchase is the reversion of a perpetuity, after 10 years, worth at 5 per cent. comp. int.? Here $m=10$, and $r=\cdot05$, $R^{-m}=1\cdot05^{-10}=\cdot613913$, which, divided by $\cdot05$, gives 12·27826 years' purchase.

[For a variety of tables to abridge the calculations, the reader is referred to "Jones's Annuities," &c., published by the Society for the Diffusion of Useful Knowledge.]

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123. **Increase of Population.**—If P represent the population of a country at any time, and if the numbers increase by an rth part of the whole each year, then Pr will be the increase at the end of the 1st year, and $P(1+r)$ the amount of the population: and, just as in compound interest (118), the amount of the population, at the end of the 1st, 2nd, 3rd.....nth year, will form the geometrical progression.

$$P(1+r),\ P(1+r)^2,\ P(1+r)^3,\ \ldots\ldots\ P(1+r)^n.$$

Calling the common ratio $1+r$, R, and the population at the end of the nth year, P_n, we have

$$P_n=PR^n \therefore \log P_n=\log P+n\log R \ldots\ldots [1] \therefore R=\left(\frac{P_n}{P}\right)^{\frac{1}{n}}=\frac{P_1}{P}.$$

Of course, n may stand for n periods of any kind, as well as for n *years;* so that if a census of the population were taken at the beginning, and also at the end of one of these periods, and the resulting value of $\frac{P_1}{P}$ found, and if this value were compared with the value of $\left(\frac{P_n}{P}\right)^{\frac{1}{n}}$, furnished by a census taken n of the periods from the aforesaid beginning, the difference between the two would show to what extent the natural law of increase had been disturbed, during the interval, by extraneous causes, such as emigration, war, epidemic diseases, &c.

If $P_n=mP$ then [1], $\log P_n=\log m+\log P=\log P+n\log R \therefore n=\frac{\log m}{\log R}$, which denotes the number of years in which the population will be increased m-fold.

Ex. The increase of the population of England and Wales from 1841

to 1851 was 12·64 per cent. At this rate of increase, in how many years will the population of 1851 be doubled?

$$n=\frac{\log m}{R}=\frac{\log 2}{\log 1{\cdot}1264}=\frac{{\cdot}30103}{{\cdot}0517}=\frac{30{\cdot}103}{5{\cdot}17}=5{\cdot}82.$$

But as the census of 1851 was taken 68 days earlier than that of 1841, n, instead of representing so many periods of 10 years, represents only so many periods of about 9·8 years: hence $5{\cdot}82\times 9{\cdot}8=57$, the number of years in which the population of 1851 may be expected to be doubled.

Examples for Exercise.

(1) At what rate, compound interest, must £400 be placed out so as to amount to £569 6*s.* 8*d.* in nine years?

(2) In how many years will £500 amount to £900 at 5 per cent. comp. int.?

(3) In what time would a sum of money treble itself at 5 per cent. comp. int.?

(4) If the compound interest be due half-yearly, what will £500 amount to in 8 years at 5 per cent.?

(5) What will an annuity of £30 amount to in 16 years at $4\frac{1}{2}$ per cent. comp. int.?

(6) What is the present value of an annuity of £30 to continue for 30 years at $3\frac{1}{2}$ per cent. comp. int.?

(7) How many years' purchase must be paid for a freehold estate, allowing 4 per cent. comp. int.?

(8) What is the present value of a freehold of £50 per ann. to a purchaser who is to take possession 4 years hence, allowing comp. int. at 5 per cent.?

(9) How many years' purchase is the reversion of an annuity worth which is to commence 7 years after the purchase is concluded and to last 14 years, allowing comp. int. at 5 per cent.?

(10) If a country contains 20 millions of inhabitants, and the annual increase is $\frac{1}{250}$th of the whole, what will be the population in 50 years?

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124. **Permutations.**—The permutations of a set of things are the different ways in which they can be arranged by interchanging their order of succession: thus, two things, *a*, *b*, admit of two permutations, *ab* and *ba*, and of no more; but three things, *a*, *b*, *c*, admit of six permutations, namely, *abc*, *acb*, *bac*, *bca*, *cab*, *cba*. A convenient notation for this is $P_{(3)}=1.2.3=6$, the bracketed 3 below the P implying that the permutations refer to *three* things taken altogether. If a fourth thing, *d*, be introduced, it may be placed in four different positions in *each* of the above sets of three: it may stand first, second, third, or fourth; so that *a*, *b*, *c*, *d*, admit of 6×4 permutations $\therefore P_{(4)}=1.2.3.4=24$. In like manner, if there are five things, *a*, *b*, *c*, *d*, *e*, the fifth *e* may be introduced in 5 different ways among each of the preceding 24 sets of four $\therefore P_{(5)}=1.2.3.4.5=120$, and so on. Hence, generally, for n things, $P_{(n)}=1.2.3.4.5\ldots\ldots n\ldots\ldots[1]$. The things permuted are here supposed to be taken altogether, but it is frequently required to find the number of permutations that can be formed with m things taken out of n things.

Problem I. *To find the number of Permutations that can be formed with m things out of n things, a, b, c, d, &c.*

1. Let $m=2$. The number of these pairs, in which any one of the letters may stand first, is obviously $n-1$, because each of the remaining $n-1$ letters may be put after it; and as *each* of the n letters may in turn stand first, the whole number of permutations, when the letters are taken two and two, must be $n(n-1)$.

2. Let $m=3$. Suppose one of the n letters, as a, to be removed, then the remaining $n-1$ letters, if taken two and two, will furnish $(n-1)(n-2)$ permutations, as just shown. Let the a be now restored, and prefixed to each of these sets of two; there will then be $(n-1)(n-2)$ different permutations of *threes*. But had any *other* of the n letters, instead of a, been removed, the same might have been said; so that, altogether, the number of different permutations of threes is $n(n-1)(n-2)$.

3. Let $m=4$. Applying similar reasoning, the number of permutations of *fours* will be n times the number of permutations of *threes* furnished by $n-1$ letters; that is, n times $(n-1)(n-2)(n-3)$. And, proceeding in this way, it is evident, generally, that the number of permutations of n things taken m together is—

$$^{n}P_{m}=n(n-1)(n-2)(n-3).....(n-m+1).....[2].$$

where the notation on the left indicates that n is the total number of things, and m the number in each set taken out of them for permutation. If $m=n$ we return to the case [1], as is plain; for, reversing the factors here written, we should then have $1.2.3.4.....n$ for the right-hand member of [2]. But the product would have been just the same if we had made $m=n-1$, instead of $m=n$, for, still reversing the factors, it would have been $2.3.4.....n$: we may, therefore, safely conclude that the number of permutations furnished by n things, by taking $n-1$ of them at a time, is the same as the number furnished by taking them altogether. Thus, in the case of three things, a, b, c, as seen above, the permutations, when they are all taken together, are 6 in number; and when taken two and two, they are ab, ba, ac, ca, bc, cb, which are also 6 in number.

If some of the n things out of which the permutations are to be formed are identical, then certain of these permutations will recur, differing in no respect from one another. In many inquiries these recurring groups must be dismissed, and only those retained which really differ from one another. We have, therefore—

PROBLEM II. *To find the number of different permutations of n things, taken altogether, when certain of these n things are identical.*

1. Beginning, as before, with the simplest case, suppose only *two* of the n things are alike; then, since these two enter every permutation of all the n things, they enter $1.2.3.....n$ times [1]. But every permutation is accompanied with another, in which the two like things are simply interchanged: hence there can be only half the whole number of permutations that are really all different; so that the foregoing product must be divided by 2, or by 1.2.

2. Suppose *three* of the n things are alike. These three enter every permutation, each being accompanied with *five* others, formed by merely interchanging the places of the three like things; so that the whole set may be divided into sets of six, in each of which there will be but one distinct permutation; hence to obtain the number of distinct permutations we must divide the entire number by 1.2.3, the number of permutations furnished by the three like things. And, continuing this reasoning, we arrive at the conclusion, that if p of the n things are alike, there must

be as many sets of recurring permutations, each set consisting of the same number of individual permutations, as there are permutations of those p things. Hence the number of different permutations of n things, taken altogether, when p of them are alike, is—

$$\frac{1.2.3.....p...n}{1.2.3...p}=\frac{P_{(n)}}{P_{(p)}}=(p+1)(p+2).....n.....[3].$$

3. After thus eliminating all the recurring permutations arising from the p like things, q other like things may be still conceived to enter. In this case, the permutations just determined divide themselves into as many sets of recurring permutations, as there are permutations of those q things. Hence the number of different permutations of n things, taken altogether, of which p of them are alike, and also q others of them are alike, is

$$\frac{1.2.3...n}{(1.2.3...p)(1.2.3...q)}=\frac{P_{(n)}}{P_{(p)}\times P_{(q)}},$$

and this reasoning may be extended to three, four, and, generally, to any number of distinct sets of recurring things. If the factors in the numerator be reversed, and the p factors, common to num. and den., p being $>q$, be suppressed, we shall have

$$\frac{P_{(n)}}{P_{(p)}P_{(q)}}=\frac{n(n-1)(n-2).....(p+1)}{1.2.3.....q}.$$

If $p+q$ make up the whole number n of things, then $q=n-p$,

$$\therefore \frac{P_{(n)}}{P_{(p)}P_{(n-p)}}=\frac{n(n-1)(n-2).....(n-p+1)}{1.2.3.....p}.....[4].$$

It is sometimes required to form permutations of m things out of n, when liberty is given to introduce into the permutations *repetitions* of any of the n things, or to solve the following problem.

PROBLEM III. *To find the number of permutations of n things, taken two and two, three and three, and so on, allowing repetitions.*

Let the n things be represented by letters connected together by the plus or minus sign, as $a+b+c+d+....$ It is obvious that if this quantity be multiplied by itself, the product will exhibit all the possible permutations, two and two, admitting the entrance of the repetitions *aa*, *bb*, *cc*, &c. If this product be multiplied by the same multiplier, all the permutations, three and three, will be exhibited, admitting repetitions such as *aaa*, *aab*, *abb*, &c.; and so on. Hence, allowing repetitions, the number of permutations of n things, taken two and two, is n^2; taken three and three, n^3; and, generally, taken m and m, the number is n^m.

(1) How many changes may be rung with a peal of 8 bells, and how many may be rung with 5 out of the 8?

$P_{(8)}=1.2.3.4.5.6.7.8=40320$, and ${}^8P_5=8.7.6.5.4=6720$.

(2) How many different permutations may be formed of the letters in the word Mississippi, taken altogether?

Here are 11 letters: i and s each occur four times, and p twice,

$$\therefore \frac{P_{(11)}}{P_{(4)}P_{(4)}P_{(2)}}=\frac{5.6.7.8.9.10.11}{(1.2.3.4)(1.2)}=5.7.2.9.5.11=34650.$$

EXAMPLES FOR EXERCISE.

(1) Nine men stand in a rank, in how many ways can their order be varied?

(2) In how many different ways may 10 persons seat themselves at table?

(3) How many different permutations may be formed of the letters in the word Salamanca, taking them altogether?

(4) How many different numbers can be formed of the figures in the number 223334444, taking them altogether?

(5) How many different numbers can be expressed by four different figures, taking them one at a time, two at a time, three at a time, and altogether; repetitions of the same figure being excluded?

(6) How many can be expressed when repetitions are allowed?

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125. **Combinations.**—Combinations of things are the different collections—each collection containing the same number—that can be formed out of a given number of things, under the restriction that in no two sets shall the things be all the same. Mere order of arrangement of the individual things in a set is not regarded here as in permutations: a combination, to differ from another, must have in it *one* thing, at least, which the other has not: thus, the combinations of *a*, *b*, *c*, taken two and two, are *ab*, *ac*, *bc*, and no others: *ba*, *ca*, *cb*, are excluded, because, though different permutations, they are the same combinations as the three above.

PROBLEM I. *To find the number of combinations that can be formed out of n different things, when p of them are taken at a time.*

1. Let the n things be taken two at a time. The number of *permutations* that may be formed of them will be $n(n-1)$, as already shown. Now each permutation, as *ab*, is accompanied with another, *ba*; so that there are only half as many combinations as permutations: hence the number of combinations is $\frac{n(n-1)}{2}$.

2. Let the n things be taken three at a time. Then, p. 89, the number of permutations that may be formed will be $n(n-1)(n-2)$. Now every combination, as *abc*, is only one out of 1.2.3 permutations of the same letters (124): hence the number of combinations is $\frac{n(n-1)(n-2)}{1.2.3}$. And following out this reasoning, it is evident that when the n things are taken p at a time, the number of combinations is

$$^nC_p=\frac{n(n-1)(n-2)\ldots(n-p+1)}{1.2.3\ldots p}\ldots[1].$$

Putting in this general expression 1, 2, 3, &c., successively for p, we have

$$^nC_1=n,\ ^nC_2=\frac{n(n-1)}{2},\ ^nC_3=\frac{n(n-1)(n-2)}{2.3},\ ^nC_4=\frac{n(n-1)(n-2)(n-3)}{2.3.4},\ \&c.,$$

which are the successive coefficients that follow the first term 1, in the development of $(1+x)^n$: hence (101)

$$1+{}^nC_1+{}^nC_2+{}^nC_3+{}^nC_4+\ldots\ldots+{}^nC_n=2^n\ldots\ldots[2],$$

so that the number of combinations of n things, formed by taking them first singly, then two at a time, then three at a time, and so on, is 2^n-1.

To find how many, out of n different things, must be taken at a time, so that the combinations may be the most numerous possible, we have evidently only to find the place of the term involving the greatest coefficient in the development of $(1+x)^n$. Not counting the first term 1, of the development, the greatest coefficient belongs to the $\frac{1}{2}$ nth term, if n be even, and to the $\frac{1}{2}(n-1)$th, or $\frac{1}{2}(n+1)$th, indifferently, if n be odd

(see art. 104). Hence, in the former case, half the proposed number of things must be taken: in the latter, half the number, which is a unit, either less or greater.

If $n-p$ be put for p in [1], then, observing that [1] is the very same as the expression [4] page 90, we shall have

$$^nC_p=\frac{P_{(n)}}{P_{(p)}P_{(n-p)}}, \text{ and } ^nC_{n-p}=\frac{P_{(n)}}{P_{(n-p)}P_{(p)}}\ldots\ldots[3],$$

which expressions are identical: consequently the *number* of combinations of n things, taken $n-p$ at a time, is the same as the number when p of those things are taken at a time. Hence, whether ^nC_p, or $^nC_{n-p}$, are computed by [1], the result is the same: these two sets of combinations (equal in number) are said to be *supplementary* to each other. It is plain that $^nC_o=^nC_n=1$.

PROBLEM II. *To find the number of combinations that may be formed out of n_1, n_2, n_3, &c., different things, by taking one thing out of each of the sets.*

1. Let there be two sets, n_1, n_2. Then, since each of the n_1 things in the first may be combined with every one of the n_2 things in the second, the whole number of combinations will evidently be n_1n_2.

2. Let now a third set of n_3 things be introduced; each of these may be combined with all the n_1n_2 combinations just deduced: therefore the whole number of combinations furnished by $n_1n_2n_3$, will be $n_1n_2n_3$; and so on. Hence, generally, the number of combinations that may be formed by taking one thing out of each of m sets of different things will be $n_1n_2n_3\ldots n_m$. If each set contain the same number n of things, this last product will, of course, be n^m.

In a similar manner it may be shown, after reference to Prob. I., that if instead of one thing, p_1 things are taken from the first set, p_2 from the second set, and so on, then the number of combinations that may be formed from the different selections is

$$^{n_1}C_{p_1}\times{}^{n_2}C_{p_2}\times{}^{n_3}C_{p_3}, \text{ \&c.}$$

(1) How many combinations may be formed with 7 things out of 12?

By equation [3] above, $^{12}C_7=^{12}C_5=\frac{12.11.10.9.8}{1.2.3.4.5}=6.11.2.3.2=792.$

(2) How many combinations may be formed out of 6 things taken 1, 2, 3, &c., at a time, and altogether? Here (art. 125), $2^n-1=2^6-1=63$.

(3) How often may a different guard of 6 men be selected out of 60, so that no two may mount guard together oftener than once?

Out of the 60 men, $\frac{60.59}{1.2}=30.59$ couples may be formed; and out of every 6 men, $\frac{6.5}{2}=15$ of those couples; but only one of these is to be admitted: hence, taking the 15th part of 30.59, we have $2.59=118$.

(4) How often may a different guard of 6 men be taken out of 60, how often would the same man be among the six, and how often would the same two men be found on guard together?

1. $^{60}C_6=\frac{60.59.58.57.56.55}{1.2.3.4.5.6}=59.58.57.14.55=50063860$ total number of guards.

2. Suppose the specified man were excluded, and that guards of *five* men only were made up out of the remaining 59 men: the number of the sets of five would be ${}^{59}C_5 = \frac{59.58.57.56.55}{1.2.3.4.5} = {}^{60}C_6 \div 10 = 5006386$, and into each of these sets must the excluded man be introduced to complete the guard: he will, therefore, be on guard 5006386 times.

3. Suppose the *two* comrades excluded, and the guard to consist of but *four* men, made up out of the remaining 58 men: then ${}^{58}C_4 = {}^{59}C_5 \times \frac{5}{59} = 5006386 \times \frac{5}{59} = 424270$; and into each of these sets must the two excluded men be introduced to complete the guard: they will ∴ be on guard 424270 times.

Examples for Exercise.

(1) How many combinations may be formed of 11 things taken 4 at a time?

(2) How many *fours* can be selected out of ten, so that the same individual may always be one of the four?

(3) What is the greatest number of combinations that can be formed out of 10 things?

(4) A person wishes to make up as many different dinner parties, each of the same number of guests, as possible, out of 8 friends: how many should he invite at a time, and how many parties can he give?

(5) How many combinations can be formed of the letters in the word *notation*, taking three at a time?

(6) Fifteen young ladies at a school walk out daily, three and three: how many walks may they take if arranged so that no two shall walk abreast twice?

(7) How often may a different guard of six men be selected out of 60, so that a *specified couple* of men may never be on guard together?

(8) From 16 privates, 6 corporals, and 4 sergeants, how many different guards can be formed, each consisting of 12 privates, 2 corporals, and one sergeant?

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126. **Probabilities.**—The doctrine of Probabilities originates in our ignorance of operating causes. If a die with six equal faces be thrown from a dice-box, we say that there is the same *chance* that any one face will turn up as any other: yet the turning up of the face which presents itself after the throw must be a necessary consequence of all the motions communicated to the die by the thrower. Of the effect of these, however, we are ignorant till after the event: and as the thrower himself has no control over the motions he communicates so as to predetermine the result, we are correct in saying that any one face is just as likely to turn up as any other. If, therefore, six persons were each to predict the turning up of a different face, the six chances would be equal. Suppose a sum of money S to become due to him whose prediction is fulfilled: it is plain that before the die is thrown, each of the six persons has the same interest in the sum S; and if these interests were all to be purchased at their exact value, the sum paid for them ought to be exactly S, because one of the six predictions must necessarily be successful, and thus the sum S be recovered. Hence the value of each of the six equal chances

is $\frac{1}{6}S$, or $\frac{1}{6}$th the value of certainty. Representing, then, certainty by 1, the chance or probability of any specified face turning up is $\frac{1}{6}$. And it is evident that in a case where there are n equal chances instead of 6, the probability of a specified one of the n events happening is $\frac{1}{n}$.

PROBLEM I. *If an event may happen in a ways, and fail in b ways, any one of these being equally probable, the chance of its happening is* $\frac{a}{a+b}$, *and the chance of its failing* $\frac{b}{a+b}$.

For the whole number of occurrences being $a+b$, that any specified one may succeed or fail, the probability is $\frac{1}{a+b}$, which may be regarded as the value of the chance; but there are a such chances of success, and b such chances of failure; therefore the value of all the chances for the event is $\frac{a}{a+b}$, and the value of all the chances against it is $\frac{b}{a+b}$: that is, these fractions express the probability of happening and of failing respectively; and the sum of them is 1, or certainty, as it ought to be.

(1) A bag contains 12 balls, 7 black and 5 white: what is the probability, in drawing two at once, that they shall both be white?

By the preceding problem, we must find all the possible ways in which two balls can be drawn; then how many of them are favourable to the proposed result:—the latter number divided by the former will be the probability of that result happening.

All the combinations of two out of 12 are ${}^{12}C_2 = \frac{12.11}{1.2} = 66$
" " 5 " ${}^{5}C_2 = \frac{5.4}{1.2} = 10$
$\therefore$ Prob^{y}. $= \frac{5}{33}$.

In like manner, $\frac{7.6}{1.2} \div \frac{12.11}{1.2} = \frac{7.6}{12.11} = \frac{7}{22}$, the probability that both shall be black.

$\therefore 1 - \frac{5}{33} = \frac{28}{33}$, and $1 - \frac{7}{22} = \frac{15}{22}$ $\therefore \frac{28}{33}$

is the probability that both shall not be white, that is, that one at least shall be black, and $\frac{15}{22}$ is the prob^{y}. that both shall not be black, that is, that one at least shall be white. The ratio of the probabilities of an event happening and failing is called the *odds*: thus, in the above example, the odds against drawing two white balls is $\frac{28}{33}$ to $\frac{5}{33}$, or 28 to 5: the odds against drawing two black balls is 15 to 7. The odds is found by merely subtracting the numerator of the probability from the denominator: if the rem. is less than the num. the odds are in favour of the event to which that probability refers; if the rem. be greater, the odds are against it: if the rem. be equal to the num. the odds are equal, or *even*.

(2) What is the probability of throwing an ace *once*, and not oftener, in four throws of a die; or, which amounts to the same thing, in a single throw with four dice?

Only *one* face of each die can enter into any combination, because two faces of the same die cannot appear at once. Hence, we have first to find the whole number of combinations of four faces, by taking 1 out of

each of the four sets of 6 faces: this number, by Prob. II. (art. 125), is $6^4=1296$. For the number of favourable combinations, each ace must combine with one out of *five* only of the faces of each of the other dice; so that each ace furnishes 5^3 combinations, and since there are four aces, the whole number of combinations favourable to the event is

$$5^3\times 4=125\times 4 \therefore \text{the prob}^{\text{y}}.\ \text{reqd. is } \frac{125.4}{1296}=\frac{125}{324}.$$

NOTE. Whether one die be thrown m times, or m dice thrown once, can make no difference; because intervals of time do not enter into consideration in determining the combinations.

PROBLEM II. *If an event* MAY *happen in any one of n different ways, and if the probability of its happening in a specified one of the n ways be p_1, the probability of its happening in a second specified way be p_2, in a third p_3, and so on: then the probability that it will happen in one or other of the n ways will be the sum of the individual probabilities, namely,*

$$p_1+p_1+p_3+\dots+p_n.$$

Suppose a sum of money S to be receivable upon the happening of the event: then the value of all the n expectations will be $p_1S+p_2S+p_3S+\dots+p_nS=(p_1+p_2+p_3+\dots+p_n)S$. But the quantity that multiplies S must be the fraction of certainty, that is, the probability, that the event *will* happen in one or other of the n ways: this probability is, therefore,

$$p_1+p_2+p_3+\dots+p_n.$$

(3) In Ex. 1, what is the probability that the two balls drawn will both be of one colour? The event may happen in two different ways: the balls may be both white or both black: the probability that both will be white has been seen to be $\frac{5}{33}$, and that both shall be black, it is $\frac{7}{22}$: hence that one or other shall happen, the probability is $\frac{5}{33}+\frac{7}{22}=\frac{31}{66}$.

The probability of *not* drawing either two white or two black balls, that is, the probability of drawing one of each, is $1-\frac{31}{66}=\frac{35}{66}$.

(4) Out of 100 mutineers, two men, drawn by lot, are to be shot: the real leaders of the mutiny are 10: what is the chance, 1. that one of these, and one only, will be shot; 2. that one, at least, will be shot; and 3. that two of them will be shot?

1. If one and one only is shot, the pair of men must consist of one man from the 10, and one from the remaining 90; the number of these combinations is 900; and the whole number of combinations, two and two, is $\frac{100.99}{1.2}=4950 \therefore \frac{900}{4950}=\frac{2}{11}$ is the probability of one and one only.

2. If the restriction to *one only* be removed, then, in addition to the 900 combinations, there will enter the combinations from the 10, taken two and two; these are $\frac{10.9}{1.2}=45$, giving the probability $\frac{45}{4950}$ that the selection will be made from these 10: hence, adding the two, $\frac{945}{4950}=\frac{21}{110}$, the probability that one at least will be shot.

3. The probability that two out of the 10 will be shot, is evidently $\frac{45}{4950}$, just determined, and which is $=\frac{1}{110}$.

PROBLEM III. *If any number of independent events can happen, in conjunction, the probability of their happening together is the product of the several probabilities of their happening separately.*

Let the individual probabilities be $\frac{a_1}{n_1}, \frac{a_2}{n_2}, \frac{a_3}{n_3}, \ldots \frac{a_m}{n_m}$, where $a_1, a_2, \ldots a_m$, are the numbers denoting the occurrences favourable to the respective events, and $n_1, n_2, \ldots n_m$, *all* the possible occurrences; then the favourable cases will be expressed by all the combinations of 1 out of a_1, with 1 out of a_2, with 1 out of a_3, &c.; and all the possible cases by all the combinations of 1 out of n_1, with 1 out of n_2, with 1 out of n_3, &c. But by Prob. II. p. 92, these several combinations are in number $a_1\, a_2\, a_3 \ldots\ldots a_m$, and $n_1 n_2 n_3 \ldots n_m$. Hence the probability of the compound event is

$$\frac{a_1 a_2 a_3 \ldots\ldots a_m}{n_1 n_2 n_3 \ldots\ldots n_m}.$$

(1) What is the probability of throwing an ace with a single die once at least in two trials? The prob^{y}. of succeeding the first time is $\frac{1}{6}$, and of failing, $\frac{5}{6}$; and the same as respects the second time. Hence, by the above, the prob^{y}. of failing *both* times is $\frac{5}{6}\times\frac{5}{6}=\frac{25}{36}$, and $\therefore$ the prob^{y}. of *not* failing both times, that is, of succeeding once at least, is $1-\frac{25}{36}=\frac{11}{36}$.

(2) What is the probability of throwing exactly *two* aces in four throws, *three* in four throws, and of ace turning up every throw? 1. The prob^{y}. that any *two* specified throws may turn up ace is $\left(\frac{1}{6}\right)^2$, (Prob. III.), and that the other two throws may fail, $\left(\frac{5}{6}\right)^2$, $\therefore$ the prob^{y}. of the concurrence of these events is $\frac{5^2}{6^4}$. But the two specified throws may be *any* two out of the six combinations of two furnished by the four dice, $\therefore\ 6\frac{5^2}{6^4}=\frac{25}{216}$ is the prob^{y}. that some two throws will each turn up ace, and the other two fail. 2. The prob^{y}. that any three specified throws will produce ace is $\left(\frac{1}{6}\right)^3$, and the prob^{y}. that the remaining throw will fail is $\frac{5}{6}$, $\therefore$ the prob^{y}. of both happening is $\left(\frac{1}{6}\right)^3\times\frac{5}{6}$; but there are 4 different ways in which three throws out of four may be combined, $\therefore\ \left(\frac{1}{6}\right)^3\times\frac{5}{6}\times 4=\frac{20}{1296}=\frac{5}{324}$ is the probability required.

(3) The prob^{y}. of an ace every throw is $\left(\frac{1}{6}\right)^4 = \frac{1}{1296}$. And $\therefore$ by (II.) the prob^{y}. of throwing either three aces exactly, or four aces, that is, the prob^{y}. of throwing three aces at least, is $\frac{20}{1296} + \frac{1}{1296} = \frac{21}{1296} = \frac{7}{432}$. But the following general problem is better suited to all questions of this kind.

PROBLEM IV. *The probability of an event happening* EXACTLY *m times in n trials is expressed by the* $(n-m+1)$*th term of the development of the binomial* $(p+q)^n$, *where p, q are the respective probabilities of happening and failing in a single trial. And the probability of happening* AT LEAST *m times, in n trials, is expressed by the sum of the first* $n-m+1$ *terms of the same development.*

Let p be the prob^{y}. of happening in a single specified trial, and $q=1-p$, the prob^{y}. of failing in that trial. Then the prob^{y}. of the specified trial succeeding, and all the remaining $n-1$ trials failing, is pq^{n-1}; and since any one of the n trials may be the trial specified, the prob^{y}. that one or other will succeed and all the rest fail, is npq^{n-1}. Again, the prob^{y}. of the event happening twice in two specified trials, and failing in the others, is p^2q^{n-2}; and since all the combinations, two and two, out of n things, are $\frac{n(n-1)}{1.2}$ in number, and that either of these may be the combination specified; the prob^{y}. that one or other of the combinations will succeed and the remaining trials fail, is $\frac{n(n-1)}{1.2}p^2q^{n-2}$. In like manner, the prob^{y}. of the event happening three times in three specified trials, and failing in the other $n-3$ trials, is p^3q^{n-3}; and since all the combinations, three and three, out of n things is $\frac{n(n-1)(n-2)}{1.2.3}$, and that either of these may be the combination specified; the prob^{y}. that one or other of them will succeed and the rest fail is $\frac{n(n-1)(n-2)}{1.2.3}p^3q^{n-3}$, and so on. Hence, generally, the prob^{y}. that one or other of the sets of m trials will succeed and the remaining $n-m$ trials fail, is

$$\frac{n(n-1)(n-2)\ldots(n-m+1)}{1.2.3\ldots m}p^mq^{n-m},$$

that is, it is ${}^nC_mp^mq^{n-m}$, or (p. 92), ${}^nC_{n-m}p^mq^{n-m}$...[1].

The prob^{y}. that the event will happen *at least* m times, in the n trials, is the same as the prob^{y}. that it will happen *either* n times, or $n-1$ times, or $n-2$ times, &c., down to m times, which (II.) is the sum of the probabilities of these happenings individually; that is, it is the sum of the first $n-m+1$ terms in the development of $(p+q)^n$.

Since $p+q=1$ $\therefore$ $(p+q)^n=1$, so that the sum of all the $n+1$ terms of the development is 1; hence the first $n-m+1$ terms will be obtained by subtracting the last m terms from 1; but (97) the last m terms of $(p+q)^n$ are the same as the first m terms of $(q+p)^n$; hence the prob^{y}. is expressed by unity minus the first m terms of $(q+p)^n$.

NOTE.—The first term p^n of the development of $(p+q)^n$ expresses the

prob^{y}. that all the n trials succeed, the last term q^n, the prob^{y}. that they all fail.

(1) A shilling is tossed up: what are the odds in favour of four heads at least turning up in ten trials?

In one trial the prob^{y}. of head is $\frac{1}{2}$, which is also the prob^{y}. of tail; hence—

$$p=\tfrac{1}{2},\ q=\tfrac{1}{2},\ \text{and}\ n=10\ \therefore\ (p+q)^n=(\tfrac{1}{2}+\tfrac{1}{2})^{10}=(\tfrac{1}{2})^{10}(1+1)^{10},$$

$$=(\tfrac{1}{2})^{10}\{1+10+\frac{10.9}{1.2}+\frac{10.9.8}{1.2.3}+\ldots\ldots+1\}.$$

The last four terms of this series is the same as the first four, the sum of which is 176; hence the prob^{y}. is $1-\frac{176}{1024}=\frac{848}{1024}=\frac{53}{64}$, and therefore the odds in favour of four heads at least turning up are 53 to 11 (see page 94).

(2) Sixty tickets are divided into six parcels of 10, each parcel consisting of 3 prizes and 7 blanks. If a person draw a ticket from every parcel, what will be the most probable result of his six drawings?

Suppose m prizes are drawn, and therefore $6-m$ blanks, the probability of this combination, by [1], is ${}^{6}C_{m}p^{m}q^{6-m}$, where p, q, are the respective probabilities of a prize or a blank in a single drawing; these probabilities are $p=\frac{3}{10}$, $q=\frac{7}{10}$ (Prob. I.). And for these values we have to find what m must be in order that the above expression for the probability of the combination may be the greatest possible; in other words, we have to find what the exponent $r-1$ of $\frac{7}{10}$ is in the *greatest* term in the development of $\left(\frac{3}{10}+\frac{7}{10}\right)^{6}$. Now (see art. 104) $\frac{7}{10}\div\frac{3}{10}=\frac{7}{3}$, and $6-(r-1)=7-r\ \therefore\ \frac{7}{3}(7-r)<r$, or $49<10r\ \therefore\ r=5=6-m+1\ \therefore\ m=2$; hence the most probable result of the six drawings will be two prizes and four blanks.

127. In the foregoing examples it is of importance to observe that the events are always understood to be wholly independent of one another, but if the first trial or the first occurrence modify the circumstances under which it is made so that in the second the conditions are changed, account must of course be taken of these modifications. The following example will sufficiently illustrate our meaning:—

(3) A lottery consists of 100 tickets, of which four are prizes and the others blanks: what is the probability that in the first three tickets drawn there shall be at least one prize? The prob^{y}. of drawing a blank the first time is $\frac{96}{100}$. If this were a certainty only 95 blanks would be left, so that the prob^{y}. of the second drawing also proving a blank would be $\frac{95}{99}$; but as the prob^{y}. of the first event happening is only the $\frac{96}{100}$th part of certainty, the prob^{y}. that both events will happen is $\frac{96}{100}\times\frac{95}{99}$.

Were these two events both certain, the prob^{y}. of the third drawing being also a blank would, in like manner, be $\frac{94}{98}$; as it is, however, the prob^{y}. of the concurrence is but $\frac{96}{100}\times\frac{95}{99}\times\frac{94}{98}=\frac{7144}{8085}$, and $\therefore$ the prob^{y}. that all the three drawings will *not* be blank; that is, that one at least will be a prize, is $1-\frac{7144}{8085}=\frac{941}{8085}$.

EXAMPLES FOR EXERCISE.

(1) A bag contains 16 balls, 10 black and 6 white: what is the prob^{y}. of drawing two white balls at once?

(2) In last ex. what is the prob^{y}. that the two balls drawn shall both be black, that one at least shall be black, and that one at least shall be white?

(3) What are the odds against throwing one ace and no more in 4 throws of a die?

(4) Two halfpence are tossed: what is the prob^{y}. that they shall both turn up heads?

(5) Five halfpence are tossed: what is the probability of their turning up 3 heads and 2 tails?

(6) In the last ex. what is the prob^{y}. that one at least will be head?

(7) A bag contains 3 white balls and 7 black; a ball is drawn and replaced: what is the prob^{y}. that each of three successive drawings shall be a white ball?

(8) If the ball drawn be not replaced, what is the prob^{y}. of drawing the 3 white balls in 3 successive trials?

(9) Drawing from the 10 balls, as in ex. 7, what, at the end of 10 trials, will be the most probable result of all?

(10) A person in the dark places two dice in contact, a face of one with the face of the other: what is the probability that two adjacent faces at least will have the same marks?

(11) In one urn there are 5 white balls and 2 black, in another 7 white balls and 3 black: a person, putting a hand into each urn, draws two balls: what is the prob^{y}. that one shall be white and the other black?

(12) *A* and *B* draw a ball alternately from an urn containing 2 white and 3 black balls; they agree that the first who draws a *white* ball shall win a certain sum: what are their respective chances, *A* having the first trial?*

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128. **Life Annuities.**—One of the most important practical purposes to which the theory of probabilities is applied is to the calculation of life annuities and life assurances. We can give here only a very brief sketch of the way in which this application is made.

The computations respecting annuities, already explained (pp. 86, 87), are all based on the condition that the annuity is *certain*, either for a

* Those who wish to go more deeply into this extensive subject, should consult the very able article on Probabilities, by Professor De Morgan, in the "Encyclopædia Metropolitana."

specified number of years or for ever, and does not depend for its duration on the life of the annuitant, or of any other person; but a *life annuity* is an annuity which is payable only during the existence of one or more lives. The data for estimating the value of such an annuity are supplied by *Tables of Mortality*, in which are registered, as the results of long and careful observation, the number of survivors left from year to year out of a large number of persons born. As it is found that in the same country the rate of mortality thus recorded, taking one year with another, approximates very closely to a uniform rate, the tables enable us to determine, with all the requisite precision, how many years persons of any given age will, *on the average*, continue in existence; and thence, in any individual case, to pronounce upon the average value of the life. The tables of mortality generally used in calculations respecting life annuities are those formed by Mr. Milne, and called the *Carlisle Tables*. A specimen of them is here given. The numbers against the age record how many persons live to that age out of 10,000 born.

CARLISLE TABLE OF MORTALITY. 10,000 PERSONS BORN ALIVE.

Age.	Living.	Age.	Living.	Age.	Living.	Age.	Living.	Age.	Living.	Age.	Living.	Age.	Living.
10	6460	21	6047	32	5528	43	4869	54	4143	65	3018	76	1515
11	6431	22	6005	33	5472	44	4798	55	4073	66	2894	77	1359
12	6400	23	5963	34	5417	45	4727	56	4000	67	2771	78	1213
13	6368	24	5921	35	5362	46	4657	57	3924	68	2648	79	1081
14	6335	25	5879	36	5307	47	4588	58	3842	69	2525	80	953
15	6300	26	5836	37	5251	48	4521	59	3749	70	2401	81	837
16	6261	27	5793	38	5194	49	4458	60	3643	71	2277	82	725
17	6219	28	5748	39	5136	50	4397	61	3521	72	2143	83	623
18	6176	29	5698	40	5075	51	4338	62	3395	73	1997	84	529
19	6133	30	5642	41	5009	52	4276	63	3268	74	1841	85	445
20	6090	31	5585	42	4940	53	4211	64	3143	75	1675	86	367

This table, which experience has shown to be the best for healthy lives, is that in reference to which annuities on lives are generally calculated in this country. The following examples will illustrate the use of it.

(1) What is the probability that a person aged 14 will attain the age of 21? Since out of 6335 persons living at the age of 14, 6047 attain the age of 21, each of the individuals at 14 has 6047 favourable chances out of 6335; hence (p. 94) the probability of living is $\frac{6047}{6335}$ ∴ the probability of dying is $\frac{288}{6335}$. The notation for the former prob^{y}. is $\frac{l_{21}}{l_{14}}$ where the symbol l_m signifies the number living aged m years.

(2) What is the probability that two persons, one A aged 14, and the other B aged 23, will both live 7 years? The prob^{y}. in favour of A is $\frac{l_{21}}{l_{14}}$, that in favour of B is $\frac{l_{30}}{l_{23}}$: hence, that both will live, the prob^{y}. (by III.) is $\frac{l_{21}}{l_{14}} \times \frac{l_{30}}{l_{23}} = \frac{6047}{6335} \times \frac{5642}{5963}$. And unity, diminished by this product, is the prob^{y}. that both will *not* live, that is, that one at least will die. That both will

die, the prob^{y}. is $\left(1-\frac{l_{21}}{l_{14}}\right)\left(1-\frac{l_{30}}{l_{23}}\right)$; and that both will *not* die, that is, that one at least will live, the prob^{y}. is unity diminished by this product, namely, $\frac{l_{21}}{l_{14}}+\frac{l_{30}}{l_{23}}-\frac{l_{21}l_{30}}{l_{14}l_{23}}$.

(3) A father wishes to provide for his daughter, aged 14, the sum of £850 on her attaining the age of 21: what present sum should be paid to secure it; interest of money being 3 per cent.? Computing the present value of £850, *certainly* receivable at the expiration of 7 years, as at (122), we find it to be £691·1278. But instead of certainty, there is here only the probability $\frac{6047}{6335}$: hence, only this fraction of the above value is to be paid, namely, $691{\cdot}1278\times\frac{6047}{6335}=659{\cdot}708=$£659 14*s*. 2*d*.

(4) If the £850 is to be paid at the end of the 7 years only on the condition that two persons, one 14, and the other 16, are both living, the calculation is $£691{\cdot}1278\times\frac{6047}{6335}\times\frac{5963}{6261}=£628{\cdot}308=$£628 6*s*. 2*d*.

Problem I. To find the present value of an annuity of £ A to be paid during the life of any proposed age.

As at (118), let R represent £1 plus its interest for 1 year; then the present value of £ A, certainly receivable n years hence, is $A\frac{1}{R^{n}}$. But that a person aged m years will live to receive it, the probability is only $\frac{l_{m+n}}{l_{m}}$, by which fraction the present value of the A must be multiplied: hence making $n=1, 2, 3,$ &c., till $m+n$ reaches the extremity of life, say 104 years, we have for the present value of all the annual payments during that life

$$\frac{A}{l_{m}}\left\{\frac{l_{m+1}}{R}+\frac{l_{m+2}}{R^{2}}+\frac{l_{m+3}}{R^{3}}+\ldots\ldots+\frac{l_{104}}{R^{104-m}}\right\}.$$

The labour of computing by this formula is superseded by means of a "Table of Annuities on Single Lives," which is constructed for all values of m, and for different rates per cent., the annuity being regarded as £1; so that the number of pounds in the table, opposite to the life l_{m}, denotes the number of years' purchase that any annuity on that life is worth.

Problem II. To find the present value of an annuity A to be paid during the joint lives of two persons aged m and k years respectively, that is, so long as both are living.

The prob^{y}. that *both* will be alive at the end of n years is (p. 95) $\frac{l_{m+n}l_{k+n}}{l_{m}l_{k}}$: hence, putting 1, 2, 3, &c., for n, the present value of all the annual payments will be

$$\frac{A}{l_{m}l_{k}}\left\{\frac{l_{m+1}l_{k+1}}{R}+\frac{l_{m+2}l_{k+2}}{R^{2}}+\frac{l_{m+3}l_{k+3}}{R^{3}}+\ldots\ldots\right\}.$$

This formula, too, like the preceding, is computed for all values of m and k, and given in tables of "Annuities on Two Joint Lives."

PROBLEM III. To find the present value of an annuity on two joint lives, to terminate with the last of those lives.

The respective ages being m and k years, the prob^{y}. that *both* will not become extinct in n years is $\frac{l_{m+n}}{l_m}+\frac{l_{k+n}}{l_k}-\frac{l_{m+n}l_{k+n}}{l_m l_k}$. (See ex. 2, p. 100).

Hence, and by the two preceding problems, we infer that the present value of an annuity on two joint lives, to continue till the last of them drops, is found by subtracting from the sum of the values of the annuity on the lives existing singly, the value on the two lives existing together.

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129. **Assurances on Lives.**—A life assurance is an engagement to pay a certain stipulated sum, either at the death of the person assured, happen whenever it may, or, only provided it take place during a specified term of years. The legal instrument, which binds the Assurance Office to the contract, is called the *Policy*, the sum paid for it by the holder is called the *Premium*: if this be paid at once, it is the *single* premium; if, as is usual, it be paid by yearly sums, it is called the *annual* premium.

PROBLEM I. To find what single premium is sufficient for a policy of £ A to be paid at the end of the year in which a person m years of age shall die.

The present value of the expectation of receiving the sum £ A, at the end of any year, is the present value of £ A, certain at that time, multiplied by the probability of dying during that year. Hence, putting d_{m+n} for the number, out of l_m of persons, who die in n years after attaining their mth year, and making $n=1, 2, 3$, &c., successively, we have for the present value of all the expectations

$$\frac{A}{l_m}\left\{\frac{d_{m+1}}{R}+\frac{d_{m+2}}{R^2}+\frac{d_{m+3}}{R^3}+\ldots\ldots\right\}$$

the series being continued till all the l_m persons have died.

For $A=$£1, this formula is calculated for different values of m and R, and the results arranged in Tables.

PROBLEM II.—To find the annual premium.

The annual premium differs from an annuity in this:—the first payment is made at the *commencement* of the year, that is, at the time of effecting the assurance; so that there is one payment more than in the case of an annuity. If $\therefore x$ represent the annual premium, the above expression, for the present value of all the annual premiums, will represent the present value of an annuity x, on the proposed life, and one x more. Now

£1+ Pres. val. of annuity of £1 : £1 :: Pres. val. of Assurance : x

$$\therefore x=\frac{\text{Pres. val. of assurance}}{\text{£1+ Pres. val. of annuity of £1}}=\text{the annual premium.}$$

Hence the annual premium is equal to the single premium divided by the present val. of an annuity of £1, increased by £1. But all calculations concerning annuities and assurances are greatly facilitated by special tables; for which the reader may consult the treatise of Mr. D. Jones, published in the Library of Useful Knowledge.

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130. **Theory of Equations in General.**—In discussing the general theory of equations involving only one unknown quantity, it is

found convenient to bring all the significant terms to one side of the sign of equality, and to leave zero only on the other side: it is necessary too to arrange these terms so that the powers of the unknown (x) may descend or ascend in regular order; and if this regular order be interrupted, on account of the absence of any term, the place of this absent term should be filled up by inserting the wanting power of x with a zero coefficient; so that the polynomial may be *complete*.

By the term *root* of an equation, is to be understood any quantity, which substituted for x in that equation, satisfies the implied condition; that is, which causes the polynomial, arranged as above directed, to become zero.

131. Any polynomial, or indeed any expression whatever, involving x is called a *function of* x, and is briefly represented by such a form as $f(x)$, or $F(x)$, or $\varphi(x)$, &c.: the initial letter f, or F, &c., prefixed to the symbol (x), which is to be the object of more special attention in the expression referred to, being, not a *factor*, but an abbreviated form for the words "expression involving x." Unless it be distinctly stated that $f(x)$ is to represent an expression, involving x, of some declared form, we could infer nothing as to the *character* of the expression from the notation $f(x)$: all that we could gather from it would be, that *an* expression involving x is thus briefly represented; for *every* such expression is a function of x. *Here*, however, $f(x)$ will always stand for the polynomial, which forms the first member of the equation

$$A_nx^n+\ldots\ldots+A_3x^3+A_2x^2+A_1x+N=0\ldots\ldots[1].$$

132. Proposition 1. If a be a root of any equation [1], then the polynomial $f(x)$, which forms the first member of it, will necessarily be divisible by $x-a$; and conversely, if $f(x)$ be divisible by $x-a$, then will a be a root of the equation $f(x)=0$.

For calling the quotient arising from the division of $f(x)$ by $x-a$, Q, and the remainder, if there be any, R, we must have $f(x)=(x-a)Q+R$: but since a is a root of $f(x)=0$, each side of this identical equation must become zero by putting a for x: the second member, upon this substitution, becomes reduced to R. $\therefore R=0$. $\therefore x-a$ accurately divides $f(x)$ without remainder. Conversely: let $x-a$ divide $f(x)$; that is, let $f(x)=(x-a)Q$, then for $x=a$ we have necessarily $f(x)=0$. $\therefore a$ is a root of this equation.

133. Proposition 2. To determine the quotient and remainder arising from dividing a polynomial [1] by any binomial of the form $(x-a)$.

Let the polynomial be of the fourth degree, namely, $A_4x^4+A_3x^3+A_2x^2+A_1x+N$, the quotient Q must evidently be of the third degree,

$$A'_3x^3+A'_2x^2+A'_1x+N'$$

$$\therefore A_4x^4+A_3x^3+A_2x^2+A_1x+N=(x-a)(A'_3x^3+A'_2x^2+A'_1x+N')+R,$$

or, by actually multiplying,

$$=A'_3x^4+(A'_2-aA'_3)x^3+(A_1'-aA'_2)x^2+(N'-aA_1')x-aN'+R,$$

which being the same as the first member, whatever be the value of x, we have, by equating the coefficients of the like powers (94),

$$A'_3=A_4$$
$$A'_2-aA'_3=A_3 \quad \therefore \quad A'_2=aA'_3+A_3$$
$$A'_1-aA'_2=A_2 \quad \therefore \quad A'_1=aA'_2+A_2$$
$$N'-aA'_1=A_1 \quad \therefore \quad N'=aA'_1+A_1$$

And $$R-aN'=N \quad \therefore \quad R=aN'+N.$$

It thus appears that the first coefficient A'_3, in the quotient Q, is the same as the first coefficient A_4 in the proposed polynomial; that the second coef. A'_2 is found by multiplying A'_3 by a, and adding the corresponding coef. of the polynomial; that the third A'_1 is found in a similar manner; and that continuing this uniform series of operations we in the same way get N' and R. And it is plain that the process is general, whatever be the degree of the polynomial.

(1) Required the quotient and remainder arising from dividing $3x^4-5x^3+2x^2-11x-17$ by $x-2$. The operation is exhibited in the margin: the first coef. we know to be 3, and every subsequent coef. is found by multiplying the preceding one by 2, and adding the product to the corresponding coef. above. The quotient of the division is therefore $3x^3+x^2+4x-3$, and the remainder is -23.

3 − 5 + 2 − 11 − 17
6 + 2 + 8 − 6
1 + 4 − 3 − 23

(2) Required the quotient and remainder arising from dividing $3x^6-6x^4+4x^3-x-45624$ by $x+5$. Supplying the absent terms, the operation is as below, the constant multiplier, $a=-5$, being placed, in evidence, on the right.

3 + 0 − 6 + 4 + 0 − 1 − 45624 (−5
−15 + 75 − 345 + 1705 − 8525 + 42630
−15 + 69 − 341 + 1705 − 8526 − 2994

$\therefore Q=3x^5-15x^4+69x^3-341x^2+1705x-8526$; $R=-2994$.

Examples for Exercise.

(1) $(3x^4-2x^3+5x^2-7x-19)\div(x-2)$.

(2) $(7x^4-13x^3+4x^2+6x-12)\div(x-3)$. (3) $(5x^4-7x^3-4x^2+6x-13)\div(x+6)$.

(4) $(7x^3-3x^2-2x+126)\div(x+5)$. (5) $(x^5+7x^4+3x^3+17x^2+10x-14)\div(x-4)$.

Note.—The student will bear in mind that the remainder R, in every example, is what the polynomial becomes when a is substituted in it for x (see art. 33); so that to calculate the value of a polynomial, for a given value of x, we have only to calculate R as above.

134. This expeditious method of arriving at the value of a polynomial for a particular value of x, or of computing the remainder R for that value, is—as we shall shortly see—of considerable practical importance in the solution of numerical equations. But the ready determination of the corresponding value of the quotient Q, is not without interest. We know, from the above, that $Q=A'_3x^3+A'_2x^2+A'_1x+N'$; and we have also seen what the conditions are which connect these coefficients with those of the original polynomial. Referring to those conditions (bottom of last page), it is easy to see that Q may be written thus:—

$$A_4x^3+(A_4a+A_3)x^2+(A_4a^2+A_3a+A_2)x+A_4a^3+A_3a^2+A_2a+A_1.$$

In this quotient, put a for x, and it becomes

$$(A_4+A_4+A_4+A_4)a^3+(A_3+A_3+A_3)a^3+(A_2+A_2)a+A_1;$$

that is, for $x=a$, $Q=4A_4a^3+3A_3a^2+2A_2a+A_1$.

And although, for convenience, we have limited the reasoning to a polynomial of the fourth degree, it is evident that it is equally applicable to one of *any* degree; so that, generally, if any polynomial

$$A_nx^n+\ldots+A_5x^5+A_4x^4+A_3x^3+A_2x^2+A_1x+N\ldots\ldots[1]$$

be divided by $x-a$, and then a be put for x in the quotient, that quotient will be

$$nA_na^{n-1}+\ldots5A_5a^4+4A_4a^3+3A_3a^2+2A_2a+A_1\ldots\ldots[2]$$

And, in like manner, if we put x for a here, and divide by $x-a$, and then replace the x by a in the quotient, we shall get

$$(n-1)nA_n a^{n-2}+\ldots+4.5A_5a^3+3.4A_4a^2+2.3A_3a+2A_2\ldots\ldots[3]$$

and so on:—interesting, and, we believe, novel deductions, which will find their practical application hereafter.

135. Proposition 3. Every numerical equation has as many roots as there are units in the exponent denoting its degree; that is, an equation [1] of the nth degree has n roots, and it has no greater number.

[By a *numerical* equation, is meant an equation in which the coefficients are all *numbers:* when they are *letters*, the equation is called a *literal*, or more frequently, though with less propriety, an *algebraic* equation.]

Let it be granted that a numerical equation has at least *one* root:* call it a_1, then the polynomial forming the first member of the equation [1] is divisible by $x-a_1$, (Prop. 1) so that, removing the coef. A_n by division, we shall have $f(x)=(x-a_1)(x^{n-1}+\ldots\ldots+A'_2x^2+A_1'x+N')$.

And since every numerical equation has a root, there must be some value a_2 of x which will satisfy the equation $x^{n-1}+\ldots\ldots+A'_2x^2+A_1'x+N'=0$, so that we shall have

$$f(x)=(x-a_1)(x-a_2)(x^{n-2}+\ldots\ldots+A_1''x+N'').$$

And, in like manner, since every equation has a root, there is some value a_3 such that the last of the above factors is divisible by $x-a_3$, giving a quotient a unit lower in degree that will be divisible by $x-a_4$, and so on. Consequently

$$f(x)=(x-a_1)(x-a_2)(x-a_3)\ldots\ldots(x-a_n).$$

Hence the equation $f(x)=0$ has n roots. It cannot have any additional root different from either of these, for if any value different from any of the values a_1, a_2, a_3,..... a_n, be put for x in the above, none of the simple factors will become zero, and neither therefore will the product become zero.

136. The last term in the first member of the equation $f(x)=0$, being $(-a_1)(-a_2)(-a_3)\ldots\ldots(-a_n)$, is the product of all the roots with their signs changed; or since the product of an *even* number of factors is the same, whether the signs of all those factors be changed or not, it follows that, in an equation of an even degree, the last term is the product of all the roots.

If one root a of an equation be found, the division of the first member $f(x)$ by $x-a$ will give a polynomial, which, equated to zero, will be the equation involving the remaining roots. For example:—

One root of the equation $x^4+9x^3+9x^2-41x-42=0$ is 2; the first member is therefore divisible by $x-2$: and performing the division as in the margin (see p. 104), we find the *depressed* equation involving the other roots to be $x^3+11x^2+31x+21=0$. A root of this also is found to be -1: hence dividing the former result by $x+1$, as in the margin, we arrive at the quadratic $x^2+10x+21=0$, of which the two roots are -3 and -7; so that the four roots of the proposed equation are 2, -1, -3, and -7; or that equation, thus decomposed, may be written

$$1+9+9-41(2$$
$$2+22+62$$
$$\overline{11+31+21(-1}$$
$$-1-10$$
$$\overline{10+21}$$
$$\overline{}$$

$$x^4+9x^3+9x^2-41x-42=(x-2)(x+1)(x+3)(x+7)=0.$$

* This is proved *à priori*, for *numerical* equations, by Cauchy and others. See the Author's General Theory and Solution of Equations of the Higher Orders, p. 33.

137. Proposition 4. If two numbers, α, β, when separately substituted for x in $f(x)$, give results with opposite signs, then at least *one* real root of the equation $f(x)=0$ must lie between α and β.

Representing the n roots of the equation $f(x)=0$ by a_1, a_2, $a_3 \ldots a_n$, we have (last prop.), regarding A_n as 1, as, of course, we always may,*

$$f(x)=(x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n).$$

Let the substitution of α for x render this product *plus*, then, whatever other substitution be made for x, the product can become *minus* only on account of an *odd* number of the factors becoming changed in sign for the new value of x, for an even number of factors changing signs has no effect on the sign of their product. Hence, since the substitution of β for x produces this change of sign in the product, *one* factor, at least, as for instance, $(x-a_1)$, must take opposite signs when α and β are successively put for x. But when the root a_1 is put for x the factor becomes zero; hence α, which gives a result *plus*, must be greater than a_1; and β, which gives a result *minus*, must be less than a_1.

It thus appears that when two numbers, substituted successively for x in the first member of an equation, give results with opposite signs, one root at least necessarily lies between those numbers; but when the results have like signs an odd number of roots cannot possibly lie between them; there must be either an even number or none at all. For instance, taking the equation in last page, if 0 and 10 be successively substituted for x, the results are successively $-$ and $+$; hence we infer that either one root or three are positive and less than 10. Also, if 0 and -10 be substituted, the results are likewise $-$ and $+$, we conclude $\therefore$ that one or three roots are negative, and that each of them is numerically less than -10.

138. If the roots of an equation are all imaginary, then, substitute whatever numbers we may for x, we shall never get results with different signs, for such results would imply the existence of at least one real root. And conversely, if we perceive that, substitute whatever numbers we may, the results must all have the same sign, we may be sure that all the roots are imaginary. We shall see presently that this unvarying sign will always be *plus*.

139. Proposition 5. Imaginary roots always enter an equation, with real coefficients, in pairs; that is, if $\alpha+\beta\sqrt{-1}$ be one root, then $\alpha-\beta\sqrt{-1}$ must be another root. [These roots, from their form, are called *conjugate roots*.]

Putting i for the imaginary $\sqrt{-1}$, as at (61), if $\alpha+\beta i$ be a root of $f(x)=0$, then

$$f(\alpha+\beta i)=(\alpha+\beta i)^n+\ldots\ldots+A_3(\alpha+\beta i)^3+A_2(\alpha+\beta i)^2+A_1(\alpha+\beta i)+N=0;$$

and if these powers be developed by the binomial theorem we know (61) that the result will be of the form $A+Bi=0$, $\therefore$ (61) $A=0$, $B=0$. Let now $\alpha-\beta i$ be written above, instead of $\alpha+\beta i$, the developed powers will differ only in the signs of the terms involving *odd* powers of βi (98), these signs will all be opposite to the signs of the like terms in the former development, but the terms themselves will be the same. The terms involving the *even* powers of βi are the same, sign and all, in each

* Hereafter, the coefficient of the first term of $f(x)$ will always be regarded as unit, unless otherwise stated.

development. Hence the result is $f(\alpha-\beta i)=A-Bi$; and since $A=0$, $B=0$ $\therefore f(\alpha-\beta i)=0$; so that if $\alpha+\beta i$ be a root of $f(x)=0$, then also $\alpha-\beta i$ must be a root. And the same conclusion follows if α be rational, and βi be a quadratic surd, and not an imaginary, provided the coefficients are rational, since the inference $A=0$, $B=0$, still holds good (p. 44).

Since the product of the factors $x-(\alpha+\beta i)$ and $x-(\alpha-\beta i)$ is real, namely, $x^2-2\alpha x+\alpha^2+\beta^2$, it follows that every polynomial, $f(x)$, with real coefficients, is composed of *real* factors of the first and second degrees.

If the roots of an equation be all imaginary, that equation must be of an even degree. Moreover, since the last term N of an equation of even degree is the product of all the roots, and since the product of every pair of imaginary roots is *plus*, inasmuch as $\alpha^2+\beta^2$, the *sum* of two squares, can never be negative, it follows that, when all the roots are imaginary, the last term of the equation must always be *plus*. By putting $x=0$ in such an equation the result therefore is plus; but the signs of the results never change, whatever be substituted for x (138); hence, when all the roots are imaginary, the results are invariably + for every substitution.

NOTE.—It is not only the imaginary and surd roots of equations that always enter in conjugate pairs, for a conjugate form may be given to any pair of roots whatever: thus, taking any two of the roots of an equation, a_1 and a_2, we may always express them thus: $a_1=\alpha+\beta$, $a_2=\alpha-\beta$, where $\alpha=\frac{1}{2}(a_1+a_2)$, and $\beta=\frac{1}{2}(a_1-a_2)$. And it is under this form that the two roots of the general quadratic are actually exhibited.

140. PROPOSITION 6. Every equation of an odd degree has at least one real root, of which the sign is opposite to that of the final term, and every equation of an even degree, provided the final term be *negative*, has at least two real roots—one positive and the other negative.

Representing the roots of the equation $f(x)=0$, as before, we have

$$f(x)=(x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n).$$

First let $N=(-a_1)(-a_2)(-a_3)\ldots(-a_n)$ be negative, and conceive a positive number, k, greater than the greatest of the real roots, to be substituted for x, then all the real factors will be positive; and since the product of all the pairs of imaginary factors, should any enter, is positive too (120), the entire product must be *positive*. But if 0 be substituted for x, the product $f(x)$ is reduced to N, which, by hypothesis, is *negative;* hence one real positive root, at least, exists between 0 and k. And this, whether the degree of the equation be even or odd. Next, let N be positive, and conceive a negative number, $-k$, numerically greater than the greatest root, to be substituted for x, so that all the real factors may become negative. These real factors must be odd in number if the equation be of an odd degree, since the imaginary factors enter in pairs, and the product of these pairs is, moreover, positive; hence the entire product, on account of the odd number of negative factors, must be *negative*. But the substitution $x=0$ reduces the product to N, which is *positive;* hence a real negative root must lie between 0 and $-k$. If, however, the equation be of an *even* degree, then the substitution of $-k$ for x renders the product positive; so that, should N be negative, the substitution of 0 for x would produce a change of sign, implying the existence of a real negative root between 0 and $-k$; hence, if the degree of the equation is even and N negative, it has at least two real roots, one positive and the other negative.

141. PROPOSITION 7. If the signs of the alternate terms, commencing

with the second term, of a *complete* equation, be changed, the signs of all the roots will be changed.

1. Let the equation be of an even degree; then the change of the alternate signs, commencing with the second, in $f(x)$, is brought about by simply writing $-x$ for x, and the roots of $f(-x)=0$ are, of course, the roots of $f(x)=0$, with their signs changed.

2. Let the equation be of an odd degree, then the changing of $f(x)$ into $f(-x)$ changes the alternate signs, commencing with the *first*; so that $-f(-x)$ restores the leading sign, and changes the alternate signs of $f(x)$, commencing with the second. But the roots of $-f(-x)=0$ are the same as the roots of $0=f(-x)$. Hence, in all cases, the signs of the roots are changed by changing the alternate signs of the terms, commencing with the second.

We thus have an easy way of changing all the positive roots of an equation into negative, and all the negative into positive. For example, changing the alternate signs of the equation considered at p. 105, it becomes $x^4-9x^3+9x^2+41x-42=0$, of which the roots are -2, 1, 3, 7.

142. Proposition 8. If the coefficient of the first term of an equation be 1, the following are the relations between the coefficients and the roots, namely:—

1. The coef. of the second term, *with its sign changed*, is equal to the sum of the roots.

2. The coef. of the third term is equal to the sum of the products of the roots, taken two and two.

3. The coef. of the fourth term, *with its sign changed*, is equal to the sum of the products of the roots, taken three and three, and so on, up to the last term N, which, with its sign changed or not, according as the degree of the equation is odd or even, is the product of *all* the roots.

These truths necessarily follow from the way in which the coefficients, in the product of the factors $(x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n)$, are constructed: this product being

$$f(x)=x^n-(a_1+a_2+a_3+\ldots)x^{n-1}+(a_1a_2+a_1a_3+a_2a_3+\ldots)x^{n-2}$$
$$-(a_1a_2a_3+\ldots)x^{n-3}+\ldots+N\ldots[1]$$

where N is the product of all the roots a_1, a_2, a_3,...with their signs changed.

Hence, if the second term of an equation be absent, then $a_1+a_2+a_3+\ldots=0$; that is, the sum of the positive roots must = the sum of the negative roots, when all are real.

If the roots be all real and negative, the signs of the terms must be all positive; and if they be all real and positive, the signs of the terms must be alternately positive and negative. Since—A_1 is the sum of all the combinations of the n factors of N, taken $n-1$ at a time, it follows that

$$-\frac{A_1}{N}=\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\ldots\ldots+\frac{1}{a_n}.$$

Similarly, $\frac{A_2}{N}=\frac{1}{a_1a_2}+\frac{1}{a_1a_3}+\frac{1}{a_2a_3}+\ldots\ldots$, $-\frac{A_3}{N}=\frac{1}{a_1a_2a_3}+\ldots\ldots$, &c.

If the coefficient A_n of x^n, in what is said above, be other than unity, then $\frac{N}{A_n}$ must be written for N in [1], that is, $\frac{N}{A_n}=(-a_1)(-a_2)(-a_3)^*\ldots\ \therefore A_n=\frac{N}{(-a_1)(-a_2)(-a_3)\ldots}$

* The *ratios*, however, $\frac{A_1}{N}$, $\frac{A_2}{N}$, &c., remain the same, because num. and denom. are divided by the same thing, namely, A_n.

and $\therefore$ by substitution, $A_nx^n+\ldots+A_2x^2+A_1x+N=A_n(x-a_1)(x-a_2)(x-a_3)\ldots$

$$=N\left(1-\frac{x}{a_1}\right)\left(1-\frac{x}{a_2}\right)\left(1-\frac{x}{a_3}\right)\ldots\ldots$$

143. **Transformation of Equations.**—1. To transform an equation into another of which the roots shall be assigned multiples or sub-multiples of those of the proposed equation.

Let it be required to transform the equation $A_nx^n+\ldots+A_3x^3+A_2x^2+A_1x+N=0$, of which the roots are a_1, a_2, $a_3\ldots a_n$, into another of which the roots shall be ka_1, ka_2, $ka_3\ldots ka_n$. Substitute $\frac{y}{k}$ for x : then the equation is

$$A_n\left(\frac{y}{k}\right)^n+\ldots+A_3\left(\frac{y}{k}\right)^3+A_2\left(\frac{y}{k}\right)^2+A_1\left(\frac{y}{k}\right)+N=0,$$

or multiplying by k^n, it is $A_ny^n+\ldots+A_3k^{n-3}y^3+A_2k^{n-2}y^2+A_1k^{n-1}y+Nk^n=0$.

The n values of y in this equation are ka_1, ka_2,$\ldots ka_n$, because $y=kx$. Hence we see that an equation is transformed into another, each of whose roots shall be k times as great, by multiplying the terms in order by 1, k, k^2,$\ldots k^n$. A ready method is thus suggested of removing fractions from the coefficients of an equation: we have only to find a common multiple k of the denominators, and then to transform the equation into another whose roots shall be k times as great. This method may be employed with advantage whenever we wish to remove the leading coefficient A_n, or to render it unity, and at the same time to exclude from the other coefficients the fractions which division by A_n usually introduces.

If $k=-1$, the multipliers 1, k, k^2, &c., become $1,-1,+1$, &c.; so that an equation is converted into another whose roots are -1 times those of the former, that is, into one whose roots are the same in value, but contrary in sign, by multiplying the terms in order by $1,-1,+1$, &c., that is, by simply changing the alternate signs commencing with the second; a truth already established otherwise at (141).

Note.—Before using the multipliers, the equation must have its absent terms, if any, supplied by zeros.

(1) $x^3+\frac{1}{2}ax^2-\frac{1}{3}bx+c=0$. Clear fractions, and keep the 1st coef. $=1$.

The *L.C.M.* of the denominators is 6 $\therefore$ the multipliers are 1, 6, 6^2, 6^3, and the transformed equation is $x^3+3ax^2-12bx+216c=0$, the roots of which are each 6 times those of the proposed equation.

(2) $x^4+\frac{1}{2}x^3-\frac{1}{3}x^2-\frac{1}{6}x+10=0$. Here also $k=6$, and the transformed equation is, therefore, $x^4+3x^3-12x^2-36x+12960=0$, the roots of which are each 6 times those of the proposed equation.

144. Prop. 2.—To transform an equation into another whose roots shall be those of the original increased or diminished by a given quantity.

For greater simplicity, suppose the equation to be of the fourth degree only, namely, $A_4x^4+A_3x^3+A_2x^2+A_1x+N=0\ldots[1]$ and that it is required to transform it into another, of the same degree, whose roots shall each of them differ from those of the proposed equation by r. In order to this, we have only to substitute $x'+r$ for x, when the resulting equation, $A_4x'^4+A'_3x'^3+A'_2x'^2+A'_1x'+N'=0\ldots[2]$ will have each of its roots greater or less than the corresponding roots of the proposed equation by r, according as r is negative or positive; for $x'=x-r$. The following is an easy way of arriving at the coefficients of the transformed

equation. Return to the original by replacing x' by $x-r$: we then have

$$A_4(x-r)^4+A'_3(x-r)^3+A'_2(x-r)^2+A'_1(x-r)+N'=A_4x^4+A_3x^3+A_2x^2+A_1x+N.$$

If we divide the first member of this identity by $x-r$, the remainder must be N': the division, therefore, of the second member must also give N' for remainder, the quotient furnished by each being

$$A_4(x-r)^3+A'_3(x-r)^2+A'_2(x-r)+A_1'.$$

In like manner, dividing this quotient by $(x-r)$, we have A_1' or remainder, and for a second quotient

$$A_4(x-r)^2+A'_3(x-r)+A'_2.$$

Again dividing, the third remainder is A'_2, and the corresponding quotient, $A_4(x-r)+A'_3$. And lastly, dividing this by $x-r$, we have for the final remainder A'_3, and for the final quotient, A_4.

We see, therefore, that the several *remainders*, arising from these successive divisions of the original polynomial [1], by $x-r$, furnish, one after another, all the coefficients of the required transformed equation [2]. A very expeditious method of executing these several divisions is explained at (133): we shall give a few applications of it to the present problem.

(1) Transform

$$x^3-2x^2-23x+60=0$$

into an equation whose roots shall be the roots of this *increased* by 2.

$$\begin{array}{l} 1-2-23+60(-2=r \\ \quad -2+\ 8+30 \\ \hline -4-15+90 \quad \therefore N'=\ 90 \\ \quad -2+12 \\ \hline -6-\ 3 \quad \therefore A'_1=-3 \\ \quad -2 \\ \hline -8 \quad \therefore A'_2=-8 \end{array}$$

Hence the transformed equation is

$$x'^3-8x'^2-3x'+90=0.$$

(2) Transform the equation

$$2x^4-15x^3+40x^2-45x+18=0$$

into one whose roots shall be the roots of this *diminished* by 4.

$$\begin{array}{l} 2-15+40-45+18(4=r \\ \quad 8-28+48+12 \\ \hline -7+12+\ 3+30 \quad \therefore N' =30 \\ \quad 8+\ 4+64 \\ \hline 1+16+67 \quad \therefore A'_1=67 \\ \quad 8+36 \\ \hline 9+52 \quad \therefore A'_2=52 \\ \quad 8 \\ \hline 17 \quad \therefore A'_3=17 \end{array}$$

Hence the transformed equation is

$$2x'^4+17x'^3+52x'^2+67x'+30=0.$$

NOTE.—The operation may be slightly modified as follows:—

(3) Transform the equation

$$6x^3-3x^2+4x-1=0$$

into one whose roots shall *exceed* the roots of this by 3.

$$\begin{array}{llll} 6 & -3 & +4 & -1(-3 \\ & -18 & 63 & -201 \\ \hline & -21 & 67 & \\ & -18 & 117 & \\ \hline & -39 & & \\ & -18 & & \end{array}$$

[formed eq.

$\therefore 6x'^3-57x'^2+184x'-202=0$, the trans-

(4) Transform

$$2x^4-13x^3+10x-19=0$$

into an equation whose roots shall be *less* than the roots of this by 3.

$$\begin{array}{llll} 2 & 0 & -13 & +10 \quad -19(3 \\ & 6 & 18 & 15 \quad 75 \\ \hline & 6 & 5 & 25 \\ & 6 & 36 & 123 \\ \hline & 12 & 41 & \\ & 6 & 54 & \\ \hline & 18 & & \\ & 6 & & \end{array}$$

$\therefore 2x'^4+24x'^3+95x'^2+148x'+56=0$, the transformed equation required.

145. By the preceding process, it is easy to transform an equation into another in which the second term shall be absent, for we see, in diminishing the roots of an equation of the nth degree by r, that r times the first coef., added n times to the second coef., gives the second coef. in the transformed equation. In order, therefore, that this may be 0, we have only to satisfy the condition $nrA_n+A_{n-1}=0 \therefore r=-\frac{A_{n-1}}{nA_n}$. Take, for instance, the equation $x^3-9x^2+20x-12=0$. The transforming multiplier that will effect the removal of the second term, is $r=\frac{9}{3}=3$, which, employed as in the preceding examples, conducts to the equation $x'^3+0x'^2-7x'-6=0$, or $x'^3-7x'-6=0$, of which the roots are those of the preceding equation, each diminished by 3. Again: take $2x^4+24x^3+95x^2+148x+56=0$. (See ex. 4 above.) The transforming multiplier is $r=-\frac{24}{4.2}=-3$, which, employed as above, furnishes the transformed equation $2x'^4-13x'^2+10x'-19=0$, the roots of which are those of the former equation, each increased by 3.

146. Prop. 3. To transform an equation into one whose roots shall be the reciprocals of the roots of the former. In order to this, it is necessary merely to put $\frac{1}{y}$ for x in the proposed equation: thus, writing

$$A_nx^n+\ldots+A_3x^3+A_2x^2+A_1x+N=0\ldots[1]$$

as follows, $N+A_1x+A_2x^2+A_3x^3+\ldots\ldots+A_nx^n=0$

we have $N+\frac{A_1}{y}+\frac{A_2}{y^2}+\frac{A_3}{y^3}+\ldots\ldots+\frac{A_n}{y^n}=0$

$$\therefore Ny^n+A_1y^{n-1}+A_2y^{n-2}+A_3y^{n-3}+\ldots\ldots+A_n=0\ldots\ldots[2]$$

the equations whose roots are the reciprocals of those of [1]. It appears, therefore, that, by simply reversing the order of the coefficients in [1], we obtain an equation [2], the roots of which are the reciprocals of those of [1].

147. Should the coefficients in [1] be the same series of numbers when taken in reverse order, as when taken in direct order, as, for instance, in the equations

$$A_5x^5+A_4x^4+A_3x^3+A_3x^2+A_4x+A_5=0,$$

$$A_6x^6+A_5x^5+A_4x^4+A_3x^3+A_4x^2+A_5x+A_6=0,$$

then the reciprocals of the roots are themselves roots; that is, if a_1 be a root, then also $\frac{1}{a_1}$ must be a root. Such equations are called *reciprocal equations*, in reference to this property of the several pairs of roots: they are also called *recurring* equations, in consequence of the recurrence of the coefficients.

148. *When the equation is of an odd degree*, like the first of the preceding, it is still a reciprocal equation, though the equal coefficients have opposite signs, for in this case the coefficients of [2] become the same, upon changing *all* their signs, as those of [1]. *Unity*, with a sign opposite to that of the last term, must always be one root of a reciprocal

equation of an odd degree; because if this value be put for x, the terms with equal coefficients will have opposite signs, and therefore the polynomial will vanish.

149. *When the equation is of an even degree*, and complete in all its terms, it can be a reciprocal equation only so long as the equal coefficients have the *same signs;* for here there is a *middle term*, so that if the equal coefficients have opposite signs, it is impossible, on account of this middle term, that the row of signs, taken in reverse order, can be the opposites of those taken in direct order; and therefore the two rows cannot become alike by changing *all* the signs in one row.

150. But if the middle term be absent, then the obstacle is removed; and the equation is reciprocal, whether the equal coefficients have the same or opposite signs. And in this case, $+1$ and -1 are both of them roots, if the last term be negative; but neither is a root if the last term be positive; for, in the former case, the signs of the equal coefficients being opposite, and the powers of x connected with them being both of an even or both of an odd degree, the terms must vanish equally for $x=1$, and $x=-1$. But for neither of these values can they vanish if the last term be positive, and therefore the signs of the equal coefficients *like*, as is obvious.

Examples for Exercise.

(1) Transform $x^3-2x^2-x+2=0$ into an equa. whose roots are 3 times as great.

(2) Transform $x^3-28x+48=0$ into an equa. whose roots are only half as great.

(3) Transform the equa. $6x^3-3x^2+4x-1$ into one with roots each less by 3.

(4) Transform the equa. $12x^3+24x^2-58x+25=0$ into one with roots each less by $\cdot 5$.

(5) Transform the equa. $19x^4-22x^3-35x^2-16x-2=0$ into one with roots less by 3.

(6) Change $x^3+6x^2+5x+1=0$ into an equa. wanting the second term.

(7) Remove the second term from the equa. $x^2+A_1x+N=0$.

(8) Remove the second term from $2x^4-16x^3+43x^2-41x+5=0$.

(9) Find the equa. whose roots are reciprocals of those of $56x^3+34x^2+x-1=0$.

(10) What equation is that whose roots are 1, 2 and 3?

(11) One root of $x^3-7x+6=0$ is 1: find the remaining roots.

(12) One root of $x^3-5x^2-18x+72=0$ is -4: find the remaining roots.

(13) Three roots of $x^5+5x^4+x^3-16x^2-20x-16=0$ are 2, -2, -4: prove that the remaining roots are imaginary.

(14) Find the equa. whose roots are the reciprocals of the roots of $x^4-12x^2+12x-3=0$ diminished by 1.

(15) Find the three cube roots of unity, that is, solve the equation $x^3-1=0$.

(16) Find the four values of $\sqrt[4]{1}$, that is, solve the equation $x^4-1=0$.

(17) Given the equation $x^6-3x^4+2x^3+7x-5=0$: write down another whose roots shall be the positive roots of the former changed into negative, and the negative into positive.

(18) If four roots of the equation $A_5x^5+A_4x^4+\ldots=0$ be a_1, a_2, a_3, a_4, what will the fifth root a_5 be equal to?

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151. **Limits of the Real Roots of an Equation.**—If in an equation in which the second member is zero, a positive number r be substituted for x, such that it, and every greater number, gives a positive result, it is plain that r will be greater than the greatest root of that equation; that is to say, r will be a *superior limit* to the positive roots. If, therefore, all the terms of an equation are plus, *all* positive numbers from $x=0$, to $x=\infty$ must give a positive result; and therefore, as indeed was already seen at (137), the equation cannot have a positive root. On this fact is founded

152. NEWTON'S METHOD of finding a superior limit. For x, in the proposed equation, he substituted $x'+r$, that is, he diminished the roots each by r, and then sought for the smallest positive integer value of r that would render the signs of the transformed equation all *positive*, and consequently its real roots all negative. Such a value of r would obviously be a superior limit to the positive roots of the proposed equation, seeing that if each be diminished by r, the remainders are all negative.

As an example of this method, let it be required to find a superior limit to the positive roots of the equation $x^3-5x^2+7x-1=0$. Substituting $x'+r$ for x, the transformed equation is that in the margin; and after a few trials, we find that 3 is the smallest integer which, put for r, causes all the coefficients to become positive; $\therefore$ 3 exceeds the greatest positive root of the equation.

$$\begin{array}{l|l|l} 0=x'^3+3r & x'^2+\ 3r^2 & x'+\ r^3 \\ \quad -5 & \quad -10r & \quad -5r^2 \\ & \quad +\ 7 & \quad +7r \\ & & \quad -1 \end{array}$$

This method of Newton readily leads to that of Maclaurin, which is free from trials.

153. MACLAURIN'S LIMIT.—When the leading coef. is unity, the greatest negative coef., taken positively, and increased by unity, will be a superior limit to the positive roots. For let r be equal to the greatest negative coef., taken positively, and increased by 1: then r is >1, and diminishing the roots by r, as already explained,

$$1 \qquad A_{n-1} \qquad A_{n-2} \qquad A_1 \qquad N \quad (r$$

it is evident that, whether A_{n-1} be the greatest negative coef. or not, the number r, added to A_{n-1}, will give a positive result not less than unity: this result, multiplied by r, will therefore give a quantity, not less than r, to be added to A_{n-2}; the result, therefore, whether A_{n-2} be the greatest negative coef. or not, is positive, and not less than unity; and so on. Hence all the results in the first row of operations are positive. Of necessity, therefore, all the results of the second row, and of the following rows, must be positive, since only positive quantities are henceforth added. The coefficients of the transformed equation are, therefore, all positive, so that r is a superior limit to the roots of the proposed equation.

154. When the *second* is the greatest negative coefficient, Maclaurin's, as a general method of finding a superior limit, is the most convenient that has been proposed. But when the third coef. is negative, whatever the preceding may be, and all that follow *positive*, a much closer superior

limit may be discovered as follows: Writing the three leading coefficients only, as in the margin, all we have to do is to find the smallest positive value of r that will render the second result *positive;* the first result, for such a value of r, will necessarily be positive, as likewise all the following results, inasmuch as all the original coefficients after the third are positive. Hence, to find a superior limit r, we have only to determine this value so as to satisfy the condition $A_n r^2 \pm A_{n-1} r =$ or $> A_{n-2}$.

$A_n \pm A_{n-1}$	$-A_{n-2}(r$
$A_n r$	$(A_n r \pm A_{n-1})r$
$A_n r \pm A_{n-1}$	positive

Take for ex. $3x^3-2x^2-11x+4=0$: then we must have $3r^2-2r>11$ $\therefore$ $r=3$ is a superior limit.

155. But even should the coefficients following the third be not all positive, this method will still be available: for ex., take the equation

$$x^4+x^3-15x^2-19x-3=0.$$

Proceeding as above, we have to determine r from the condition $r^2+r>15$ $\therefore$ $r=$ an integer >3. Trying 4, and proceeding with the subsequent coefficients as in the margin, observing that for $r=4$, $r^2+r-15=5$, we see that the results which follow will all become positive: hence 4 is a superior limit. The superior limit, determined by Maclaurin's method, would be 20.

-19	-3	(4
20	4	
1	1	

Lastly, let the equation be $2x^7+11x^6-10x^5-26x^4+31x^3+72x^2-230x-348=0$. The condition $2r^2+11r>10$, gives $r=1$, and $2r^2+11r=13$, and since $13-10$, or 3, multiplied by 1, does not amount to 26, r is too small. For $r=2$, $2r^2+11r-10=20$; trying, therefore, this 2, in reference to the coefficients after the third, we have

-26	$+31$	$+72$	-230	-348	(2
40	28	118	380	300	
14	59	190	150	-48	

From these results, it is plain that 3 is a superior limit, and that the first figure of the greatest positive root of the equation is 2 (art. 137).

It is always easy to see at a glance, when the left-hand member of the inequality is computed, whether the result, under the negative coef. with which we are dealing, is sufficiently great for its product by r to exceed the next coefficient, should this be negative.

The foregoing remarks and illustrations will sufficiently prepare the student for the following general rule for finding a close superior limit to the positive roots of a numerical equation.

156. General Rule. 1. Take the *first* negative coef.—that in the term $-A_k x^k$ suppose, and subtract k from the exponent, and from all the preceding exponents; then, disregarding all the following terms, find the least integer value r of x that will satisfy the condition $A_n x^{n-k}+\ldots.>A_k$, which it will not be difficult to do, since *all* the terms on the left are positive.

2. Use the value of x thus found for a transforming multiplier, and get the first row of results: if these are all positive, the number used will be a superior limit; but if not, complete the transformation: if the results of *this* are all positive, the number will be a superior limit.

3. If negative signs occur in these last results—which, however, can happen only in the more advanced of them, proceed exactly as before—taking the *first* of the negative coefficients, and determining the least integer r' for x'; and so on, till a row of positive results is obtained; then the number $r+r'+\ldots$, which is the sum of all the transforming multipliers, will be a close superior limit.

Note. 1. As exemplified above, it will always be best, when the *third*

coef. is negative, even should the second be negative also, still to determine the transforming multiplier from the condition $A_n x^2 \pm A_{n-1}x > A_{n-2}$. The equation $A_n x^2 \pm A_{n-1}x - A_{n-2} = 0$ has necessarily but one positive root (68), so that if this condition be satisfied for $x=r$, r must be such as to render the first two results positive.

2. When the second coef. is negative, and the third *positive*, it will still be better to deduce r from the like condition $A_n x^2 - A_{n-1}x + A_{n-2} > 0$. But the value r of x must not be taken less than $\frac{1}{2}(A_{n-1} \div A_n)$, otherwise the first result in the second row will be negative.

Should, however, the roots of the quadratic be imaginary, that is, should $4A_nA_{n-2} > A^2_{n-1}$, then we may disregard the above condition, using at once the transforming multiplier $r = \frac{1}{2}(A_{n-1} \div A_n)$, since, for this value of r, the first term in the second row of results is 0; and the second in the first row is + for *all* values of r (68). It may be noticed too here, that in the foregoing conditions for determining r, the symbol $>$ may always be replaced by $=$.

(1) Let the equation be

$$x^4 + 11x^2 - 25x - 67 = 0.$$

Here the condition is

$$x^3 + 11x > 25 \therefore x = 2$$

1	+0	+11	−25 −67(2
	2	4	30 10
	2	15	5 −57
	2	8	46
	4	23	51
	2	12	
	6	35	
	2		
	8		

$\therefore x'^4 + 8x'^3 + 35x'^2 + 51x' > 57 \therefore x' = 1.$

Hence 2+1=3 is a superior limit, and the greatest root lies between 2 and 3. And this we might have inferred as soon as we had reached the positive results 4+23+51, for these alone exceed 57.

(2) Let the equation be

$$x^5 + 7x^4 - 12x^3 - 49x^2 + 52x - 13 = 0.$$

Here the condition is

$$x^2 + 7x > 12 \therefore x = 2$$

1	+7	−12	−49	+52	−13(2
	2	18	12	−74	−44
	9	6	−37	−22	−57
	2	22	56	38	
	11	28	19	16	

Now, without proceeding further, we see that, as the sum of the positive results already arrived at exceeds 57, the completed results would give $x'=1$: hence 3 is a superior limit, and the greatest root of the equation must lie between 2 and 3, inasmuch as for $x=2$, the polynomial is negative, and for $x=3$, it is positive (137).

(3) Let the equation be $x^4 + 16x^3 - 2x^2 - 12x - 6 = 0 \therefore x^2 + 16x > 2$ $\therefore x = 1$; and proceeding as in the margin, we see that all the results are positive, except the last, and that their sum exceeds that last. It is unnecessary therefore to complete the transformation, which would necessarily give $x'=1$: we conclude, therefore, that the greatest root lies between 1 and 2.

1	+16	−2	−12	−6(1
	1	17	15	3
	17	15	3	−3

(4) Let the equation be $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0$. Here the condition is $x^2 - 8x + 14 > 0$, and taking x not less than 4, (156) $\therefore x = 6$, and proceeding as in the margin, we soon have evidence that 6 is abundantly large. Transforming again by 5, the final result of the first row is −13; we conclude, therefore, that the greatest root lies between 5 and 6. As the smallest integer fulfilling the first condition here is 6, the restriction "x not less than 4" is superfluous.

1	−8	+14	+4	−8(6	
	6	−12	12	96	
	−2	2	16	88	
	6				
	4				

1	−8	+14	+4	−8(5	
	5	−15	−5	5	
	−3	−1	−1	−13	

(5) Take the equation $x^4-8x^3+20x^2-14x-9=0$. Here $4\times20>8^2$, therefore (156) $r=4$. Proceeding as in the margin, we see that as the positive results $4+18$ exceed 1, 5 will be a superior limit: hence the greatest root lies between 4 and 5. And this we might have inferred even from the first row of results.

1	−8	+20	−14	−9	(4
	4	−16	16	8	
−4	4	2	−1		
4	0	16			
0	4	18			

On account of the importance of the present problem in the solution of numerical equations, we shall give one example more. Required the first figure of the greatest positive root of the equation

$$x^4-10x^3+4x^2-20x-78=0.$$

Now, whether we take the condition $x=10$ by the rule (for $>$ may always be changed for $=$), or the condition in the NOTE, the same value is suggested, and the operations are as in the margin, where it is seen that for $x=10$, the last result is $+$, and that for $x=9$, it is $-$: the greatest root lies therefore between 9 and 10, that is, the first figure of it is 9.

1	−10	+4	−20	−78	(10
	10	0	40		
0	4	20			

1	−10	+4	−20	−78	(9
	9	−9			
−1	−5				

157. **Inferior Limit.**—To find an inferior limit to the positive roots of $f(x)=0$, all we have to do is to take the reciprocal equation, that is, to reverse the coefficients (146), and to proceed as above to find the *superior* limit, the reciprocal of which will of course be the inferior limit to the positive roots of the proposed equation. And in this way we may always discover the extreme limits within which all the real positive roots of an equation lie.

158. **Limits to the Negative Roots.**—As to the negative roots, we know (141) that they are converted into positive roots by simply changing the alternate signs of the terms: if therefore we make this change, and then determine the limits of the positive roots, as above explained—these limits, with the negative sign prefixed, will be the limits of the real negative roots, or the extremes within which they all lie.

If a superior limit of the positive roots of $f(x)=0$ be 0, that is, if for $x=0$, and for all higher positive values, $f(x)$ be positive, we infer that there are *no* positive roots. In like manner, if 0 be a (negatively) superior limit to the negative roots, that is, if for $x=0$, and for all higher *negative* values, $f(x)$ be positive, the equation can have no negative roots, so that if both conditions have place, we conclude that all the roots are imaginary. But more precise information about the number of real roots may be obtained from arts. (163), (164), &c., following.

159. PROP.—Let

$$A_nx^n+A_{n-1}x^{n-1}+A_{n-2}x^{n-2}+\ldots+A_3x^3+A_2x^2+A_1x+N=0\ldots[1]$$

be any equation, and let the following, of inferior degree, be derived from it by the uniform process of multiplying each term by the exponent of x, and then diminishing such exponent by unity,

$$nA_nx^{n-1}+(n-1)A_{n-1}x^{n-2}+(n-2)A_{n-2}x^{n-3}+\ldots+3A_3x^2+2A_2x+A_1=0\ldots[2].$$

Then the real roots of this latter equation will all lie between the real roots of the former. [The latter is called the *limiting* equation to the former:—it is also called the *derived* equation.]

The polynomial forming the first member of [1] is

$$f(x)=(x-a_1)(x-a_2)(x-a_3)(x-a_4)\ldots(P)\ldots[3]$$

where P represents the product of the quadratic factors involving the imaginary roots, should any enter, which product we know (139) to remain positive whatever value be given to x; we shall also regard the real roots a_1, a_2, &c., to be arranged in the order of their magnitude, a_1 being the *greatest* root. Now we have seen (pa. 103), that if any polynomial $f(x)$ be divided by $x-a$, and if a be put for x in the quotient, Q, we shall have

$$Q=nA_na^{n-1}+(n-1)A_{n-1}a^{n-2}+(n-2)A_{n-2}a^{n-3}+\ldots+3A_3a^2+2A_2a+A_1\ldots[4]$$

whatever be a*. Let $a=a_1$, then [3], since, when a_1 is put for x, Q is also

$$Q=(a_1-a_2)(a_1-a_3)(a_1-a_4)\ldots(P), \text{ in which, by hypothesis,}$$

all the factors are positive, it follows that when a_1 is substituted for a in [4], or which is the same thing, for x in [2], the result is *positive*. Let $a=a_2$, then [3], for $x=a_2$, Q is

$$Q=(a_2-a_1)(a_2-a_3)(a_2-a_4)\ldots(P), \text{ in which, by hypothesis,}$$

the first factor is negative, and all the others positive: hence, when a_2 is substituted for a in [4], or which is the same thing, for x in [2], the result is *negative:* hence a real root of [2] must lie between a_1, and a_2. Again let $a=a_3$, then [3], $Q=(a_3-a_1)(a_3-a_2)(a_3-a_4)\ldots\ldots(P)$, in which the first two factors are negative and all the others positive: the product is $\therefore$ positive: hence when a_3 is substituted for a in [4], or for x in [2], the result is *positive*, $\therefore$ a real root of [2] must lie between a_2, and a_3. And so on: therefore the *derived* or *limiting* equation [2] has at least as many real roots, wanting one, as the primitive equation [1], so that if all the roots of [1] be real, all the roots of the derived equation [2] must be real. It follows, moreover, that if imaginary roots enter [2], the same number, at least, must enter [1].

160. If a positive number greater than a_1 be substituted in [1], the result will be positive; and if a root of [2], lying between a_1 and a_2, be substituted, the result will be negative; if again a root of [2], lying between a_2 and a_3, be substituted, the result will be positive; and so on. Consequently, if the real roots of [2] be substituted, one after another, in [1], the changes of sign in the results will make known, not only the exact number of real roots in [1], but also between what two numbers each root lies.

161. We may add here that the function of x derived from $f(x)$, as above, is usually called the *first derived function*, and is marked $f_1(x)$, or $f'(x)$; the *second* derived function, deduced in like manner from this, is marked $f_2(x)$, or $f''(x)$; and so on. The function [2] above is $f_1(x)$, which, if we put $x=0$, reduces simply to A_1; the function $f_2(x)$ (see [3] p. 105), for $x=0$, reduces to $2A_2$, the next function $f_3(x)$, to $2.3A_3$; and so on: consequently, the polynomial

$$f(x)=N+A_1x+A_2x^2+A_3x^3+A_4x^4+\ldots\ldots+A_nx^n$$

---

* We may arrive at this conclusion a little differently, thus:—Taking for convenience a polynomial of the fourth degree only, namely, $A_4x^4+A_3x^3+A_2x^2+A_1x+N$, and dividing by $x-a$, after the manner already explained, we have

$$\begin{array}{lllll} A_4 & +A_3 & +A_2 & +A_1(a \\ & A_4a & A_4a^2+A_3a & A_4a^3+A_3a^2+A_2a \end{array}$$

$$Q=A_4x^3+(A_4a+A_3)x^2+(A_4a^2+A_3a+A_2)x+A_4a^3+A_3a^2+A_2a+A_1.$$

Putting a for x, this becomes $4A_4a^3+3A_3a^2+2A_2a+A_1$; and similarly, whatever be the degree of the polynomial.

may be written thus:—

$$f(x)=f(0)+\frac{f_1(0)}{1}x+\frac{f_2(0)}{1.2}x^2+\frac{f_3(0)}{1.2.3}x^3+\frac{f_4(0)}{1.2.3.4}x^4+\ldots+\frac{f_n(0)}{1.2.3\ldots n}x^n.$$

If the polynomial $f(x)$ be transformed into another, $F(x')$, by putting $a+x'$ for x, we shall of course have a similar expression, F being put for f, and x' for x; but since

$$F(x')=f(a+x'),\ F_1(x')=f_1(a+x'),\ F_2(x')=f_2(a+x'),\ \&c.$$

$$\therefore F(0)=f(a),\ F_1(0)=f_1(a),\ F_2(0)=f_2(a),\ \&c.$$

$$\therefore f(a+x')=f(a)+f_1(a)x'+\frac{f_2(a)}{2}x'^2+\frac{f_3(a)}{2.3}x'^3+\frac{f_4(a)}{2.3.4}x'^4+\ldots+\frac{f_n(a)}{2.3\ldots n}x'^n$$

$$=N'+A'_1x'+A'_2x'^2+A'_3x'^3+A'_4x'^4+\ldots+A'_nx'^n.$$

Should the student wish for a verification of this, he may complete the transformation in the foot-note at page 117. The above form for $f(a+x')$ will find its application hereafter.

162. **Equal Roots.**—If the proposed equation $f(x)=0$, have p equal roots, the derived equation $f_1(x)=0$, must have $p-1$ of those roots. For whatever be a, we know (134) that the quotient $Q=f(x)\div(x-a)$, for $x=a$, is $f_1(a)$. But there are $p-1$ equal values of a, for each of which this quotient vanishes, therefore the equation $f_1(a)=0$ must furnish these $p-1$ values; or, which is the same thing, the equation $f_1(x)=0$ has $p-1$ of the equal roots. If besides the p roots, each equal to a_1, there be q roots, each equal to a_2, and so on, the equation $f_1(x)=0$ must evidently involve just $p-1$ of the former, just $q-1$ of the latter, and so on, besides the roots peculiar to itself.

It follows therefore that if an equation $f(x)=0$ have equal roots, the functions $f(x)$, and $f_1(x)$ must have a common divisor, and that the *greatest* common divisor will be the product of the equal factors, each raised to a power a unit lower than in $f(x)$.

(1) What must the value of N be, so that the equation $f(x)=x^4-x^3+N=0$ may have two equal roots? The limiting equation is

$$f_1(x)=4x^3-3x^2=0\ \therefore 4x-3=0\ \therefore x=\frac{3}{4}$$

$$\therefore f(x)=\left(\frac{3}{4}\right)^4-\left(\frac{3}{4}\right)^3+N=0\ \therefore N=\left(\frac{3}{4}\right)^3-\left(\frac{3}{4}\right)^4=\left(\frac{3}{4}\right)^3\frac{1}{4}=\frac{27}{256}.$$

Hence the equation $x^4-x^3+\frac{27}{256}=0$ has two roots, each equal to $\frac{3}{4}$.

(2) In the equation $f(x)=x^4+A_3x^3+A_2x^2+A_1x+A_0=0$, in which A_3, A_2, are supposed to be known, what values must A_1, A_0, have in order that three of the four roots may be equal? The first derived equation, $f_1(x)=4x^3+3A_3x^2+2A_2x+A_1=0$, must involve two of the equal roots; and therefore the second derived equation, $f_2(x)=12x^2+6A_3x+2A_2=0$, must involve one. Let a_1, a_2, be the roots of this quadratic; then substituting a_1 for x in $f(x)$, we get the value of A_0, and making the same substitution in $f_1(x)$, we get the value of A_1. For these values of A_1, A_0, the equation will have two roots each equal to a_1. In like manner, substituting a_2 for x in $f(x)$ and $f_1(x)$, we also get values for A_0, A_1, for which the equation will have two roots each equal to a_2. If the given coefficients A_3, A_2, be such as to render the roots of the quadratic imaginary, it will be impossible for three roots of $f(x)=0$ to be equal.

163. **The Rule of Descartes.**—A complete equation cannot have a *greater number* of positive roots than it has changes of sign from + to −, and from − to + ; nor a *greater number* of negative roots, than continuations of the same sign.

To prove this we have first to notice, that if we write the two signs +, −, or −, +, with an interval between them, thus, +.....−, or −.....+, then with whatever signs we fill up that interval, be they few or many, the row must exhibit *at least* one change of sign. It is obvious, in fact, that there must always be an *odd number* of changes. Take, now, the row of signs furnished by any polynomial $f(x)$, as, for instance, the row in the margin, and let a new positive root a be introduced into the equation $f(x)=0$, that is, let $f(x)$ be multiplied by $x-a$. The multiplication by x will of course give the *same* row of signs, but the multiplication by $-a$ will give a row of *opposite* signs, pushed one place to the right, as in the margin. Now it is clear that every change of a *horizontal* variation of sign, in the first row, to a permanency in the second, gives rise to a *vertical* permanency, and that these vertical permanencies—taken one after another in order—necessarily supply *changes of sign only* in the results. Hence, including the leading sign +, the vertical permanencies alone supply, in the product, just as many changes of sign as $f(x)$ has. But whatever may be the intervening signs in the complete product, the changes here referred to cannot be *diminished* by their insertion—as seen above; but the final sign in the product is always *opposite* to the final sign in $f(x)$; this sign produces therefore an additional change. Hence the introduction of one additional positive root into an equation always introduces one additional change of sign *at least;* consequently there cannot be a *greater number* of positive roots in an equation than there are changes of sign in its terms.

```
+++−−++++−
 −−−+++−−−+
+  −  +   −+
```

Suppose now the alternate signs in $f(x)$ to be changed; then the permanencies will become variations, and the variations permanencies; and the negative roots will be converted into positive (141). Consequently, from what is proved above, there cannot be a *greater number* of negative roots in an equation than there are permanencies, or continuations of the same sign.

164. When the roots of the complete equation are *all real*, the positive roots are exactly equal in number to the variations, and the negative roots to the permanencies; for since the variations and permanencies together amount in number to the entire number of roots, if there were fewer positive roots than variations, there would be more negative roots than permanencies, and if there were fewer negative roots than permanencies, there would be more positive roots than variations; which contradicts what is proved above.

165. It follows from this that when the roots of a complete equation are all real, and that by diminishing those roots by any positive number r, we find that m variations are lost in passing from the primitive to the transformed equation, we may be sure that m of the positive roots have been converted into negative roots, that is, that exactly m positive roots of the primitive must lie between 0 and r.

166. Whether the roots of the complete equation are all real or not, by diminishing the roots by any positive number r, as many variations *at least* will be lost as there are positive roots between 0 and r.

To prove this it will be necessary to show, first, that whatever be the positive number r, variations can never be *gained* by diminishing the

roots. Now it has been shown above, that by multiplying a polynomial by $x-r$, one variation at least is gained: consequently, by dividing a polynomial by $x-r$, provided there be no remainder—one variation at least must be lost. In transforming by r, as at (144) the row of results, excluding the last result under the final term N of the equation, is the same, whether this last result, which is the remainder, be 0 or not. Consequently, omitting the last result, one variation, at least, is lost in the row; so that, including the last result, it is impossible, whatever be its sign, that a variation can be *gained*.

Like reasoning applies to the next row of results:—one variation at least is lost from the results above them, so that including the last of the former results, a variation cannot be gained; and so on. Hence, a variation can never be gained by diminishing the roots of an equation.

Suppose now that one real root of the equation $f(x)=0$, and one only lies between the positive numbers p and q, p being $<q$: the last term of the transformed equation $f(x-q)=0$ must be opposite in sign to the last term of $f(x-p)=0$ (137), so that if the extreme signs be like in one of these transformations, they must be unlike in the other. But when the signs of the first and last terms of a polynomial are *like*, the total number of variations must evidently be *even*, whatever be the intermediate signs; and when the extreme signs are *unlike*, the number of variations must be *odd*: hence $f(x-q)$ must have either a variation more or a variation less than $f(x-p)$: it cannot *gain* a variation, as proved above: therefore, in proceeding from $f(x-p)=0$ to $f(x-q)=0$, one variation, at least, must be lost.

Now, whether we diminish the roots of an equation at once by r, or by the smaller numbers, a, b, c, &c., in succession, till $a+b+c+\&c.=r$, the final transformation is, of course, the same; and we see that, in our progress to this final transformation, one additional variation, at least, is lost every time a real root is passed over. Hence, if the equation has m real roots between 0 and r, m variations, at least, will be lost in passing from $f(x)=0$ to $f(x-r)=0$.

And, consequently, if no variations be lost, there cannot be a root between 0 and r.

167. Should the number by which we diminish the roots be a_1, one of those roots, then the last term of the transformed will be 0; and if there be two roots equal to a_1, the last two terms will be 0. We have seen that whatever signs we give to these zeros, the transformed into which they enter cannot *gain* variations; and it is plain that such signs may be given as to furnish, with the sign of the preceding term, *two* variations: these may be converted into two permanencies by changing the middle sign, so that then two variations, at least, will be lost in passing from the primitive to the transformed equation; and as none are restored in continuing to diminish the roots, it follows that in passing over two roots, whether equal or unequal, two variations, at least, will be lost. And similar reasoning applies if three or more equal roots are passed over.

168. Should two variations be lost, and yet no real roots be passed over, then, on the supposition that all the roots are real, the transformed equation would have fewer variations than there are positive roots, which contradicts the rule of Descartes: the above supposition is, therefore, untenable: hence the loss of variations must indicate the presence of two imaginary roots. In a similar manner, if m variations be lost, and yet only $m-k$ roots be passed over, the equation must have at least k imaginary roots, and, therefore, k must be an even number. The foregoing conclusions may be summed up in the following:—

169. **Theorem of Budan and Fourier.**—1. If, in diminishing the roots of an equation by a positive number r, m variations be lost, the equation *may* have m real roots between 0 and r; but it cannot have a greater number.

2. If the number of real roots be less than m, it can differ from m only by an *even* number k: and the additional loss of variations must indicate the existence of k imaginary roots, at least, in the proposed equation.

This theorem, however, does not enable us to decide upon the precise character of the roots to which the loss of variations spoken of may be due: the following criteria for testing the existence of imaginary roots will, therefore, be found of especial service.

170. **Imaginary Roots:—Criterion of De Gua.**—If one of the terms of an equation be absent, and if the immediately preceding and succeeding terms have both the same sign, the equation will have two imaginary roots at least.

For the sign of the zero, thus occurring between two like signs, being arbitrary, we may give to it a sign opposite to the like adjacent signs, and thus get two variations; or we may give to it a like sign, and thus get two permanencies:—results which, on the supposition that all the roots are real, are contradictory: hence the equation must have at least one pair of imaginary roots.

171. **Other Criteria** may be deduced as follows: If the general equation

$$A_n x^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + \ldots\ldots + A_2x^2 + A_1x + A_0 = 0 \ldots\ldots [1]$$

be transformed into another by putting $x+r$ for x, and then r be determined so that the second coefficient may vanish, as explained at (145), the third coef. will be $\frac{A_{n-2}}{A_n} - \frac{n(n-1)}{2}\left(\frac{A_{n-1}}{nA_n}\right)^2$. Consequently, if the sign of this third coef. be the same as the sign of the first, the first three terms of the transformed equation will satisfy the criterion of De Gua, and, therefore, imply the existence of a pair of imaginary roots: hence, multiplying this third coef. by the positive quantity $2nA^2_n$, a pair of imaginary roots will be indicated, provided the first three coefficients of the equation satisfy the condition

$$2nA_{n-2}A_n > (n-1)A^2_{n-1} \ldots\ldots [1']$$

or, indeed, provided they satisfy it when $=$ is put for $>$.

If the order of the coefficients be reversed, we shall have an equation the roots of which will be the reciprocals of those of the original: the existence of a pair of imaginary roots in either implies, therefore, the existence of a corresponding pair in the other. The criterion [1'] may, therefore, be applied to the *last three* coefficients of [1], as well as to the first three; so that a pair of imaginary roots will be also indicated, provided that

$$2nA_0A_2 > (n-1)A_1^2 \ldots\ldots [2'].$$

172. Now we know (p. 117), that if a limiting or derived equation have imaginary roots, the primitive equation must also have imaginary roots: taking therefore the several limiting equations derived one after another, each from the preceding, [1] being the primitive, and

$$\begin{aligned}
2nA_0A_2 &> (n-1)A^2_1 \ldots\ldots [1]\\
3(n-1)A_1A_3 &> 2(n-2)A^2_2 \ldots\ldots [2]\\
4(n-2)A_2A_4 &> 3(n-3)A^2_3 \ldots\ldots [3]\\
5(n-3)A_3A_5 &> 4(n-4)A^2_4 \ldots\ldots [4]\\
&\vdots\\
2nA_{n-2}A_n &> (n-1)A^2_{n-1} \ldots [n-1]
\end{aligned}$$

applying the criterion [2′] to the last three coefficients of each, we shall arrive at the series of conditions in the margin, and the existence of any one of which will imply the existence of at least one pair of imaginary roots in the equation [1].

The criteria here established pave the way for the proof of an important rule, first given by Newton, but given without demonstration; it may be established from the following considerations.

173. **Newton's Rule for Imaginary Roots.**—From the general equation

$$A_n x^n + \ldots\ldots + A_4 x^4 + A_3 x^3 + A_2 x^2 + A_1 x + A_0 = 0,$$

the following series of derived equations are deduced, namely,

$$nA_n x^{n-1} + \ldots\ldots + 4A_4 x^3 + 3A_3 x^2 + 2A_2 x + A_1 = 0$$

$$n(n-1)A_n x^{n-2} + \ldots\ldots + 4.5A_5 x^3 + 3.4A_4 x^2 + 2.3A_3 x + 2A_2 = 0$$

$$n(n-1)(n-2)A_n x^{n-3} + \ldots\ldots + 4.5.6A_6 x^3 + 3.4.5A_5 x^2 + 2.3.4A_4 x + 2.3A_3 = 0.$$

&c. &c. &c.

And if any of these have imaginary roots, we know that as many, at least, must enter the primitive: the same remark applies if we reverse the coefficients of each, as also if we take the limiting equations derived from them when the coefficients are thus reversed. Reversing, then, the coefficients of each equation, commencing with the primitive, we readily see that the derived *cubic* equations, or equations of the third degree, are as follows:—

$$4.5\ldots\ldots nA_0 x^3 + 3.4\ldots\ldots n-1A_1 x^2 + 2.3\ldots\ldots (n-2)A_2 x + 2.3\ldots\ldots (n-3)A_3 = 0$$
$$4.5\ldots(n-1)\ A_1 x^3 + 3.4\ldots\ldots (n-2)2A_2 x^2 + 2.3\ldots\ldots (n-3)3A_3 x + 2.3\ldots\ldots (n-4)4A_4 = 0$$
$$4.5\ldots(n-2)2A_2 x^3 + 3.4\ldots(n-3)2.3A_3 x^2 + 2.3\ldots(n-4)3.4A_4 x + 2.3\ldots(n-5)4.5A_5 = 0.$$

&c. &c.

Or, expunging the common numerical factors, they are

$$n(n-1)(n-2)A_0 x^3 + 3(n-1)(n-2)A_1 x^2 + 2.3(n-2)A_2 x + 2.3A_3 = 0$$
$$(n-1)(n-2)(n-3)A_1 x^3 + 3(n-2)(n-3)2A_2 x^2 + 2.3(n-3)3A_3 x + 2.3.4A_4 = 0$$
$$(n-2)(n-3)(n-4)2A_2 x^3 + 3(n-3)(n-4)2.3A_3 x^2 + 2.3(n-4)3.4A_4 x + 2.3.4.5A_5 = 0.$$

&c. &c.

Now, if any of these limiting cubics indicate imaginary roots, when tested by the criteria [1′], [2′], at p. 121, we may be sure that imaginary roots exist also in the primitive equation. But since only one pair of imaginary roots can enter a cubic equation, it follows that whether the criterion of imaginary roots be satisfied by the three leading terms of any of the above cubics, or by the three final terms, or by both sets of three—one pair of imaginary roots, and one pair only, is implied.

Now, upon examining the coefficients of the above cubics, we see that the first three terms of each have a common factor, so that in applying the criterion to these, this common factor may be suppressed. Let us suppress the common factor, namely $(n-3)$, in the second cubic: then the product of the first and third coefficients will be $2.3^2(n-1)(n-2)\,A_1 A_3$, and the square of the middle coef. will be $2^2.3^2(n-2)^2\,A_2^2$. But these are the same as we should get by employing, in like manner, the *last* three coefficients of the *preceding* cubic: and we see the same to be true of the following cubics, that is, if the criterion of imaginary roots is satisfied by the last three terms of one cubic, it must be satisfied by the first three of the next, and *vice versâ*; so that the fulfilment of the condition by two

consecutive sets of three terms, implies but a single pair of imaginary roots.

We thus arrive at the following conclusions, namely:—

1. If the first three terms in the first cubic satisfy the criterion, we infer the existence of one pair of imaginary roots.

2. If the next set, the three final terms of the same cubic, also satisfy it, the preceding condition merely recurs, and supplies no additional information. In this case the following set of three—the leading terms of the next cubic—must of necessity furnish the same concurring condition, and so on, till we arrive at a set of three terms for which the condition *fails*, thus putting a stop to the series of concurring indications, and preparing the way for new and distinct conditions.

3. So soon as the criterion is again satisfied, the condition being entirely independent of, and non-concurring with, the former, must imply another and distinct pair of imaginary roots. And so on, to the end of the series of cubics.

The criterion which we have here supposed to be applied to the terms, taken three at a time, of the successive cubics—namely, either [1′] or [2′], at p. 121, supplies, one after another, the entire series of criteria there given, as we shall presently see. But as the three final terms of any cubic always furnish the same condition as the three leading terms of the next, the repetitions may be omitted. Attending to this, and applying the criterion to each of the foregoing cubics in succession, we have the following conditions for imaginary roots, namely:—

1st. $2^2.3^2n(n-1)(n-2)^2A_0A_2>2.3^2(n-1)^2(n-2)^2A^2_1$,

or, suppressing com. factors, $2nA_0A_2>(n-1)A_1^2$ ..... [1].

2nd. $2^2.3^3(n-1)(n-2)A_1A_3>2^3.3^2(n-1)^2A_2^2$;

or, suppressing com. factors, $3(n-1)A_1A_3>2(n-2)A^2_2$ ..... [2].

3rd. $2^3.3^3.4(n-2)(n-3)A_2A_4>2^3.3^4(n-3)^2A^2_3$;

or, suppressing com. factors, $4(n-2)A_2A_4>3(n-3)A^2_3$ ..... [3].

4th. $2^3.3^4.4.5(n-3)(n-4)A_3A_5>2^3.3^4.4^2(n-4)^2A^2_4$;

or, suppressing com. factors, $5(n-3)A_3A_5>4(n-4)A^2_4$..... [4].

&c. &c.

And thus, as stated above, are we led to the series of criteria at page 121; and which we now know to be such that if, when proceeding from one set of three terms in an equation to the three next in order, the consecutive criteria both have place, the concurrence is to be regarded merely as a second indication of the same thing—the existence of a single pair of imaginary roots; but that so soon as the condition fails, preparation is made for a new and independent indication, and so on, till all the sets of three have been submitted to the tests. And this is substantially the rule of Newton.*

174. In employing the criteria [1], [2], &c., at page 121, we may consider the symbols A_o, A_1, A_2, &c., to represent the coefficients taken in order either from the beginning or from the end. As an example, let the equation $x^4-4x^3+8x^2-16x+20=0$ be submitted to examination. Taking $-4x^3$ for the middle term, x^4 and $8x^2$ being the adjacent terms, the condition [1] is satisfied. Taking $8x^2$ for the middle term, the condition [2] fails. And lastly, taking $-16x$ for the middle term, the condition

* The above investigation of Newton's rule, here slightly modified, was first published by the author of this work in 1844 in his "Researches on Imaginary Roots," forming an Appendix to the "Theory and Solution of Equations of the Higher Orders."

[3], or which is the same [1], is satisfied: hence all the four roots are imaginary.

NOTE.—The student must bear in mind that although we may always be certain of the existence of imaginary roots when any of the criteria are satisfied, yet that we cannot be sure of the non-existence of such roots when the criteria are not satisfied. If imaginary roots are indicated, we may safely infer that so many, at least, enter the equation:—if the criteria are all unsatisfied, we cannot affirm anything as to the character of the roots:—the *quadratic* equation alone excepted. The series of criteria at (172) may be very readily written down, without reference, if the first only be remembered, and this will be suggested by the well-known criterion for the quadratic. In those which follow the first, the *numerical* factors regularly increase by unity, and the *literal* factors as regularly decrease. It is plain, too, that any set of three coefficients may be deprived of any factor common to them.

175. **Theorem.**—If in the biquadratic, or equation of the fourth degree,

$$A_4x^4+A_3x^3+A_2x^2+A_1x+N=0\ldots[1]$$

the condition $(4A_4A_2-A^2_3)N>A_4A^2_1$ is satisfied, then all the roots of the equation are imaginary.

Multiply by A_4, and extract the square root:—

$$A^2_4x^4+A_4A_3x^3+A_4A_2x^2+A_4A_1x+A_4N\Big(A_4x^2+\frac{1}{2}A_3x$$

$$2A_4x^2+\frac{1}{2}A_3x\Big)\overline{\begin{array}{l}A^2_4x^4\\ \\ \hline A_4A_3x^3+A_4A_2x^2\\ A_4A_3x^3+\frac{1}{4}A^2_3x^2\\ \hline \\ \Big(A_4A_2-\frac{1}{4}A^2_3\Big)x^2+A_4A_1x+A_4N.\end{array}}$$

Consequently the first member of [1] is made up of the two compound terms

$$\Big(A_4x^2+\frac{1}{2}A_3x\Big)^2+\Big\{\Big(A_4A_2-\frac{1}{4}A^2_3\Big)x^2+A_4A_1x+A_4N\Big\}.$$

Now if that within the braces be *always positive*, whatever real value be given to x, the whole expression must be always positive, because the first compound term is a square. But the second *is* always positive for real values of x, if

$$4\Big(A_4A_2-\frac{1}{4}A^2_3\Big)A_4N>A^2_4A^2_1, \text{ or if } (4A_4A_2-A^2_3)N>A_4A^2_1\ldots[2].$$

Hence, when this condition is satisfied, N being positive, [1] cannot become zero for any real value of x; that is to say, all its roots are imaginary.

If $A_4=1$, the condition is $(4A_2-A^2_3)N>A^2_1$, and if, in addition, $A_3=0$, it is simply $4A_2N>A^2_1\ldots[3]$.

176. It is well worthy of note that, whether A_3 be absent or present, and whatever be the degree of the equation, that equation cannot have a real root within any of the intervals throughout which the portion of the equation preceding A_2x^2 is *positive*, whenever the condition [3] is satisfied; and this condition, for N positive, may always be brought about by increasing A_2x^2 by some multiple of itself, and then diminishing the preceding terms by the same quantity. If N is *negative*, the condition may

be made to have place by diminishing A_2x^2, instead of increasing it; but the second portion of the polynomial will then be *negative* for all real values of x, so that real values will be excluded from the intervals within which the *first* portion is always negative also. Should the two portions of the polynomial, when separated by the negative sign, be both always positive, it will in general be easy to see within what limits one always exceeds the other:—there can be no real roots in the interval. (See the examples at page 133.)

NOTE.—Although in the above we have considered the condition to be uniformly secured for the last three coefficients, yet it may be equally brought about for the first three; and this will sometimes be found to be the more effective. (See ex. 10, p. 133.)

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177. **Solution of Numerical Equations in General.**—Although the process about to be explained applies equally to all numerical equations, as will readily be perceived, yet for simplicity sake we shall at present confine ourselves to equations of the fourth degree. Let the successive figures of one of the roots of the equation

$$A_4x^4+A_3x^3+A_2x^2+A_1x+N=0 \ldots [1]$$

be r, r', r'', &c. When the first figure r is discovered (see art. 156), we may readily obtain the transformed equation

$$A_4x'^4+A'_3x'^3+A'_2x'^2+A'_1x'+N'=0 \ldots [2]$$

involving the remaining portion of the root, that is, the root of which the figures are r', r'', &c. When the first figure r' of this is found, we can arrive at a second transformed equation

$$A_4x''^4+A''_3x''^3+A''_2x''^2+A''_1x''+N''=0 \ldots [3]$$

in which the first figure of the root we are tracing is r''; and so on. Now the leading figure r of the root having been found, and the transformation [2] obtained, we may avail ourselves of r to discover r', the next figure, much in the same way as, in the extraction of the square and cube roots of numbers, the leading figure of the root supplies a *trial divisor* for the discovery of the *next* figure. Thus, transposing N' to the right, and regarding it as a dividend, the first figure r' of the root of [2] must be such that when the dividend is divided by $A_1r'^3+A'_3r'^2+A'_2r'+A'_1$, the leading figure of the quotient may be r', for the finding of which we are assisted by the known value A'_1—the partial or *trial divisor*. That this, in general, forms an important part of the true or complete divisor above, will appear from reflecting that the composition of

$$A'_1 \text{ is } A'_1=f_1(r)=4A_4r^3+3A_3r^2+2A_2r+A_1 \text{ (p. 118)},$$

and that r is a place higher in the arithmetical scale than r'. The trial divisors A''_1, A'''_1, &c., for finding the subsequent figures r'', r''', &c., of the root, as in the case of the operation for the cube root already alluded to, will become more and more efficient as the work proceeds. An example or two will make this apparent.

(1) Required a root of the equation $x^3+8x^2+6x-75\cdot9=0$. This equation has a positive root, because the last term is negative (140), and it is easily seen that this root lies between 2 and 3 (156). The first transforming factor, or the first figure of the root, is therefore $r=2$; and avail-

ing ourselves of the several trial divisors A'_1, A''_1, &c., the successive steps of the work, for finding r', r'', &c., are as follows:—

A_3	A_2	A_1	N	
1	8	6	$-75\cdot9(2\cdot4257$	$r=2$
	2	20	52	
	10	26	$-23\cdot900=N'$	$r'=\cdot4$
	2	24	22·304	
	12	$50=A'_1$	$1\cdot596000=N''$	$r''=\cdot02$
	2	5·76	1·239688	
	$14=A'_2$	55·76	$\cdot356312000=N'''$	$r'''=\cdot005$
	·4	5·92	·311827625	
	14·4	$61\cdot68=A''_1$	$44484375000=N''''$	$r''''=\cdot0007$
	·4	·3044	43716797593	
	14·8	61·9844	$767577407=N'''''$	
	·4	·3048		
	$15\cdot2=A''_2$	$62\cdot2892=A'''_1$		
	·2	76325		
	15·22	62·365525		
	2	76350		
	15·24	$62\cdot441875=A''''_1$		
	2	1069299		
	$15\cdot26=A'''_2$	62·45256799		
	5	1069348		
	15·265	$62\cdot46326147=A'''''_1$		
	5			
	15·270			
	5			
	$15\cdot275=A''''_2$			
	7			
	15·2757			
	7			
	15·2764			
	7			
	$15\cdot2771=A'''''_2$			

The transformed equation at which we have now arrived is

$$x'''^3+15\cdot2771x'''^2+62\cdot46326147x''''=\cdot000767\ldots\ldots$$

And by regarding the small decimal ·000767... as 0, we have the depressed quadratic equation

$$x''''^3+15\cdot2771x'''''^2+62\cdot463=0$$

involving the remaining roots diminished by 2·4257. But since $4\times62\cdot463>15\cdot2771^2$, these two remaining roots are imaginary.

The depressed quadratic, obtained as above, ought not be regarded as involving any practical error because we have replaced the small value N'''' by 0. We have stopped the development of the root of the cubic at the fourth place of decimals, considering 2·4257 to be an approximation to that root sufficiently close to be regarded as *the* root: it is, however, in strictness, the complete root of $x^3+8x^2+6x=75\cdot9-\cdot000767\ldots$ Consequently, in thus putting a termination to the development, we have tacitly assumed $\cdot000767\ldots=0$; we are, therefore, quite consistent in adhering

to this assumption; but the roots of the depressed quadratic, had they been real, could not have been obtained true beyond the fourth decimal place.

For the extent to which the root of the cubic has been carried, it is plain that several of the right-hand figures in each column of the work are superfluous: we may preclude the entrance of these useless figures thus: Having arrived as above at that absolute number N', or N'', &c., which contains as many figures plus one as there are root-figures still to be determined, work, as in contracted division, so as to provide against the entrance of figures in the last column beyond the limit thus fixed for the absolute number. It is scarcely necessary to mention that the *carryings* from rejected figures are not to be neglected. Taking the preceding example, limiting the number of root-figures to five, the contracted form of the work will stand thus:—

1 8	6	75·9(2·4257
2	20	52
—	—	——[1]
10	26	23·900
2	24	22·304
—	—[1]	——[2]
12	50	1·596
2	5·76	1·240
—[1]	——	——[3]
14	55·76	·356
·4	5·92	·312
——	——[2]	——
14·4	61·68	44
·4	·30	43
——	——	—
14·8	61·98	1
·4	'	
——[2]	·3	
15·2	——[3]	
' '	62·3	
	·1	
	——	
	62·4	
	' '	

The numbers [1], [2], &c., mark the successive complete transformations. The three figures 15·2, at the bottom of the first column, and 62·4 at the bottom of the second, remain constant for all subsequent transformations: therefore the approximate quadratic

$$x^2+15{\cdot}2x+62{\cdot}4=0$$

would guide us sufficiently to the leading figure of each of the remaining roots, diminished by 2·4257, of the proposed cubic, if those remaining roots were real.

(2) Find a negative root of the equation $x^3-3x+6=0$ to six or seven decimals. This equation in a complete form is $x^3+0x^2-3x+6=0$; and changing alternate signs (141), we have to find a positive root of $x^3-0x^2-3x-6=0$. For the superior limit r we have (156), $x^2-3>0$ $\therefore r=2$; and transforming by this 2, as in the margin, we see that as $4+9$ exceeds 4, the first figure of the root is 2.

1	0	−3	−6(2
	2	4	2
	—	—	—
	2	1	−4
	2	8	
	—	—	
	4	9	

Hence, arranging the given coefficients as in the preceding example, and checking the unnecessary accumulation of decimals as above, we arrive at the successive figures of the root, and the corresponding transformed equations, as follows:—

1	0	−3	6(2·3553014
	2	4	2
	−	−	−[1]
	2	1	4
	2	8	3·267
	−	−[1]	———[2]
	4	9	·733
	2	1·89	·660875
	−[1]	———	———[3]
	6	10·89	72125
	·3	1·98	68014
	———	———[2]	———[4]
	6·3	12·87	4111
	·3	·3475	4092
	———	———	———
	6·6	13·2175	19
	·3	·3500	14
	——[2]	———[3]	—
	6·9	13·5675	5
	5	353	5
	———	———	—
	6·95	13·6028	
	5	'	
	———	35	
	7·00	———[4]	
	5	13·638	
	——[3]	'	
	7·05	2	
	''	———	
		13·640	
		''''	

Hence the negative root of the proposed equation is $-2{\cdot}3553014$. The approximate quadratic is $x^2+7{\cdot}05x+13{\cdot}64=0$, the roots of which are imaginary.

178. In each of the preceding examples, it is the greatest root that has been sought for; but it sometimes happens that a number greater than the greatest positive root of an equation may, nevertheless, be much too small to render all the terms of the equation which results from transforming by it *positive*, for variations may still exist which no longer imply real roots, but imaginary pairs. On account of such imaginary roots entering, the superior limit to the real positive roots, as determined by the rules already given—and which limit must overstep all the imaginary, as well as all the real indications—may be a number considerably greater than the greatest real root. The following is an example:—

(3) Required a positive root of the equation $x^3-5x^2+7x-1=0$. A glance at the terms here is sufficient to show that if 0 be put for x the result is -1, and that if 1 be put, the result is $+2$, so that a root lies between 0 and 1. But the lowest transforming number that would give results all positive is $r=3$, as appears by the work in the margin, which shows, however, that a root lies between 0 and 2, on account of the change of sign in the final term. Now the two variations still presented by the transformed results $x'^3+x'^2-x'+1$, in the last three terms, imply a pair of imaginary roots, as Newton's rule, (p. 123), at once shows.

1	−5	7	−1(2
	2	−6	2
	—	—	—
	−3	1	1
	2	−2	
	—	—	
	−1	−1	
	2		
	—		
	1		

The real root of the equation has been shown to lie between 0 and 1: its leading figure is, therefore, a decimal; we may consequently use the

7 ($=A_1$) as a trial divisor, and proceed to develope the root as follows:—

1	−5	7	1 (·160713
	·1	−·49	·651
	−4·9	6·51	·349
	·1	−·48	·345096
	−4·8	6·03	3904
	·1	−·2784	3832
	−4·7	5·7516	72
	6	−·2748	55
	−4·64	5·4768	17
	6	−32	16
	−4·58	5·4736	1
	6	−32	
	−4·52	5·47	

The approximate quadratic is $x^2-4\cdot52x+5\cdot47=0$, which verifies the rule of Newton as to the remaining roots being imaginary. But whenever the approximate quadratic is such that 4 times the product of the extreme terms is very nearly *equal* to the square of the middle term, it would be unsafe to pronounce confidently on the character of the remaining roots, seeing that, on account of the contractions introduced, the quadratic is only an approximation to the true quadratic. Take, for instance, the following example:—

(4) Find all the roots—if real—of the equation $x^3-7x+7=0$. Supplying the absent term $0x^2$, and changing the alternate signs, in order to convert the negative into a positive root (141), we have $x^3+0x^2-7x-7=0$, and since for $x^2-7>0$, we have $r=3$, we proceed as in the margin. And as $1+3+2>1$, we conclude that the positive root lies between 3 and 4, and we then develope that root as in next page.

1	0	−7	−7(3
	3	9	6
	3	2	−1

[The student is recommended to pay special attention to the remarks subjoined to the developments at pages 130, 131: to notice first, how the determination of one root of an equation, by the process exhibited, conducts to an equation, a unit lower in degree, from which the character of the remaining roots, or the first figure of each of them, when they are real, may be ascertained; and second, when any doubt exists, as to whether a pair of these roots is real or imaginary, to observe that the true character of the doubtful roots may always be discovered by proceeding to approximate to one of them as if it were real, by aid of the first figure, which, on the hypothesis of its reality, that root must have.

There is a theorem—Sturm's Theorem—which enables us to discover the character of roots indicated in any interval, with unerring certainty; but the numerical operations necessary for this purpose, although very simple in themselves, sometimes become laborious on account of their length. The author of the present work has endeavoured to reduce this numerical labour to its smallest amount; and the inquiring student is recommended to make himself acquainted with this improved form of Sturm's Theorem from the chapters on that subject in the "Analysis

and Solution of Cubic and Biquadratic Equations." All discussion of the theorem has been excluded from this work, because—not only of the additional space which such discussion would require, but because the character of the roots of an equation may be determined, as hinted at above, without its aid, and, usually, with less labour.]

1	0	−7	7(3·048917
	3	9	6
	–	–	–[1]
	3	2	1
	3	18	·814464
	–	–[1]	———[2]
	6	20	·185536
	3	·3616	·166383
	–[1]	———	———[3]
	9	20·3616	19153
	·04	·3632	18791
	———	———[2]	———
	9·04	20·7248	362
	4	73024	209
	———	———	———
	9·08	20·797824	153
	4	'	146
	——[2]	73	———
	9·12	———[3]	7
	8	20·871	
	———	8	
	9·128	———	
	8	20·879	
	———	'''	
	9·136		
	' 8		
	———[3]		
	9·14		
	'		

Hence the negative root of $x^3-7x+7=0$ is $-3{\cdot}048917$; and the approximate quadratic is $x^2+9{\cdot}14x+20{\cdot}879=0$. Four times the prod. of the extreme coefficients is 83·51... and the square of the middle one is 83·56..., results which imply a pair of real roots differing but little from each other. But, on account of the above curtailments of the decimals, it is possible that the roots, which thus appear to be nearly equal, *may* be imaginary. We proceed, therefore, to develope the remaining roots on the presumption that they are real and nearly equal.*

Since the sum of the nearly equal positive roots of $x^3+0x^2-7x+7=0$ must be equal to the negative root taken positively, seeing that the sum of all three is 0, (142), half of 3·048... must be an approximation to each positive root; so that both roots must lie between 1 and 2. A like inference is deducible from the approximate quadratic; for the half of $-9{\cdot}14$ is $-4{\cdot}57$, which increased by 3·04..., the quantity by which each root has been diminished, gives $-1{\cdot}53...$ as an approximation to each root when thus rendered *negative*. The developments are as follows:—

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* As observed in the foot-note at the bottom of next page, we could settle the doubt at once by anticipating the deductions at page 138, since the present equation is only of the third degree: for as $\left(\frac{7}{3}\right)^3>\left(\frac{7}{2}\right)^2$, we conclude with certainty, from the page referred to, that the roots are all real and unequal.

1	0	−7	−7(1·356895	1	0	−7	−7(1·692021
	1	1	−6		1	1	−6
	–	—	—[1]		–	—	—[1]
	1	−6	−1		1	−6	−1
	1	2	−·903		1	2	−1·104
	–	—[1]	———[2]		–	—[1]	———[2]
	2	−4	−97		2	−4	·104
	1	·99	−86625		1	2·16	·100809
	–[1]	———	———[3]		–[1]	———	———[3]
	3	−3·01	−10375		3	−1·84	3191
	·3	1·08	−9049		·6	2·52	3157
	—	———[2]	———[4]		—	———[2]	——[4]
	3·3	−1·93	−1326		3·6	·68	34
	·3	·1975	−1186		·6	·4401	32
	—	———	——		—	———	—
	3·6	−1·7325	−140		4·2	1·1201	2
	·3	·2000	−133		·6	·4482	2
	——[2]	———[3]	—		——[2]	———[3]	–
	3·9	−1·5325	−7		4·8	1·5683	
	5	24336	−7		9	10144	
	—	———	–		—	———	
	3·95	−1·508164			4·89	1·578444	
	5	\			9	\	
	—	24			—	10	
	4·00	———[4]			4·98	———[4]	
	5	−1·484			9	1·588	
	——[3]	2			——[3]	\ \	
	4·05	———			5·07		
	6	−1·482			2		
	—	\ \ \			—		
	4·056				5·072		
	6				2		
	—				—		
	4·062				5·074		
	\ 6				\ 2		
	——[4]				——[4]		
	4·068				5·076		
	\						

In proceeding from the transformation [1] to [2], in the second development, we see that one variation disappears, and that a real root is passed; showing that this root lies between 1 and 1·6:—it is the root 1·356..., previously developed. If the roots assumed to be real had been imaginary, we should have discovered the fact after a few steps: the several *remainders*, in the last column, instead of continually tending towards zero, or evanescence, would have approached towards some constant finite number, as a limit, which the diminishing remainders could never pass—a decided indication of imaginary roots. If the roots had been equal, the remainders, in the second and third columns, would have simultaneously converged to zero.*

The cubic equation here discussed is justly regarded by Lagrange as one of more than ordinary difficulty.

(5) Required a positive root of $2x^4-3x^3+6x-8=0$. To find a superior limit, we have to make $2x-3>0$, $\therefore r=2$; and proceeding as in the margin, we see that the greatest root lies between 1 and 2, for the sum of the given coefficients is −3.

2	−3	0	6	−8(2
	4	2	4	20
	—	–	—	—
	1	2	10	12

* As far as *cubics* are concerned, criteria for determining the character of the roots with the same certainty that the character of the roots of a quadratic is determined,

2	−3	0	6	8(1·414214
	2	−1	−1	5
	—	—	—	-[1]
	−1	−1	5	3
	2	1	0	2·8512
	-	—	-[1]	———[2]
	1	0	5	·1488
	2	3	2·128	·10422
	-	-[1]	———	———[3]
	3	3	7·128	4458
	2	2·32	3·184	4231
	-[1]	———	———[2]	——[4]
	5	5·32	10·312	227
	·8	2·64	110022	212
	———	———	———————	—
	5·8	7·96	10·422022	15
	·8	2·96	\	10
	—	———[2]	111	—
	6·6	10·92	———[3]	5
	·8	822	10·533	4
	———	———————	45	-
	7·4	11·0022	———————	
	·8	824	10·578	
	——[2]	———————	\	
	8·2	11·0846	4	
	2	\	———[4]	
	———	826	10·62	
	8·22	——— [3]	\ \ \	
	2	11·2		
	———	\ \		
	8·24			
	2			
	———			
	8·26			
	2			
	——[3]			
	8·3			

Applying the second criterion at (172) to the three coefficients marked [1] above, we see that two of the roots are imaginary: and since the last term of the proposed equation is negative, we conclude (140) that the other real root is negative. In order to find the leading figure of it, we change the alternate signs, writing the equation thus, $2x^4+3x^3+0x^2-6x-8=0$, and seek the value r of x which satisfies the condition $2x^3+3x^2-6>0 \therefore r=2$; and since for $r=1$ a change of sign takes place in the final term, we infer that the positive root of the changed equation lies between 1 and 2, $\therefore$ the leading figure of the negative root of the proposed is -1.

The foregoing process for finding the real roots of numerical equations was first proposed by Mr. Horner, and is known by the name of "Horner's Method." But for a more ample discussion of the general theory and solution of Equations, the inquiring student is referred to the Author's octavo work on that subject, as also to his smaller volume on "The Analysis and Solution of Cubic and Biquadratic Equations."

(6) Required the character of the roots of $x^4-4x^3+8x^2-16x+20=0$. The rule of Newton applied to this equation detects the existence of two imaginary roots, and two only: but employing the criterion at (175), we find, since $(4\times 8-4^2)20>16^2$, that *all* the roots are imaginary.

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will be given hereafter. (See the subsequent article, 183.) But the remarks and directions above apply generally.

(7) Required the nature of the roots of $x^5-3x^4+2x^3+3x^2-2x+2=0$.

This equation being of an odd degree, with its last term positive, has one negative root (140); the remaining roots, if real, are positive (164).

Newton's rule, applied to the last three coefficients, detects the existence of two imaginary roots. But writing the equation thus $(x^2-3x+2)x^3+(3x^2-2x+2)=0$, we know (176) that no real positive roots can exist in the intervals within which x^2-3x+2 is *plus;* and if a_1, a_2, be the roots of $x^2-3x+2=0$, the first member will be plus throughout the intervals $[0, a_1]$ and $[a_2, \infty]$.

These roots are $a_1=1$, $a_2=2$; and transforming by 1, as in the margin, we see that the indications of positive roots all lie in the interval $[0, 1]$: hence the equation has one real negative root and four imaginary roots.

1	−3	2	3	−2	2(1
	1	−2	0	3	1
	−	−	−	−	−
	−2	0	3	1	3
	1	−1	−1	2	
	−	−	−	−	
	−1	−1	2	3	
	1	0	−1		
	−	−	−		
	0	−1	1		
	1	1			
	−	−			
	1	0			
	1				
	−				
	2				

(8) Required the nature of the roots of $x^3+0x^2-4x+6=0$.

This equation has one real negative root (140): the other roots, if real, are positive (164). Adding and subtracting x^2, to bring about the condition (176), we have $(x-1)x^2+(x^2-4x+6)=0$; and since the first member is positive throughout the interval $[1, \infty]$, and from the results in the margin, it is in this interval that the indications of positive roots lie, seeing that none lie in the interval $[0, 1]$, we at once conclude the roots indicated in the positive region to be imaginary.

1	0	−4	6(1
	1	1	−3
	−	−	−
	1	−3	3
	1	2	
	−	−	
	2	−1	
	1		
	−		
	3		

(9) Required the nature of the roots of $2x^4-3x^3+6x-8=0$.

Here we know (140) that there are two real roots with opposite signs, and that (164) the remaining roots, if real, are positive. Subtracting and adding $2x^2$, we have $(2x^2-3x+2)x^2-(2x^2-6x+8)=0$, where each term is positive for every value of x (138). Now from $x=0$ to $x=1$, it is plain that $(2x^2-6x+8)>(2x^2-3x+2)$, or $(8-6x)>(2-3x)$, $\therefore$ within this interval, $(2x^2-6x+8)>(2x^2-3x+2)x^2$; hence no roots exist in the interval $[0, 1]$, (176). But proceeding as in the margin, we see that two variations are lost in this interval. Consequently the two remaining roots are imaginary. (Ex. 5, p. 131.)

2	−3	0	6	−8(1
	2	−1	−1	5
	−	−	−	−
	−1	−1	5	+
	2	1	0	
	−	−	−	
	1	0	5	

(10) Required the nature of the roots of $x^3+8x^2+6x-75\cdot9=0$.

Changing alternate signs, the equation may be written (adding and subtracting $10x$), $(x^2-8x+16)x+(75\cdot9-10x)=0$, which is positive for every value of x from $x=0$ to $x=7$, as we know from regarding the second term only. Hence no root exists between these limits. But transforming by 7, as in the margin, we see that the two variations are lost, $\therefore$ the equation has two imaginary roots. (Ex. 1, p. 125.)

1	−8	6	75·9(7
	7	−7	−7
	−	−	−
	−1	−1	+
	7	42	
	−	−	
	6	+	

179. Examples for Exercise.

Required a positive root of each of the following equations by Horner's method:—

(1) $3x^2+22x-102=0$.
(2) $17x^2-23x-45=0$.
(3) $x^3-21x+21=0$.
(4) $x^3+x-13=0$.
(5) $x^3+5x^2+3x-48=0$.
(6) $x^4+x^2-8x-15=0$.
(7) $2x^4-16x^3+40x^2-30x+1=0$.
(8) $x^3-673373097125^*=0$.
(9) The cube root of 9 to nine decimals.
(10) $x^4-3x^2+75x-10000=0$.
(11) $x^4-4x^3-4x^2-11x+4=0$.
(12) $25x^4-298x^3+576x^2-281x-26=0$.

Required a negative root of each of the following equations:—

(13) $x^3-5x^2+3x+48=0$.
(14) $x^3-21x+21=0$.
(15) $x^4-12x^2+12x-3=0$.
(16) $x^4-2x^3+3x-20=0$.
(17) $x^4+x^3-8x-15=0$.
(18) $4x^3+240x^2+3996x+14937=0$.

---

(19) Find all the roots of $x^3-49x^2+658x-1379=0$.
(20) Find all the roots of $x^4-12x^2+12x-3=0$.
(21) Required the character of the roots of $x^4-4x^3+7x^2-4x+1=0$.
(22) Required the nature of the roots of $x^4-4x^3-3x+23=0$.
(23) Required the nature of the roots of $x^4-2x^3+2x^2-x+3=0$.
(24) Required the nature of the roots of $x^5+x^4+x^3-2x^2+2x-1=0$.

Note.—The student must be careful to remember that the non-detection of imaginary roots, by any of the methods now given, does not necessarily imply their non-existence: whenever the character of the roots still remains doubtful, we have only to proceed to develope them on the supposition that they are real, when all doubt will be speedily removed, as noticed at page 131.

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180. **Newton's Method of Approximation.**—The method of Newton, like that which has just been exolained, requires that the leading figure of the root to be approached to, be previously found; and not only this, but also that the interval within which it lies be so narrowed, by preparatory transformations, that the extreme limits of that interval may not differ from one another by more than the small fraction $\frac{1}{10}$ or ·1, in order that each limit may be within this small amount of the value of the root. Calling the partial value of x thus obtained r, and the remaining portion of the root x', we shall have $x=r+x'$, and therefore (161)

$$f(x)=f(r+x')=f(r)+f_1(r)x'+\frac{f_2(r)}{2}x'^2+\frac{f_3(r)}{2.3}x'^3+\ldots=0,$$

where, since x' is less than ·1, x'^2 must be less than ·01, x'^3 less than ·001, &c.; hence, rejecting the terms into which these small factors enter, we have for a first approximation to the correct value of x' the

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* This is simply to find the cube root of the above number. We imagine the wanting terms $0x^2+0x$ to be supplied, write the top row 1 0 0 673373097125, pointing off every three figures—as in the common method—and then proceed with the leading figure 8, of the root, as in Horner's process.

expression $x'=-\frac{f(r)}{f_1(r)}$, which will usually give the value true to two places of decimals. Adding, therefore, this correction to r, we obtain a nearer value r' to the root x, the error being usually below ·01. We now have $x=r'+x''$, and proceeding as before, $x''=-\frac{f(r')}{f_1(r')}$, which will usually give x'' true to four decimal places. And continually repeating the operation, the approximation to the true value of x may be pushed to any extent.

Newton illustrated his method by only one example, certainly a most favourable one for the purpose: this is it, namely, $x^3-2x-5=0$; so that we have

$$f(x)=x^3-2x-5 \therefore f_1(x)=3x^2-2 \ldots [1].$$

A root of $f(x)=0$ lies between 2 and 3: to narrow these limits, transform successively by 2, 2·1, 2·2, &c.: the root is thus found to lie between 2 and 2·1, so that 2·0 is the preparatory approximation, $\therefore x=2\cdot0+x'$.

$$\therefore x'=-\frac{f(2)}{f_1(2)}=-\frac{-1}{10}=\cdot1 \therefore x=2\cdot1+x'' \therefore x''=-\frac{f(2\cdot1)}{f_1(2\cdot1)}=-\frac{\cdot061}{11\cdot23}=-\cdot0054$$

$$\therefore x=2\cdot0946+x''' \therefore x'''=-\frac{f(2\cdot0946)}{f_1(2\cdot0946)}=-\frac{\cdot0005416}{11\cdot16205}=-\cdot00004852$$

$$\therefore x=2\cdot09455148.$$

Even in this, one of the most simple examples that could be chosen, the process of Newton, requiring the successive substitutions of r, r', r'', &c. in [1], becomes very irksome. It is entirely superseded by the elegant and effective method of Horner. It will be seen that each of Newton's divisors $f_1(r)$, $f_1(r')$, &c., is Horner's *trial divisor*, which, by the continuous series of uniform operations peculiar to his mode of approximation, becomes converted into the *true divisor*, by aid of previously-computed results.

181. **Approximation by Double Position.**—Let a be one of the real roots of an equation, and let $a+x'$, $a+x''$, be two values near to that root, either both less, both greater, or one less and the other greater. Then, transposing the absolute number N to one side, and writing the equation thus, $f(x)=N$, we shall have

$$f(a)=N, \text{and (161)}, f(a+x')=f(a)+f_1(a)x'+\frac{f_2(a)}{2}x'^2+\ldots$$

$$f(a+x'')=f(a)+f_1(a)x''+\frac{f_2(a)}{2}x''^2+\ldots$$

Now supposing that the errors x', x'' are so small that the squares and higher powers of them may be disregarded, we have

$$f(a+x')-f(a+x'')=(x-x'')f_1(a) \therefore f_1(a)=\frac{f(a+x')-f(a+x'')}{x'-x''}.$$

$$\text{But } f_1(a)=\frac{f(a+x')-f(a)}{x'}, \text{ or } =\frac{f(a+x'')-f(a)}{x''} : \text{ therefore}$$

$$\frac{f(a+x')-f(a+x'')}{x'-x''}=\frac{f(a+x')-f(a)}{x'}=\frac{f(a+x'')-f(a)}{x''},$$

where, although x' and x'' are unknown, yet $x'-x''$ is known, being the difference of the suppositions, namely, $(a+x')-(a+x'')$. Hence this Rule. Transpose the absolute number N to one side of the equation, and find by trials two numbers, which, when separately substituted for x,

give results each very near to N. Then, as the difference of these results is to the difference of the suppositions, so is the difference of the true result N and either of the former, to the error involved in the corresponding supposition.

Correcting for this error we shall have a first approximation, which substitute for one of the suppositions, and with this and the other supposition, or with a new supposition still nearer the truth, proceed as before; and so on till an approximation sufficiently near is obtained. Ex.: A value of x in the equation $x^3+3x^2=500$ is found by trials to lie between 7 and 7·1. Making, then, these substitutions, we find the approximate errors thus:—

$x=7 \therefore x^3=343$		$x=7{\cdot}1 \therefore x^3=357{\cdot}911$
$3x^2=147$		$3x^2=151{\cdot}23$
490.....	Results	509·141
509·141		$N=500$
Diff. of results 19·141		9·141

$$\therefore 19{\cdot}141 : {\cdot}1 :: 9{\cdot}141 : {\cdot}047 = \text{error of } 7{\cdot}1$$
$$\therefore x = 7{\cdot}1 - {\cdot}047 = 7{\cdot}053.$$

Again: $x=7{\cdot}05 \therefore x^3=350{\cdot}4026$, $x=7{\cdot}06 \therefore x^3=351{\cdot}8958$

$3x^2=149{\cdot}1075$		$3x^2=149{\cdot}5308$
499·5101.....	Results.....	501·4266
$N=500$		499·5101
·4899	Diff.	1·9165

$$\therefore 1{\cdot}9165 : {\cdot}01 :: {\cdot}4899 : {\cdot}00255 = \text{error of } 7{\cdot}05$$
$$\therefore x = 7{\cdot}05 + {\cdot}00255 = 7{\cdot}05255.$$

As an example for exercise, take the equation $x^3+x^2+x=100$, the positive root of which lies between 4 and 5. By the above rule it will be found that $x=4{\cdot}26443$ very nearly.

NOTE.—The preceding method may be applied with success to the approximate solution of any equation, whether its terms be rational or irrational, or even exponential, provided only that we can find two near values of the unknown to begin with.

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182. **Cardan's Method for Cubics.**—The foregoing methods apply exclusively to numerical equations, and we see that they are perfectly general. But it is far otherwise with equations whose coefficients are letters instead of numbers; not only is the general solution of such equations beyond the powers of algebra, but even literal equations of so low a degree as the third or fourth have hitherto proved unmanageable, except under certain restrictions. The quadratic, or equation of the second degree, is the only literal equation, after a simple equation, of which algebra, in its present state, can furnish a perfectly general and satisfactory solution. We shall here investigate the formula at which Cardan arrived for the general solution of a cubic. Let the general cubic equation $A_3x^3+A_2x^2+A_1x+N=0$ be deprived of its second term by (145), and reduced to the form $x^3+px+q=0$....[1]. For x substitute $y+z \therefore y^3+z^3+3yz(y+z)+p(y+z)+q=0$....[2]. Assume $3yz=-p$, which we are at liberty to do, since two unknowns (y, z) may always be

made to satisfy two conditions; the two conditions here are $y+z=x$, $3yz=-p$, and *therefore* [2], $y^3+z^3=-q$. From the second of these we have $y^3z^3=-\frac{p^3}{27}$; hence, taking this in conjunction with the third, we have the sum of two quantities, y^3+z^3, and their product y^3z^3, given to determine those quantities. Now we know (68) that the quantities sought will be the roots of the quadratic $v^2+qv-\frac{p^3}{27}=0$....[3]. Solving, therefore, this equation, we have

$$y^3=-\frac{q}{2}+\sqrt{\left\{\frac{q^2}{4}+\frac{p^3}{27}\right\}},\ z^3=-\frac{q}{2}-\sqrt{\left\{\frac{q^2}{4}+\frac{p^3}{27}\right\}},$$

therefore since $x=y+z=y-\frac{p}{3y}$, we have, by taking the cube root,

$$x=\left[-\frac{q}{2}+\sqrt{\left\{\frac{q^2}{4}+\frac{p^3}{27}\right\}}\right]^{\frac{1}{3}}-\frac{\frac{1}{3}p}{\left[-\frac{q}{2}+\sqrt{\left\{\frac{q^2}{4}+\frac{p^3}{27}\right\}}\right]^{\frac{1}{3}}},$$

which is called "Cardan's formula," although, strictly speaking, it is due to Tartalea. This formula, like the general formula for the solution of a quadratic, has the appearance of presenting the three roots (for every cube root has *three* values, see p. 105) as a finite function of the coefficients of the equation. But if it should happen that those coefficients are so related that $\frac{q^2}{4}+\frac{p^3}{27}<0$...[4], each of the two expressions which make up the value of x will be imaginary; so that, for such values of p and q, it will be impossible to compute a real value for x— and *one* real root a cubic must always have —, except by developing each expression by the binomial theorem in an infinite series, and expunging the imaginaries, which, entering both with opposite signs, destroy one another.

That all the three roots are real when the condition [4] is satisfied, may be proved thus:—

Let a be one real root, and let the other roots be $b+\sqrt{c}$, $b-\sqrt{c}$; these will be real if c is positive, and imaginary if c is negative. The quadratic involving the two latter roots will be $x^2-2bx+b^2-c=0$, and since the third root a is such as to cause the second term in the resulting cubic to vanish, the factor by which the quadratic must be multiplied to produce that cubic will be $x+2b$, for a must be equal to $-2b$. The product being $x^3-(3b^2+c)x+2b^3-2bc=0$, we must have

$$p=-(3b^2+c),\ q=2b^3-2bc\ \therefore\ \left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3=(b^3-bc)^2-\left(b^2+\frac{c}{3}\right)^3=-3b^4c+\frac{2}{3}b^2c^2-$$

$$\left(\frac{c}{3}\right)^3=\left(9b^4-2b^2c+\frac{c^2}{3^2}\right)\times-\frac{c}{3}=\left(3b^2-\frac{c}{3}\right)^2\times-\frac{c}{3},$$

which is necessarily negative when c is positive; that is, when all the roots of the cubic are *real*, and positive when c is negative; that is, when two of the roots are *imaginary*.

When $\frac{q^2}{4}+\frac{p^3}{27}=0$, then evidently c must be 0, and conversely; hence, in this case, two of the roots must be real and equal, and the formula of Cardan would determine them; but when the roots are real and unequal the formula is said to be in the *irreducible case*, and for practical purposes is useless.

183. Nevertheless, the investigation just given is not barren of noteworthy results, for we learn from it the following interesting THEOREM. In the cubic equation $x^3+px+q=0$; whatever be the signs of p and q, the following are sure and complete tests of the character of the roots:—

1. If $\left(-\frac{p}{3}\right)^3>\left(\frac{q}{2}\right)^2$, the roots are all real and unequal.

2. If $\left(-\frac{p}{3}\right)^3<\left(\frac{q}{2}\right)^2$, two of the roots are imaginary.

3. If $\left(-\frac{p}{3}\right)^3=\left(\frac{q}{2}\right)^2$, the roots are real, and two of them are equal.

The student is recommended to apply these tests to the cubic equations already discussed, after freeing each from its second term, and dividing by the leading coefficient. Whenever p is positive, there is no occasion to refer to these tests, since, by the rule of De Gua (170), two roots are then necessarily imaginary. As an example, let us take the equation $x^3-3x+6=0$, solved at page 128. Since $\left(\frac{3}{3}\right)^3<\left(\frac{6}{2}\right)^2$, two roots are imaginary: applying, therefore, Cardan's formula, we have

$$x=[-3+\sqrt{\{9-1\}}]^{\frac{1}{3}}+\frac{1}{[-3+\sqrt{\{9-1\}}]^{\frac{1}{3}}}=(-3+2\sqrt{2})^{\frac{1}{3}}+\frac{1}{(-3+2\sqrt{2})^{\frac{1}{3}}}$$

$$=-\cdot555\ldots+\frac{1}{-\cdot555\ldots}=-2\cdot355\ldots, \text{ which is the real root.}$$

The two imaginary cube roots of unity are $\frac{-1+\sqrt{-3}}{2}$, and $\frac{-1-\sqrt{-3}}{2}$, (ex. 15, p. 112), and each is the reciprocal of the other (147);* therefore, introducing each of these separately, as factors of $-\cdot555\ldots$, the two imaginary roots are

$$\frac{1}{2}\left(\cdot555\ldots+\frac{1}{\cdot555\ldots}\right)\pm\frac{1}{2}\left(\cdot555\ldots-\frac{1}{\cdot555\ldots}\right)\sqrt{-3}.$$

184. From what has now been shown in reference to Cardan's formula, it is evident that it cannot be considered as a general solution of an algebraic cubic. Every general solution of an algebraic, or literal equation—to deserve that designation—while it exhibits the several roots, as functions of the literal coefficients, should also be competent to supply, in a finite form, the numerical values of those roots when the letters are replaced by numbers. We have brought it under the notice of the student, partly on account of its celebrity, and because a more complete solution of a general cubic has never been discovered; but chiefly because of the criteria given above, to which the investigation has conducted us, and which criteria preceding writers have usually passed over without directing that special attention to them which they certainly deserve. Up to the present point, the only numerical equation, the character of whose

* Moreover, either one of these imaginaries is the square of the other, so that if we call the *real* cube root of any number $N^{\frac{1}{3}}$, and put α for either of the two imaginary values of $\sqrt[3]{1}$, all three of the roots of $\sqrt[3]{N}$ will be expressed by either $N^{\frac{1}{3}}$, $\alpha N^{\frac{1}{3}}$, $\frac{1}{\alpha}N^{\frac{1}{3}}$, or $N^{\frac{1}{3}}$, $\alpha N^{\frac{1}{3}}$, $\alpha^2 N^{\frac{1}{3}}$. This is obvious, because if $\alpha^3=1$, then the third power of either of these expressions must be N.

roots may always be ascertained, without any tentative operations, is the quadratic: we may now add the cubic.

Formulæ for the general solution of a biquadratic have also been proposed by several of the older algebraists, Ferrari, Descartes, Euler, Simpson, and others; but as they all presuppose the general solution of the cubic, they cannot be regarded as fully effecting the object sought. It would be little better than waste of time to enter upon these early attempts to accomplish for *literal*, what Horner has so satisfactorily effected for *numerical* equations; since the roots of the former class of equations are available in practice only when numerical values are given to the letters.*

It remains for us now to present to the student a brief view of a method of dealing with certain biquadratic equations, which will often be found of successful application. It differs from all the foregoing processes in this peculiarity:—the object sought is, not at once to develope the roots, but to decompose the equation into two quadratics, without the aid of any subsidiary equation.

185. **Decomposition of a Biquadratic into a Pair of Quadratics.**—The way of doing this, when practicable, is to extract the square root of the polynomial, conducting the operation with this aim, namely, to secure a *remainder* which shall be either a monomial of the form mx^2, or a mere number, or else a trinomial square. If the quadratic remainder, instead of being a trinomial square, should have four times the product of its extreme terms—these being positive—greater than the square of the middle term, we may conclude that all the roots are imaginary. The following are examples of this mode of decomposition:—

(1)

$$x^4+4x^3-3x^2-8x+4=0$$

$$\begin{array}{l|l} x^4+4x^3-3x^2-8x+4 & x^2+2x-2 \\ x^4 & \\ \hline 2x^2+2x) & 4x^3-3x^2 \\ & 4x^3+4x^2 \\ \hline 2x^2+4x-2) & -7x^2-8x+4 \\ & -4x^2-8x+4 \\ & \hline & -3x^2 \end{array}$$

The number -2 is suggested for the final term of the root, as it is seen to be the number which, when applied to the divisor, causes the terms $-8x+4$ to disappear from the subsequent remainder. The two component quadratic factors of the polynomial are thus found to be

$$\{(x^2+2x-2)+\sqrt{3}.x\}\{x^2+2x-2)-\sqrt{3}.x\}$$

and consequently the two quadratics themselves are

$$x^2+(2+\sqrt{3})x-2=0 \text{ and } x^2+(2-\sqrt{3})x-2=0.$$

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* Able and interesting investigations, respecting literal equations of the higher degrees, will be found in Mr. Jerrard's "Researches," and in various papers by Mr. Cockle, and Mr. Harley, in the "Philosophical Magazine," and in the "Manchester Memoirs." The advanced student is also referred to the writings on these subjects of Sir W. R. Hamilton and M. Abel: those of the former distinguished analyst, in the "Transactions of the Royal Irish Acadamy," and those of the latter, in his "*Œuvres Complètes.*"

(2) $x^4+2x^3+2x^2-6x-15=0$

$$x^4+2x^3+2x^2-6x-15(x^2+x+1$$
$$\underline{x^4}$$
$$2x^2+x) \quad 2x^3+2x^2$$
$$\underline{2x^3+\ x^2}$$
$$2x^2+2x+1) \quad x^2-6x-15$$
$$\underline{2x^2+2x+\ 1}$$
$$-x^2-8x-16=-(x+4)^2$$

The polynomial is therefore $(x^2+x+1)^2-(x+4)^2$, so that the equation is

$$\{(x^2+x+1)+(x+4)\}\{(x^2+x+1)-(x+4)\}=0$$

and the two component quadratics are $x^2+2x+5=0$ and $x^2-3=0$.

(3) $x^4+2x^3+3x^2+x+3=0(x^2+x$

$$\underline{x^4}$$
$$2x^2+x) \quad 2x^3+3x^2$$
$$\underline{2x^3+\ x^2}$$
$$2x^2+x+3$$

As four times 6 exceeds 1^2, the roots of the equa. are all imaginary, for the equa. is

$$(x^2+x)^2+(2x^2+x+3)=0,$$

where each of the terms forming the first member is always positive, and the sum of two real *positive* quantities cannot be zero.

(4) $x^3+7x^2+11x-4=0$

Multiply by x:—

$$x^4+7x^3+11x^2-4x\Big(x^2+\frac{7}{2}x-\frac{1}{2}$$
$$\underline{x^4}$$
$$2x^2+\frac{7}{2}x\Big) \quad 7x^3+11x^2$$
$$\underline{7x^3+\frac{49}{4}x^2}$$
$$2x^2+7x-\frac{1}{2}\Big) \quad -\frac{5}{4}x^2-4x$$
$$\underline{-x^2-\frac{7}{2}x+\frac{1}{4}}$$
$$-\frac{1}{4}x^2-\frac{1}{2}x-\frac{1}{4}=-\Big(\frac{1}{2}x+\frac{1}{2}\Big)^2$$

Hence the polynomial of the fourth degree is

$$\Big\{\Big(x^2-\frac{7}{2}x-\frac{1}{2}\Big)+\Big(\frac{1}{2}x+\frac{1}{2}\Big)\Big\}\Big\{x^2+\frac{7}{2}x-\frac{1}{2}\Big)-\Big(\frac{1}{2}x+\frac{1}{2}\Big)\Big\}$$

that is, $(x^2+4x)(x^2+3x-1)$, and therefore the proposed cubic is $(x+4)(x^2+3x-1)=0$.

In the foregoing example, the proposed equation is multiplied by x, in order to render the leading term a complete square; and for a similar purpose, the equation in the next example following is multiplied by $2x$. When the factors of the first member, thus multiplied, are discovered, the extraneous multiplier is expunged from their product.

(5) $2x^3+8x^2+9x+9=0$

Multiply by $2x$:—

$$4x^4+16x^3+18x^2+18x\Big(2x^2+4x+\frac{3}{2}$$

$$\underline{4x^4}$$

$$4x^2+4x\Big)\quad 16x^3+18x^2$$

$$\underline{16x^3+16x^2}$$

$$4x^2+8x+\frac{3}{2}\Big)\quad 2x^2+18x$$

$$6x^2+12x+\frac{9}{4}$$

$$\underline{\qquad\qquad}$$

$$-4x^2+\ 6x-\frac{9}{4}=-\Big(2x-\frac{3}{2}\Big)^2$$

$\therefore$ the polynomial of the fourth degree is

$$\Big\{2x^2+4x+\frac{3}{2}+\Big(2x-\frac{3}{2}\Big)\Big\}\Big\{2x^2+4x+\frac{3}{2}-\Big(2x-\frac{3}{2}\Big)\Big\}$$

that is, $(2x^2+6x)(2x^2+2x+3)$, and dividing by $2x$, the cubic is $(x+3)(2x^2+2x+3)=0$.

Examples for Exercise.

(1) Find the four roots of $x^4+4x^3-3x^2-8x+4=0$.
(2) Find all the roots of $x^4-8x^3+15x^2+4x-8=0$.
(3) Show that the roots of $5x^4+4x^3+3x^2-4x+2=0$ are imaginary.
(4) Find the three roots of the cubic $x^3-6x^2+6x+9=0$.
(5) Find all the roots of $x^4-4x^3+7x^2-4x+1=0$.
(6) Resolve $x^4+ax^3+bx^2+cx+\Big(\frac{c}{a}\Big)^2$ into its component quadratic factors.

186. **Recurring Equations.**—Whenever the biquadratic is a recurring equation (147), the above method of decomposition always applies, as the following general example shows:—

$$x^4+ax^3+bx^2+ax+1=0\Big(x^2+\frac{1}{2}ax+1$$

$$\underline{x^4}$$

$$2x^2+\frac{1}{2}ax\Big)\quad ax^3+bx^2$$

$$ax^3+\frac{1}{4}a^2x^2$$

$$\underline{\qquad\qquad}$$

$$2x^2+ax+1\Big)\Big(b-\frac{1}{4}a^2\Big)x^2+ax+1$$

$$\underline{\qquad\qquad 2x^2+ax+1}$$

$$\Big(b-\frac{1}{4}a^2-2\Big)x^2$$

Consequently the first member of the equation is

$$\Big(x^2+\frac{1}{2}ax+1\Big)^2-\Big(\frac{1}{4}a^2-b+2\Big)x^2$$

and therefore the component quadratics are

$$x^2+\Big\{\frac{1}{2}a-\sqrt{\Big(\frac{1}{4}a^2-b+2\Big)}\Big\}x+1=0,\text{ and }x^2+\Big\{\frac{1}{2}a+\sqrt{\Big(\frac{1}{4}a^2-b+2\Big)}\Big\}x+1=0.$$

187. But the usual way of treating recurring equations in general is the following. Since $\left(x^n+\frac{1}{x^n}\right)\left(x+\frac{1}{x}\right)=x^{n+1}+\frac{1}{x^{n+1}}+x^{n-1}+\frac{1}{x^{n-1}}$, by putting z for $1+\frac{1}{x}$ we have $x^{n+1}+\frac{1}{x^{n+1}}=\left(x^n+\frac{1}{x^n}\right)z-\left(x^{n-1}+\frac{1}{x^{n-1}}\right)$, from which general relation we see that from any two consecutive forms $x^{n-1}+\frac{1}{x^{n-1}}$, $x^n+\frac{1}{x^n}$, all that follow may be derived one after the other, thus:—

$$x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)z-\left(x^0+\frac{1}{x^0}\right)=z^2-2$$

$$x^3+\frac{1}{x^3}=\left(x^2+\frac{1}{x^2}\right)z-\left(x+\frac{1}{x}\right)=z^3-3z$$

$$x^4+\frac{1}{x^4}=\left(x^3+\frac{1}{x^3}\right)z-\left(x^2+\frac{1}{x^2}\right)=z^4-4z^2+2$$

$$x^5+\frac{1}{x^5}=\left(x^4+\frac{1}{x^4}\right)z-\left(x^3+\frac{1}{x^3}\right)=z^5-5z^3+5z$$

&c. &c. &c.

which shows that every recurring equation in x may be reduced to another in z of only half its degree if that degree be even. If the degree be odd, we know that one of the roots will be $+1$ or -1, according as the last term is $-$ or $+$ (140); so that eliminating the root $x=1$, or $=-1$, the depressed equation will be a recurring one of an even degree, and this may always be reduced by the above conditions to an equation of half that degree—as in the following example:—

(1) $x^5-11x^4+17x^3+17x^2-11x+1=0$.

Since one root of this is $x=-1$, we divide by $x+1$, and get the coefficients of the depressed biquadratic, as in the margin.

1	−11	17	17	−11	(−1
	− 1	12	−29	12	
−12	29	−12	1		

This biquadratic is therefore $x^4-12x^3+29x^2-12x+1=0$, or dividing by x^2, and bringing the equidistant terms together, it is $\left(x^2+\frac{1}{x^2}\right)-12\left(x+\frac{1}{x}\right)+29=0$; so that putting z for $x+\frac{1}{x}$, we have $(z^2-2)-12z+29=0$, or $z^2-12z+27=0$, which is only half the degree of the proposed. The roots of it are $z=9$, and $z=3$, that is, $x+\frac{1}{x}=9$, and $x+\frac{1}{x}=3$; and thus the two quadratics for determining the four remaining roots of the proposed equation, are $x^2-9x=-1$, and $x^2-3x=-1$, of which the roots are $\frac{9\pm\sqrt{77}}{2}$, and $\frac{3\pm5}{2}$, where one of each pair of roots is the reciprocal of the other of that pair, for the four roots may be written thus: $\frac{9+\sqrt{77}}{2}$, $\frac{2}{9+\sqrt{77}}$; $\frac{3+\sqrt{5}}{2}$, $\frac{2}{3+\sqrt{5}}$, as will at once be seen by rationalizing the denominators. The preceding quadratics are

of course the same as those at (186), when -6 is put for $\frac{1}{2}a$, and 29 for b.

188. It may be here noticed that the recurring biquadratic, when put in the form $\left(x^2+\frac{1}{x^2}\right)+a\left(x+\frac{1}{x}\right)+b=0$, is lowered in degree by one-half by simply adding 2 to the first term, and then subtracting it from the last, that is, by writing the equation thus, $\left(x^2+2+\frac{1}{x^2}\right)+a\left(x+\frac{1}{x}\right)+b-2=0$, which is $\left(x+\frac{1}{x}\right)^2+a\left(x+\frac{1}{x}\right)+b-2=0$, or $z^2+az+b-2=0$. And if the form be $\left(x^2-\frac{1}{x^2}\right)+a\left(x-\frac{1}{x}\right)+b=0$, we effect a similar reduction by subtracting 2 from the first term and adding it to the last; for we shall then get $\left(x-\frac{1}{x}\right)^2+a\left(x-\frac{1}{x}\right)+b+2=0$; but here the original is not a reciprocal equation, since it does not remain the same when $\frac{1}{x}$ is put for x.

In the general forms at (187), it is assumed that the signs of the terms paired together are *like;* but they still answer every requisite purpose: for when the equation is of an odd degree, with its last term negative, we may divide by $x-1$, and thus get a reduced reciprocal equation of an even degree with its last term positive: and if the equation be of an even degree, with its last term negative, we may divide by x^2-1 (150), and thus also get a reduced reciprocal equation of even degree with its last term positive. The extreme terms will thus have like signs; and in a reciprocal equation, if the extremes have like signs, all the intermediate pairs must have like signs (149).

189. **Binomial Equations.**—A binomial equation is an equation of two terms of the form $y^n \pm a^n=0$, where a^n is a known quantity. By putting ax for y, it becomes $a^n x^n \pm a^n=0$, or, more simply, $x^n \pm 1=0$, to which form, therefore, every binomial equation may be reduced. And it is plain that, after this reduction, the equation is a reciprocal equation in which all the terms between the first and last are so many zeros.

THEOREM 1. If α be one of the imaginary roots of $x^n-1=0$, then every power of α will also be a root.

For since α is a root, therefore $\alpha^n=1$. $\therefore \alpha^{2n}=1$, $\alpha^{3n}=1$, &c.; also $\alpha^{-n}=1$, $\alpha^{-2n}=1$, $\alpha^{-3n}=1$, &c. Consequently α, α^2, α^3, ..., α^{-1}, α^{-2}, α^{-3}, ..., all satisfy the conditions of the equation, and are therefore roots of it. The roots may, therefore, be exhibited under an infinite variety of forms, but of course only n of them can be essentially different. No two of the roots can be equal, because the limiting equation $nx^{n-1}=0$ has no divisor in common with $x^n \pm 1=0$.

THEOREM 2. If α be one of the imaginary roots of $x^n+1=0$, then every *odd* power of α will also be a root.

For since α is a root, therefore $\alpha^n=-1$, and every odd power of -1, is also -1. Consequently α, α^3, α^5, ..., α^{-1}, α^{-3}, α^{-5}, ..., all satisfy the

equation, and are therefore roots of it, though no more than n of them can be essentially different.

It is plain that if n be even, the equation $x^n-1=0$, or $x^n=1$, has but two real roots, namely, $+1$ and -1; since of no other real values can an even power be 1.

If n be odd, the equation $x^n-1=0$ has but one real root, namely, $+1$, this being the only real value the odd powers of which produce 1.

If n be even, the equation $x^n+1=0$, or $x^n=-1$ has no real root, since no even power of a real value can produce -1.

If n be odd, the equation $x^n+1=0$, or $x^n=-1$ has one real root, and one only, namely, -1; this being the only real value of which an odd power is -1.

The actual determination of the imaginary roots of a binomial equation is best effected by the formulæ of TRIGONOMETRY.

EXAMPLES FOR EXERCISE.

(1) Required the four roots of the reciprocal equation $x^4+x^3-4x^2+x+1=0$.

(2) Required the four roots of the recurring equation $26x^4-85x^3+102x^2-85x+26=0$.

(3) Required the six roots of the recurring equation $4x^6-24x^5+57x^4-73x^3+57x^2-24x+4=0$.

(4) Required the three roots of the binomial equation $x^3+1=0$.

(5) Required the four roots of $x^4+1=0$, or $x^2+\frac{1}{x^2}=0$, or $x^2+2+\frac{1}{x^2}=2$.

(6) Required the five roots of the equation $x^5+x^4-x-1=0$.

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190. **Vanishing Fractions.**—It sometimes happens, upon putting a particular value for x in a fraction involving this general symbol, that both numerator and denominator vanish, the fraction, in this particular case, assuming the form $\frac{0}{0}$. This is called a *vanishing fraction*, and can arise only from num. and denom. of the general fraction having a common factor which becomes zero for the particular value of x mentioned. We shall here show how such common factors may be eliminated, and thence the value of the vanishing fraction determined.

Let the general fraction be $\frac{F(x)}{f(x)}$, the numerator and denominator being rational polynomials, or capable of being developed into such. If for the same value a of x, we have $F(x)=0$, and $f(x)=0$, we shall know that a is a root common to both of these equations, and therefore that each of the polynomials must be divisible by $x-a$. If the division of num. and denom. by this were actually performed, the result would be an equivalent fraction free from the vanishing factor $x-a$. There might, however, still remain a second factor $x-a$, equally evanescent for $x=a$: a second division by $x-a$ would then be necessary to free the fraction from vanishing factors, and so on. But when all are removed, the substitution of a for x in the reduced fraction—the general equivalent of the original—would be a particular fraction in the ordinary form.

Now, the successive divisions here alluded to may be avoided thus: we

have proved (134), that if a be put for x in the quotient $F(x)\div(x-a)$, the result will be $F_1(a)$; and that if a be put for x in the quotient $f(x)\div(x-a)$, the result will be $f_1(a)$.

Similarly, for the same substitution of a for x, $F_1(x)\div(x-a)$ becomes $F_2(a)$, and $f_1(x)\div(x-a)$ becomes $f_2(a)$, and so on. Consequently we have only to replace the fraction $\frac{F(x)}{f(x)}$ by $\frac{F_1(x)}{f_1(x)}$, $\frac{F_2(x)}{f_2(x)}$, &c., till we arrive at a fraction which, upon putting a for the x, assumes the ordinary intelligible form. The student will remember that, in the following examples, the notation $F_1(a)$, $F_2(a)$, &c., means what the derived functions $F_1(x)$, $F_2(x)$, &c., become when a is put for x:—

(1) Required the value of $\frac{x^2-a^2}{x-a}$ when $x=a$. Here $\frac{F_1(x)}{f_1(x)}=\frac{2x}{1}$ $\therefore$ $\frac{F_1(a)}{f_1(a)}=\frac{2a}{1}=2a$.

Consequently, in this case, $\frac{0}{0}=2a$.

(2) Required the value of $\frac{2x^3-7x^2+12}{x^3-7x+6}$ when $x=2$. Here $\frac{F_1(x)}{f_1(x)}=\frac{6x^2-14x}{3x^2-7}$

$\therefore$ $\frac{F_1(2)}{f_1(2)}=-\frac{4}{5}$, the value required.

(3) Required the value of $\frac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2}$ when $x=1$. Here

$$\frac{F_1(x)}{f_1(x)}=\frac{n(n+1)x^n-n(n+1)x^{n-1}}{-2(1-x)}, \text{ which for } x=1 \text{ is still } \frac{0}{0}.$$

$$\frac{F_2(x)}{f_2(x)}=\frac{n^2(n+1)x^{n-1}-n(n+1)(n-1)x^{n-2}}{2} \therefore \frac{F_2(1)}{f_2(1)}=\frac{n(n+1)}{2}.$$

Examples for Exercise.

(1) Required the value of $\frac{x^2-1}{x^2+2x-3}$ when $x=1$.

(2) ,, ,, $\frac{x^3-2x^2-x+2}{x^3-7x+6}$ when $x=2$.

(3) ,, ,, $\frac{x^3-5x^2+3x+9}{x^3-x^2-21x+45}$ when $x=3$.

(4) ,, ,, $\frac{b(a-x\sqrt{a})}{a-x^2}$ when $x=\sqrt{a}$.

(5) ,, ,, $\frac{x^3+2x^2-4x-8}{x^3+3x^2-4}$ when $x=-2$.

(6) ,, ,, $\frac{2x^3-5x^2-4x+12}{x^3-12x+16}$ when $x=2$.

When the general symbol x is under a radical sign, the following method may be employed to get the value of the vanishing fraction for $x=a$. Instead of a, substitute $a+h$ for x in numerator and denominator; develope the terms containing $a+h$, by the binomial theorem; expunge from num. and denom. the highest power of h common to all the terms, and then put $h=0$, as in the following examples:—

(1) Required the value of $\frac{\sqrt[4]{(5x-1)}-\sqrt{2}}{x-1}$, when $x=1$. Putting $1+h$ for x, the fraction is changed into $\frac{(4+5h)^{\frac{1}{4}}-2^{\frac{1}{2}}}{h}=\frac{4^{\frac{1}{4}}+\frac{1}{4}.4^{-\frac{3}{4}}5h-\text{ \&c. }-2^{\frac{1}{2}}}{h}=\frac{\frac{1}{4}.4^{-\frac{3}{4}}5h-\text{ \&c.}}{h}$. Divid-

ing num. and denom. by h, and then making $h=0$, we have $\frac{1}{4}.4^{-\frac{3}{4}}5=\frac{5}{4\sqrt[4]{64}}$ for the value of the fraction when $x=1$.

(2) Required the value of $\frac{\sqrt{(a+x)}-\sqrt{2a}}{\sqrt{(a+2x)}-\sqrt{3a}}$, when $x=a$. Putting $a+h$ for x, we have $\frac{(2a+h)^{\frac{1}{2}}-(2a)^{\frac{1}{2}}}{(3a+2h)^{\frac{1}{2}}-(3a)^{\frac{1}{2}}}=\frac{(2a)^{\frac{1}{2}}+\frac{1}{2}(2a)^{-\frac{1}{2}}h-\text{ \&c. }-(2a)^{\frac{1}{2}}}{(3a)^{\frac{1}{2}}+\frac{1}{2}(3a)^{-\frac{1}{2}}2h-\text{ \&c. }-(3a)^{\frac{1}{2}}}$. This, when num. and den. are divided by h, and then h put $=0$, is $\frac{1}{2}\left(\frac{2a}{3a}\right)^{-\frac{1}{2}}=\frac{1}{2}\sqrt{\frac{3}{2}}=\frac{1}{4}\sqrt{6}$, the value of the fraction when $x=a$.

EXAMPLES FOR EXERCISE.

(1) $\frac{\sqrt{(3x+1)}-2}{x-1}$ when $x=1$. (2) $\frac{x-a+\sqrt{(2ax-2a^2)}}{(x^2-a^2)}$, when $x=a$.

(3) $\frac{x^n}{a-\sqrt[n]{(a^n-x^n)}}$, when $x=0$. (4) $\frac{\sqrt{(a^2+ax+x^2)}-\sqrt{(a^2-ax+x^2)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}$, when $x=0$.

191. **Scholium.**—From the foregoing examples we see how a consistent and perfectly intelligible interpretation of $\frac{0}{0}$ may be arrived at when we previously know the general form of the fraction which, for a particular value of x, assumes this appearance. To ask the value of $\frac{0}{0}$, without supplying any information as to the original fraction of which the numerator and denominator have thus disappeared, would be absurd: in *itself*, $\frac{0}{0}$ may mean anything, and is therefore properly regarded as the *symbol of indeterminateness.* It is equally so regarded when two variable or arbitrary quantities, x and y, enter the fraction, and the form arises from putting $x=a$, and $y=b$, provided no such relation between x and y exists as to render one of these hypotheses a necessary consequence of the other.

The zero symbol 0, which, throughout the earlier stages of algebra, all writers agree in regarding as what it really is, *nothing*, is interpreted by many, in the more advanced parts of the subject, in a different sense. "When we write $\frac{1}{0}=\infty$," says a very successful modern author, "we are not to suppose the denominator actually *zero*, but only a very small quantity." The conclusion from this is, that ∞ is not infinite, but only a very large quantity. But if it be really necessary to symbolize a very small quantity and a very large quantity, it would surely be no difficult matter to devise marks expressly for the purpose, without appropriating symbols previously employed for a different purpose. Another writer of repute says, "That mistakes will constantly arise from considering 0 as an absolute quantity, is easily seen: thus, it has been shown that $a^0=1=b^0$, therefore we might say, if 0 is a quantity, that $a=b$." This is strange reasoning: it has been shown that if α be one of the imaginary cube roots of unity that $1^3=1=\alpha^3$, and therefore we might say that $1=\alpha$. We can scarcely imagine a student who knows anything of

algebra to fall into such an error as to infer that, because the same powers of two different things produce the same result, that *therefore* those different things are equal. The symbol a^0 is merely an extension of notation to mean $\frac{a}{a}$, and, of course, $\frac{a}{a}=1=\frac{b}{b}$ justifies the conclusion that $\frac{a}{a}=\frac{b}{b}$; but nothing more.

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192. **Maxima and Minima.**—It is often of importance to discover for what real values of the unknown quantities involved in it, an algebraic expression becomes the greatest or least possible.

For such values of the unknowns, the expression is said to be a *maximum*, or a *minimum*. But a meaning is given to each of these terms, which is still more comprehensive: Let $f(x)$ be any function of x, then if a value r be found for x, such that $f(r)$ is greater than each of the neighbouring values $f(r+\delta)$, $f(r-\delta)$, by however small a quantity, δ, r be increased or diminished, the value $x=r$ is said to render $f(x)$ a maximum; but if $f(r)$ is less than each of these neighbouring values, then $x=r$ is said to render the function a minimum.

It must be observed that a maximum has place for $x=r$, only when for *every other value of* x, between r and $r+\delta$, and between r and $r-\delta$, the function is less than it is for $x=r$; and that a minimum has place for $x=r$, only when for every other value of x, between r and $r+\delta$, and between r and $r-\delta$, the function is greater than it is for $x=r$; but there is no limitation as to the smallness of δ.

A function may therefore have a variety of maxima and minima values: thus, take an ordinary polynomial $f(x)$ with numerical coefficients: for a particular value r, of x, let it become $=u$; then r will be a real root of the equation $f(x)-u=0$. Now if u be such that this equation has two roots, *each* equal to r, then, taking δ so small that $r+\delta$, $r-\delta$ may neither of them reach another root, we know (137) that the substitution of each of these values for x will give numerical results with the same sign, for no real root will have been passed over. If the results be both positive numbers, $f(r+\delta)$, and $f(r-\delta)$ will each be greater than u, and since $f(r)$ is equal to u, we conclude that the value $x=r$ renders the function $f(x)$ a *minimum*: but if the results be both negative numbers, $f(r+\delta)$, and $f(r-\delta)$ will each be less than u, so that the value $x=r$ renders the function $f(x)$ a maximum. It is possible therefore that a minimum value may be numerically greater than a maximum. Again: another value u' may exist for u such that the equation $f(x)-u'=0$ may have two roots each equal to r', in which case the value $x=r'$ will also render the function $f(x)$ a maximum, or a minimum; and so on.

193. **Quadratic Functions.**—When the function is of the second degree, as ax^2+bx+c, whatever be the signs of a, b, c, the determination of the value to be given to x, in order that the expression may be the greatest or least possible, is very easy. Let u represent the greatest or least value of the expression; then, since $ax^2+bx+c-u=0$, we shall merely have to find a value for u such that this equation may have two equal roots:—the corresponding value r of x will render the function the greatest or least possible. For since no root of the equation exists between r and $+\infty$, nor between r and $-\infty$, whatever numbers, other than

the number r, be substituted for x, the results will always have the same sign; that is to say, for all such other values of x, ax^2+bx+c, will be either uniformly greater than u, or uniformly less than u, so that the value $x=r$, which renders the expression equal to u, must be that which makes it either the least or greatest possible.

If therefore we determine x and u from the conditions (68), $x=-\frac{b}{2a}$, and $4(c-u)=b^2$, we shall obtain the value r of x which will cause ax^2+bx+c, or u, to be the greatest or least possible, and shall also find what this maximum or minimum value u is. To ascertain which of the two states r corresponds to, we shall merely have to notice whether for $x=0$ (or indeed for any other value different from $x=r$), the expression $ax^2+bx+c-u$ becomes $+$, or $-$. If the former, u is a minimum: if the latter, it is a maximum; as in the following examples.

(1) What value must be given to x so that x^2-4x+8 may be the least possible? Putting $x^2-4x+8=u \therefore x^2-4x+8-u=0 \therefore x=\frac{4}{2}=2$, and $4(8-u)=16 \therefore u=4$. Substituting this for u, the first member of the equation is x^2-4x+4, which, for $x=0$, is $+4$: it is $\therefore +$ for every value of x, except $x=2$. Consequently this value renders the expression the least possible.

(2) For what value of x is $4x-x^2-3$ the greatest possible? Here $4x-x^2-3-u=0 \therefore x^2-4x+3+u=0 \therefore x=2$. Also $3+u=4 \therefore u=1$. For this value of u, the first member of the original equation is $4x-x^2-4$, which for $x=0$ is negative, it is $\therefore$ always negative, whatever value, other than 2, be put for x: hence $x=2$ renders the expression the greatest possible.

When the equation $f(x)-u=0$ is not a rational quadratic, but becomes one only after certain reductions, the value of u is still found as above from the known condition of equal roots, namely:—the square of the middle coef. minus 4 times the product of the extremes $=0$. And the character of u—as to its being a maximum or a minimum—is ascertained by covering the first member of this condition by the radical sign ($\surd$), and then observing whether this radical quantity becomes imaginary for $u+\delta$, or for $u-\delta$. If for the former, u is a maximum, if for the latter, u is a minimum. This is plain, for the value of x, given by the solution of the quadratic, is of the form $x=r\pm\surd U$, where U involves the u, which is determined in value from the condition $U=0$, as noticed above: if an increase, however small, of this value renders $\surd U$ imaginary, u must have reached a maximum state, and if a diminution, however small, of u renders $\surd U$ imaginary, u must have arrived at a minimum state.

(3) For what value of x is $x+\surd(a^2-x^2)$ a maximum or a minimum? Here $x+\surd(a^2-x^2)-u=0 \therefore 2x^2-2ux+u^2-a^2=0 \therefore x=\frac{1}{2}u$. Also $\surd U=\surd(u^2-2u^2+2a^2)=0 \therefore u^2=2a^2 \therefore u=a\surd 2 \therefore x=\frac{1}{2}a\surd 2$. And it is plain that if any value greater than $2a^2$ be put for u^2, then $\surd U$ will be imaginary: hence for $x=\frac{1}{2}a\surd 2$, the value of the expression, namely, $u=$

$a\sqrt{2}$, is a maximum; and since for *every* greater value $\sqrt{U}$ is imaginary, the value $a\sqrt{2}$ is the greatest possible the expression can have.

(4) For what values of x is $\frac{x^2-14x+44}{8-2x}$ a maximum or a minimum?

Here $\frac{x^2-14x+44}{8-2x}-u=0 \therefore x^2+2(u-7)x+44-8u=0 \therefore x=7-u$. Also $\sqrt{U}=\sqrt{\{(u-7)^2+8u-44\}}=0 \therefore \sqrt{(u^2-6u+5)}=0 \therefore u=1$, or 5. Now for $u=0$, $U=5$: hence for $u=1+\delta$, U must be negative, since the root $u=1$ lies between 0 and $1+\delta$, $\therefore u=1$ is a maximum, the corresponding value of x being $x=7-u=6$. Again: for $u=$a large number (100 say), U is positive; hence for $5-\delta$, it is negative, since 5 lies between 100 and $5-\delta$; $\therefore u=5$ is a minimum, the corresponding value of x being $7-u=2$.

(5) For what values of x is $\frac{2x^2-10x+8}{2x^2-2x+1}$ a max. or a min.? Here

$$2x^2-10x+8-2ux^2+2ux-u=0 \therefore (1-u)x^2+(u-5)x+\frac{8-u}{2}=0$$

$$\therefore x=\frac{u-5}{2(u-1)}. \quad \text{Also } \sqrt{U}=\sqrt{\{u^2-10u+25-2(1-u)(8-u)\}}=$$

$\sqrt{(-u^2+8u+9)}=0 \therefore u=9$, or -1. Now for $u=10$, U is $-$, and therefore for $u=9-\delta$, it must be $+$, and must continue so down to $u=-1$, so that for $u=-1-\delta$, U is $-$. Hence $u=9$ is a maximum, and $u=-1$ is a minimum, the corresponding values of x being $x=\frac{u-5}{2(u-1)}=\frac{1}{4}$, and $\frac{3}{2}$.

NOTE.—If the two roots of $U=0$ be equal, there cannot be either a maximum or a minimum, for U will then preserve the same sign whatever be put for u; and for the same reason neither a maximum nor a minimum can exist if the two roots are imaginary. It is scarcely necessary to observe that when u^2 is absent from U, as in examples (1) and (2) above, the max. or min. may be found in the same way.

194. **Polynomials in General.**—We have seen (161) that if $f(x)$ be any rational and integral polynomial, and that if x be replaced by $x+\delta$, we shall have $f(x+\delta)=f(x)+f_1(x)\delta+\frac{f_2(x)}{2}\delta^2+\frac{f_3(x)}{2.3}\delta^3+\ldots\ldots[A]$, where δ may be either positive or negative. Now if r be a value of x such as to render $f(x)$ a maximum, then a value for δ exists, so small, that for it, and for all values of δ still smaller, we must have $f(x)>f(x+\delta)$, both when δ is positive, and when it is negative. In other words, we must have $f(x+\delta)-f(x)=$a *negative* quantity. But (see Note below) a value so small may be given to δ as to cause $f_1(x)\delta$ to become numerically greater than the sum of all the terms that follow it; for this small value therefore the entire series, after $f(x)$, must amount to a positive quantity for one sign $\pm$ of δ and to a negative quantity for the other. In the case of $f(x)$ a maximum however, we must have always $f(x+\delta)-f(x)=$a negative quantity: hence for such to have place, for any value r of x, $f_1(x)$ must vanish for that value, that is, we must have $f_1(x)=0$, and moreover for the value r of x, determined by this condition, we must further have $\frac{f_2(x)}{2}=$a negative quantity.

By similar reasoning, if r be a value of x such as to render $f(x)$ a

minimum, we must have $f(x+\delta)-f(x)$=a positive quantity; also, as in the former case, we must have $f_1(x)=0$; and $\frac{f_2(x)}{2}$ must be a positive quantity.

Hence—

If $f(x)$=maximum, then $f_1(x)=0$, and $f_2(x)=-$. If $f(x)$=min., then $f_1(x)=0$, and $f_2(x)=+$. When therefore $f(x)$ is a rational and integral polynomial, the following is the mode of proceeding. RULE 1. Find the real roots of $f_1(x)=0$. 2. Substitute each of these in $f_2(x)$: those of them which give a *minus* result will make $f(x)$ a *maximum*: those which give a *plus* result, will make it a *minimum*.*

[NOTE.—That a value so small may be given to δ, as to cause the first term $A\delta$, of the polynomial $A\delta+B\delta^2+C\delta^3+D\delta^4+\ldots\ldots$, to exceed the sum of all the terms after the first, may be proved thus. Divide the polynomial by δ: the quotient is $A+(B+C\delta+D\delta^2+\ldots\ldots)\delta$, and however great may be the numerical values of B, C, D, &c., it is plain that the multiplier δ, without the brackets, may be made so small, that the product, by the quantity within, may become less than any assigned value A: so that for this small value of δ, we must have $A\delta>(B\delta^2+C\delta^3+D\delta^4+\ldots\ldots)$].

(1) Let $f(x)=x^2-4x+8$ $\therefore f_1(x)=2x-4=0$ $\therefore x=2$; also $f_2(x)=+2$. As this is *plus*, $x=2$ makes $f(x)$ a minimum, namely, $f(x)=4$, which is the least possible, and there is no maximum.

(2) Let $f(x)=4x-x^2-3$ $\therefore f_1(x)=4-2x=0$ $\therefore x=2$; also $f_2(x)=-2$, which being *minus*, $x=2$ makes $f(x)$ a maximum, which is the greatest possible, and there is no minimum.

(3) Let $f(x)=x^3-18x^2+96x-20$ $\therefore f_1(x)=3x^2-36x+96=0$, or $x^2-12x+32=0$ $\therefore f_2(x)=2x-12$. The roots of $f_1(x)=0$ are $x=4$ and $x=8$. The first makes $f_2(x)$ *negative*, the second makes it *positive*; hence for $x=4$, $f(x)$ is a *maximum*, namely, 140, and for $x=8$, it is a *minimum*, namely, 108. But these are not the greatest and least values possible, for the equation $f(x)=0$ has all its roots real (183), and each of them makes $f(x)$ *zero*; also, by giving a sufficiently great value to x, we may make $f(x)$ as great as we please.

EXAMPLES FOR EXERCISE.

(1) Divide a given number a into two parts such that their product may be the greatest possible.

(2) Divide a given number a into two parts such that the sum of the squares of those parts shall be the greatest possible.

(3) What two factors of the number a are such that the sum of their squares are the least possible?

(4) What fraction is that which exceeds its square by the greatest number possible?

(5) When is $3x^2-5x+6$ the least, and $6+5x-3x^2$ the greatest possible?

(6) For what value of x is $\frac{x^2-5x+4}{4x^2-12x+9}$ a max. or min?

(7) Does $\frac{4x^2+4x-3}{2x+1}$ admit of a maximum or a minimum?

(8) For what values of x is $2x^3-9x^2+12x$ a maximum and a minimum?

(9) Divide 12 into two parts such that the less multiplied by the square of the greater may be the greatest possible.

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* It would be out of place to enter more deeply into this subject here. See the DIFFERENTIAL CALCULUS.

(10) Find a fraction such that its square diminished by its cube may be the greatest possible.

(11) Given the equation $y^2-xy+x^2=a^2$: determine x so that y may be a maximum.

(12) Solve the equations $ax^3-x^4=$ max., and $\frac{(x+a)^4}{x}=$ min.

(For further applications of the theory of Maxima and Minima, see MENSURATION.)

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195. **Indeterminate Equations.**—An equation is said to be indeterminate when it admits of an unlimited number of solutions, which it will always do when it involves more than one unknown quantity; but if the restriction be imposed that only *integer* values of the unknowns are to be admitted, the number of solutions often becomes limited. We shall here show how the integer solutions of a simple equation with two unknowns may be found whenever they are limited in number.

Prime Number. A prime number is an integer that cannot be decomposed into factors, the only divisor of it being the integer itself and unity. Such are the numbers 3, 5, 7, 11, 13, &c. Two or more integers are said to be *prime to each other* when they do not admit of a common divisor; thus 5, 8, and 9 are all prime to each other. Numbers not prime are called *composite* numbers.

THEOREM I. If a, b be any two numbers prime to each other, then if each of the terms of the series b, $2b$, $3b$, $4b$,.....$(a-1)b$, be divided by a, the resulting remainders will all be different. For let it be supposed that some two of the above terms, as mb, nb, leave the same remainder, r. Let the respective quotients be q, q', then we must have $qa+r=mb$, and $q'a+r=nb$ $\therefore$ $a(q-q')=b(m-n)$, so that $b(m-n)\div a$ is an integer. But neither b nor $m-n$ is divisible by a since the b is prime to a, and $m-n$ less than a, for m, n are both less; hence $b(m-n)\div a$ cannot be an integer, and $\therefore$ the supposition of two equal remainders is inadmissible, and the following inferences are established:—

1. Since the remainders are all different, and are $a-1$ in number, they must include all integer numbers from 1 to $a-1$.

2. Therefore some one of the remainders must be 1, so that some integer x, less than a, may be found that will make $bx-1$ exactly divisible by a; in other words, the equation $bx-ay=1$ is always possible in positive integers, when a, b are prime to each other. But if a, b be not prime to each other the equation will be impossible in integers; for, in this case, a, b, having a common measure, one side of the equation $bx-ay=1$ would be divisible by it, and the other not.

3. Since $bx-ay=1$ is always possible in positive integers, it follows that $ay-bx=-1$ is equally possible $\therefore$ $ax-by=\pm 1$ is always possible in positive integers when a, b are prime to each other.

THEOREM II. If a, b be prime to each other, the equation $ax-by=\pm c$ will admit of an unlimited number of solutions in positive integers. For since (Cor. 3) $ax'-by'=\pm 1$ is possible $\therefore$ $acx'-bcy'=\pm c$ is also possible; or, putting x for cx', and y for cy', the equation $ax-by=\pm c$ is possible. Suppose one solution to be $x=p$, $y=q$, then $ap-bq=ax-by$, or $ax-ap=by-bq$ $\therefore$ $\frac{x-p}{y-q}=\frac{b}{a}=\frac{mb}{ma}$, or $x-p=mb$, and $y-q=ma$ $\therefore$ $x=p+mb$, and $y=q+ma$. And since m may be any integer whatever, the number of positive integer values of x and y is unlimited.

If a, b have a common divisor, which is not also a divisor of c, there cannot be any solution in integers; for, dividing by this common measure, the second member of the equation would be a fraction and the first an integer, which is impossible.

PROBLEM I. To find all the positive integer values of x and y in the equation $ax \pm by = c$.

Since $x = (by+c) \div a$ is a positive integer, it follows that, if the division by a be actually performed, the remainder, $py+d$, must also be divisible by a; hence, if the difference between ay and the nearest multiple to it of $py+d$ be taken, the remainder, $qy+e$, must be divisible by a, and q will be less than p. Again, if the difference between $py+d$, and the nearest multiple to it of $qy+e$, be taken, the remainder, $ry+f$, must be divisible by a, and r will be less than q. Proceeding in this way, we shall evidently at length arrive at a remainder of the form $y+k$, which will be divisible by a. Now the least positive integer value that y can have, in order that $(y+k) \div a$ may be a positive integer, when k is negative, will evidently be equal to the remainder arising from the division of k by a, but when k is positive, the least integer value will be a minus this remainder; hence the following rule:—

RULE. Having reduced the proposed equation to the form $x = \dfrac{by+c}{a}$, perform the division. Take the difference of ay and the nearest multiple to it of the remainder, $py+d$, then the difference of $py+d$ and the nearest multiple to it of this last remainder, and so on, till we get a remainder of the form $y-k$ or $y+k$. In the former case the least positive integer value of y will be the remainder R, arising from dividing k by a: in the latter case it will be a *minus* R.

(1) Given $21x+17y=2000$, to find all the positive integer values of x and y. Here $x = \dfrac{2000-17y}{21} = 95 - \dfrac{17y-5}{21}$, and the operation, by the rule, will be as in the margin.

$$\begin{array}{r} 1y \\ 17y-5\ldots[1] \\ \hline 4y+5 \\ 4 \\ \hline 16y+20\ldots[2] \\ \hline [1]-[2]=y-25=y-k \end{array}$$

Now $25 \div 21$ gives a remainder $R=4$, the least integer value of y: this substituted in the expression for x, gives $x=92$, the greatest value of x. And by adding 21, the coef. of x, to each successive value of y, and subtracting 17, the coef. of y, from each successive value of x, we shall have all the possible integer solutions, as in the margin below.

$x=$	92	75	58	41	24	7
$y=$	4	25	46	67	88	100

(2) Show the number of different ways in which it is possible to pay £20 in half-guineas and half-crowns only. Let x be the number of half-guineas, and y the number of half-crowns: then reducing to sixpences, we have $21x+5y=800$ $\therefore$ $x = \dfrac{800-5y}{21} = 38 - \dfrac{5y-2}{21}$, we proceed therefore as in the margin, and thus find that $R=8$, and $\therefore$ $21-8=13$, the least value of y; and $\therefore$ 35 is the greatest value of x. Consequently if we add 21 to the least value of y, and subtract 5 from the greatest value of x, and continue thus to add and subtract, we shall have all the possible integer values of x and y. It thus appears that the payment may be made in seven different ways.

$$\begin{array}{r} 5y-2 \\ 4 \\ \hline 20y-8 \\ 21y \\ \hline y+8 \end{array}$$

$x=$	35	30	25	20	15	10	5
$y=$	13	34	55	76	97	118	139

PROBLEM II. To determine the *number* of solutions that the equation $ax+by=c$ will admit of in positive integers, zero being excluded. Find x', y' so as to satisfy the condition $ax'-by'=1$ (Theo. 1 Cor. 2), then

$$acx'-bcy'=c \therefore ax+by=acx'-bcy'.$$

Consequently, putting $x=cx'-mb$, and $y=ma-cy'$, m may be any integer, taken at pleasure, that will make these values of x and y positive integers: but if no such value of m exist, it will be a proof that the proposed equation is impossible in positive integers. On the contrary, as many suitable values of m as can be found, so many solutions will the equation admit of, and no more. Hence, because we must have $cx'>mb$, and $cy'<ma$, the entire number of solutions will be expressed by the difference between the integral parts of $\frac{cx'}{b}$ and $\frac{cy'}{a}$. For as m must be less than the first of these fractions, and greater than the second, the difference of their integral parts must express the number of different integer values of m, except when $\frac{cx'}{b}$ is itself an integer, in which case, since $m<\frac{cx'}{b}$, the difference of the integral parts would be *one more* than the number of different values of m: therefore, when $\frac{cx'}{b}$ is an integer, we must treat $\frac{b}{b}$ as a fraction, and reject it therefrom; but this must not be done with the other quantity $\frac{cy'}{a}$, because $m>\frac{cy'}{a}$.

EX. How many solutions in positive integers will the equation $9x+13y=2000$ admit of? From the equation $9x'-13y'=1$, we have $x'=\frac{13y'+1}{9}=y'+\frac{4y'+1}{9}$; therefore, proceeding as in the margin, we see that $y'=2$, and $\therefore x'=3$; hence the number of solutions is the integral part of $\frac{2000\times 3}{13}$ *minus* the integral part of $\frac{2000\times 2}{9}$, which is 17.

$$\begin{array}{r|l} 4y'+1 & 2 \\ \hline 8y'+2 & \\ 9y' & \\ \hline y'-2 & \end{array}$$

EXAMPLES FOR EXERCISE.

(1) Given $5x+11y=254$ to find all the integer values of x and y.

(2) Given $19x-117y=11$ to find the least integer values of x and y.

(3) In how many ways can 12 guineas be paid with half-guineas and half-crowns?

(4) Is it possible to pay £50 by means of guineas and three-shilling pieces only?

(5) In how many ways can £1000 be paid in crowns and guineas?

(6) How many ounces of gold, of 17 and 22 carats fine, must be mixed with 5 ounces of 18 carats fine, to produce a composition of 20 carats fine?

PROBLEM III. To find the positive integer values of x, y, z in $ax+by+cz=d$. Take the greatest coefficient:—suppose c, then since x and y can neither of them be less than 1, z cannot be greater than $(d-a-b)\div c$.

If, therefore, we ascertain this limit, and then proceed as in Prob. I., we shall at length arrive at a remainder of the form $y \pm z \pm k$, in which, if 1, 2, 3, &c., up to the limit, be successively substituted for z, all the integer values of x and y corresponding to these values of z will be found.

Ex. Given $3x+5y+7z=100$, to find all the integer positive values of x, y, z, zero being excluded. Here z cannot exceed $(100-3-5) \div 7 = 13\frac{1}{7}$, so that 13 is the integer limit of z; or, rather, 12 is the limit, since the above division by 7 cannot give 13 without a fraction.

$$x = \frac{100-5y-7z}{3} = 33-y-2z-\frac{2y+z-1}{3}; \text{ and } 3y-(2y+z-1=y-z+1.$$

Now, by taking $z=1$, y becomes 0, and $x=31$: but the value $y=0$ is inadmissible. Adding therefore 3, the coef. of x, to this value of y, and subtracting 5, the coef. of y, from the corresponding value of x, we obtain the solution $z=1$, $y=3$, $x=26$. And continuing thus to add and subtract, we shall find the number of solutions in positive integers, for $z=1$ to be *six*. In like manner for $z=2$, the number of solutions will be also *six*: for $z=3$, the number will be *five*; and so on up to $z=12$, for which there will be but a single solution, namely $z=12$, $y=2$, $x=2$. The entire number of solutions will be *forty-one*. The number of solutions due to any particular value of z is found by dividing the *greatest* corresponding value of x by 5, the coef. of y; the integral part of the quotient, increased by 1, will be the number sought.

If we have *two* simple equations involving x, y, z, we may eliminate one of these, and thus get a single equation involving the other unknowns only. And this may be treated as in Prob. I. For example—

Given $\left\{\begin{matrix} 2x+5y+3z=51 \\ 10x+3y+2z=120 \end{matrix}\right\}$ to find all the positive integer values of x, y, z.

Multiplying the first by 5, and subtracting the second, $22y+13z=135$. $\therefore z = \frac{135-22y}{13} = 10-y-\frac{9y-5}{13}$. And proceeding as in the margin, we find $y=2$. $\therefore z=7$. $\therefore x=10$. These are the only positive integral values of x, y, z: but when z and y admit of several values, *some* of the corresponding values of x *may* be fractional: it is necessary, therefore, to ascertain these by actual substitution, and to reject them.

$$\begin{array}{r} 13y \\ 9y-5 \ldots [1] \\ \hline 4y+5 \\ 2 \\ \hline 8y+10 \ldots [2] \\ \hline [1]-[2]=y-15 \end{array}$$

Examples for Exercise.

(1) How many solutions in positive integers does the equa. $17x+19y+21z=400$ admit of?

(2) How many solutions does $5x+7y+11z=144$ admit of in positive integers?

(3) Exhibit all the solutions of $17x+19y+21z=200$ in positive integers.

(4) Give all the solutions of $2x+7y+5z=27$ in positive integers.

(5) How must gold of 14, 11, and 9 carats fine be combined so that the mixture may make 20 ounces of 12 carats fine?

(6) Given $\left\{\begin{matrix} 3x+5y+7z=560 \\ 9x+25y+49z=2920 \end{matrix}\right\}$ to find all the positive integer values of x, y, z.

196. Problem IV. To find the least positive integer which, being divided by given numbers, shall leave given remainders.

Let $a_1, a_2, a_3 \ldots$ be the given divisors, $b_1, b_2, b_3 \ldots$ the respective remainders, and N the required number: then putting x, y, y', &c., for the respective quotients, we have $N = a_1x + b_1 = a_2y + b_2 = a_3y' + b_3$, &c., $\therefore a_1x - a_2y = b_2 - b_1$. Find the least values of x and y that satisfy this condition, then will $a_1x + b_1$ be the least integer that fulfils the first two of the required conditions. Call this integer c; then it is plain that this, and every other integer fulfilling the same conditions will be furnished by the expression $a_1a_2x' + c$, x' being 0, 1, 2, &c., successively. [See Note below.] We have $\therefore a_1a_2x' + c = a_3y' + b_3$ to find the least values of x' and y', for which values, $a_1a_2x' + c$, or $a_3y' + b_3$, will be the least integer fulfilling the first three conditions. Call this integer d, and deduce in like manner the least integer values of x'' and y'', which will satisfy $a_1a_2a_3x'' + d = a_4y'' + b_4$, for which values either of these expressions will fulfil the first four conditions; and so on.

NOTE.—If a_1, a_2, &c., have a common factor, it should be expunged from a_1a_2, $a_1a_2a_3$, &c.; that is, these products should each be only the least common multiple of the given divisors here represented as forming them.

Required the least positive integer, which, when divided by 11, 19, and 29, shall leave for remainders 3, 5, and 10 respectively. Here $N = 11x + 3 = 19y + 5 = 29y' + 10 \therefore y = \frac{11x - 2}{19}$, and from the annexed work it appears that $x = 14$; and since $c = 11x + 3 = 157 \therefore 11.19x' + 157 = 29y' + 10 \therefore y' = \frac{209x' + 147}{29} = 7x' + 5 + \frac{6x' + 2}{29}$.

$$\begin{array}{r} 11x - 2 \\ 2 \\ \hline 22x - 4 \ldots [1] \\ 19x \\ \hline 3x - 4 \\ 7 \\ \hline 21x - 28 \ldots [2] \\ \hline [1] - [2] = \quad x + 24 \end{array}$$

And the operation annexed shows that $x' = 19$, and consequently that $209x' + 157 = 4128$ is the number required.

$$\begin{array}{r} 6x' + 2 \\ 5 \\ \hline 30x' + 10 \\ 29x' \\ \hline x' + 10 \end{array}$$

EXAMPLES FOR EXERCISE.

(1) Required the least whole number which, when divided by 14 and 5, will leave 1 and 3 respectively for remainders.

(2) Required the smallest integer which, when divided by 7, 8, 9, respectively, shall leave 6, 7, 8 for remainders.

(3) Find the least integer which, when divided by 3, 5, 7, 2, respectively, shall leave 2, 4, 6, 0 for remainders.

(4) Find the least integer which, when divided by 28, 19, 15, respectively, shall leave 19, 15, 11 for remainders.

[For more extensive information on the INDETERMINATE ANALYSIS, see "Barlow's Theory of Numbers."]

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197. **Indeterminate Coefficients.**—The principle of indeterminate coefficients has already been applied in the investigation of the Binomial and Exponential Theorems (94): we shall here show how the

same method may be employed to develope other algebraic forms, as also to effect useful changes in such forms.

(1) Required the development of $\frac{a}{a'+b'x}$. [It is plain here that the "&c.," in the assumption below, vanishes for $x=0$].

$$\text{Assume } \frac{a}{a'+b'x}=A+Bx+Cx^2+Dx^3+\&c.$$

then multiplying each side by $a'+b'x$, and transposing, we have

$$\left.\begin{matrix}Aa'+Ba'\\-a+Ab'\end{matrix}\right\}x \quad \left.\begin{matrix}+Ca'\\+Bb'\end{matrix}\right\}x^2 \quad \left.\begin{matrix}+Da'\\+Cb'\end{matrix}\right\}x^3+\&c.=0.$$

As this is true whatever be the value of x, the several coefficients must each be $=0$; hence, we have the following conditions for determining A, B, C, &c., namely,

$$Aa'-a=0,\ Ba'+Ab'=0,\ Ca'+Bb'=0,\ Da'+Cb'=0,\ \&c.$$

$$\therefore A=\frac{a}{a'},\ B=-\frac{b'}{a'}A,\ C=-\frac{b'}{a'}B,\ D=-\frac{b'}{a'}C,\ \&c.$$

$$\therefore \frac{a}{a'+b'x}=\frac{a}{a'}-\frac{b'}{a'}Ax-\frac{b'}{a'}Bx^2-\frac{b'}{a'}Cx^3-\&c.=\frac{a}{a'}\left\{1-\frac{b'}{a'}x+\left(\frac{b'}{a'}\right)^2x^2-\left(\frac{b'}{a'}\right)^3x^3+\&c.\right\}$$

It is desirable, in the summation of certain series, that the general term of the series should be such that the factors involving x may be in arithmetical progression. Whenever the general term is a rational polynomial, it may always be changed into the desired form by the method of indeterminate coefficients, as in the following examples:—

(1) Let the expression be ax^2+bx+c. Assume $ax^2+bx+c=Ax(x+1)+Bx+C=Ax^2+(A+B)x+C\therefore A=a,\ B=b-a,\ C=c\therefore ax^2+bx+c=ax(x+1)+(b-a)x+c$.

(2) Let x^3 be proposed, and put $x^3=Ax(x+1)(x+2)+Bx(x+1)+Cx+D=Ax^3+(3A+B)x^2+(2A+B+C)x+D\therefore A=1,\ 3A+B=0,\ 2A+B+C=0\therefore A=1,\ B=-3,\ C=1,\ D=0\therefore x^3=x(x+1)(x+2)-3x(x+1)+x=x(x+1)(x+2-3)+x=(x-1)x(x+1)+x$.

(3) As another application, let it be required to find the two partial fractions by the addition of which the fraction $\frac{2x+3}{x^2+x-2}$ results.

$$\text{Put } \frac{2x+3}{x^2+x-2}=\frac{2x+3}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}:\ \text{then } \frac{2x+3}{x+2}=A+\frac{B(x-1)}{x+2}.$$

For $x=1$, the second member of this becomes simply A, and the first becomes

$$\frac{5}{3}\therefore A=\frac{5}{3}.\quad \text{Also } \frac{2x+3}{x-1}=B+\frac{A(x+2)}{x-1}.$$

For $x=-2$, the second member becomes simply B, and the first,

$$\frac{1}{3}\therefore B=\frac{1}{3}\therefore\frac{2x+3}{x^2+x-2}=\frac{5}{3(x-1)}+\frac{1}{3(x+2)}.$$

Or thus: In the proposed fraction put 1st, $x=0$, 2nd, $x=2$;

$$\therefore -\frac{3}{2}=-A+\frac{1}{2}B,\ \text{and } \frac{7}{4}=A+\frac{1}{4}B.\therefore\frac{1}{4}=\frac{3}{4}B.\therefore B=\frac{1}{3}.\therefore A=\frac{5}{3}.$$

(4) Let $\frac{3x+2}{x(x+1)^3}$ be proposed. Put it $=\frac{A}{x}+\frac{X}{(x+1)^3}.\therefore\frac{3x+2}{(x+1)^3}=A+\frac{Xx}{(x+1)^3}$.

$$\text{Put } x=0.\therefore 2=A.\therefore\frac{X}{(x+1)^3}=\frac{3x+2}{x(x+1)^3}-\frac{2}{x}=-\frac{2x^2+6x+3}{(x+1)^3}.$$

Put this, disregarding the minus sign, $=\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{D}{(x+1)^3}$, then $\frac{2x^2+6x+3}{(x+1)^2}=$

$B+\frac{C}{x+1}+\frac{D}{(x+1)^2}$. . . . . [1]. Multiply by $(x+1)^2$, then we have

$$2x^2+6x+3=Bx^2+2Bx+B+Cx+C+D=Bx^2+(2B+C)x+B+C+D$$

$$\therefore B=2,\ 2B+C=6.\ \therefore C=2,\ B+C+D=3.\ \therefore D=-1.$$

Hence, introducing the neglected minus sign, $\frac{3x+2}{x(x+1)^3}=\frac{2}{x}-\frac{2}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{(x+1)^3}$.

Or thus: In [1] put $x=0$, 1 and 2, in succession, we shall then have the three conditions

$3=B+C+D$, $\frac{11}{4}=B+\frac{1}{2}C+\frac{1}{4}D$, $\frac{23}{9}=B+\frac{1}{3}C+\frac{1}{9}D$, from which we readily get

$$B=2,\ C=2,\ D=-1, \text{ as before.}$$

NOTE.—In thus decomposing a rational fraction into its component partial fractions, the numerator should be rendered of lower degree than the denominator by reducing the proposed to a mixed quantity, when such is not the case. And it must be observed that when, as in the last example above, the denominator is a *power*, that power, and all the inferior powers, must each form a denominator of a partial fraction. There is a difference therefore in the assumptions when the factors of the denominators are unequal, and when they are equal:

Thus, $\frac{P}{(x+a)(x+b)}$ is assumed $=\frac{A}{x+a}+\frac{B}{x+b}$, but if $a=b$, then the assumption must be $\frac{P}{(x+a)^2}=\frac{A}{x+a}+\frac{B}{(x+a)^2}$; for, clearing fractions, $P=Ax+Aa+B$, which supplies two conditions—and two are necessary—for determining the unknowns A and B. But for the above form of decomposition, we should have only one condition: and similarly in all such cases.

If the denom. of the proposed fraction involve a quadratic factor X, of which the roots r, r' are imaginary, then, instead of decomposing in the form $\frac{P}{x-r}+\frac{Q}{x-r}+$ &c., it is better to unite the two fractions, with imaginaries in their denominators, into one, which will be of the form

$$\frac{Ax+B}{X}.$$

Thus, suppose we have to decompose $\frac{x}{(x^2+4x+5)(x-4)}$, we assume it $=\frac{Ax+B}{x^2+4x+5}+\frac{C}{x-4}$. Now put 0, 1, -1, successively for x; then we have the following equations of condition, namely $\frac{B}{5}-\frac{C}{4}=0$, $\frac{A+B}{10}-\frac{C}{3}=-\frac{1}{30}$, $\frac{B-A}{2}-\frac{C}{5}=\frac{1}{10}$, therefore

$4B-5C=0$, $3A+3B-10C=-1$, $5B-5A-2C=1$, from which we readily find

$$A=-\frac{4}{37},\ B=\frac{5}{37},\ C=\frac{4}{37},$$

$$\therefore \frac{x}{(x^2+4x+5)(x-4)}=\frac{4}{37(x-4)}-\frac{4x-5}{37(x^2+4x+5)}.$$

EXAMPLES FOR EXERCISE.

(1) Write $(2x-1)^2$ as an expression in which the factors involving x shall be in arithmetical progression.

(2) Decompose $\frac{x+1}{x^2+5x+6}$ into its partial fractions.

(3) Decompose $\frac{5x+1}{x^2+x-2}$, and $\frac{x^2+1}{x^3-1}$ into partial fractions.

(4) Decompose $\frac{4x^3+6x^2-8x+4}{x^4+x^3-2x^2}$ into partial fractions.

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198. **Summation of Series.**—The general term of a numerical series is such a function of n, that when n is put $=1, 2, 3, \ldots, n$, the expression becomes the first, second, third ... , nth term of that series. Suppose the general term of a series to consist of factors in arithmetical progression, that is, let

$$\text{General term} = n'(n'+p)(n'+2p)\ldots(n'+mp)\ldots[A].$$

Where n' is either equal to n, or differs from it, or a multiple of it, only by some constant number. It is easily seen that the expression $[A]$ may be got thus:—

From $n'(n'+p)(n'+2p)(n'+3p)\ldots\ldots\ldots(n'+mp)(n'+[m+1]p)\ldots\ldots[B]$

Subtract $(n'-p)n'(n'+p)(n'+2p)\ldots\ldots\ldots(n'+mp)\ldots\ldots\ldots\ldots[C]$

There remains $\{n'+[m+1]p-(n'-p)\}$ times $[A]$, that is, $(m+2)p$ times $[A]$.

And from this it follows that any series, of which the general term is $[A]$, is equal to $\frac{1}{(m+2)p} \times$ the difference between two series, each of the same extent as the former series, and of which the general terms are respectively $[B]$ and $[C]$.

(1) Let the sum S of n terms of $1.2+2.3+3.4+\ldots+n(n+1)$ be required,

$[B]= 1.2.3+2.3.4+3.4.5+\ldots+(n-1)n(n+1)+n(n+1)(n+2)$

$[C]=0+1.2.3+2.3.4+3.4.5+\ldots+(n-1)n(n+1)$

$\therefore$ Taking the difference (and observing that $p=1$, $m=1$), we have

$$S=\frac{n(n+1)(n+2)}{3}.$$

The proposed series is double the series in the example at (89), so that the number of shot in a triangular pile may be investigated as above.

NOTE.—The series $[C]$ is written so that the leading term of $[B]$ may have its equal immediately under it, and as the number of terms is always the same in each series, the lower series extends towards the left beyond the upper, to as many terms as the upper extends beyond the lower, to the right. It is plain, therefore, that the intermediate terms, common to both series, may be suppressed, and account be taken only of those terms thus projecting to the left and right, since these are the only terms which actually appear in the resulting remainder. The projecting terms here spoken of are always p in number, because p units must be added to n to increase $(n-p)$ till it reaches n.

(2) Find the sum of n terms of $1.3+2.4+3.5+\ldots+n(n+2)$. Here $p=2$, and $m=1$, so that the series $[B]$, $[C]$, are

$[B]= 1.3.5+\ldots\ldots(n-1)(n+1)(n+3)+n(n+2)(n+4)$

$[C]=-1.1.3+0+1.3.5+\ldots(n-2)(n)(n+2)$. The difference of these is

$$3+(n-1)(n+1)(n+3)+n(n+2)(n+4) \therefore S=\frac{(2n^2+9n+7)n}{6}=\frac{n(n+1)(2n+7)}{6}.$$

(3) Find the sum of n terms of $1^2+2^2+3^2+\ldots+n^2$. Here $n^2=(n-1)n+n$, so that the series is composed of two others, whose general terms are $(n-1)n$ and n respectively. And by the foregoing principle, the sums of these latter are

$$\frac{1}{3}(n-1)n(+1), \text{ and } \frac{1}{2}n(n+1)$$

$$\therefore S=\frac{1}{3}(n-1)n(n+1)+\frac{1}{2}n(n+1)=\frac{1}{6}n(n+1)(2n+1).$$

In this way, therefore, may the formula for the number of shot in a square pile be readily found.

(4) Let it be required to find the sum of n terms of $1^3+2^3+3^3+\ldots+n^3$. Here since $n^3=(n-1)n(n+1)+n$ (ex. 2, p. 156), we have by the preceding principle,

$$S=\frac{1}{4}(n-1)n(n+1)(n+2)+\frac{1}{2}n(n+1)=\frac{1}{2}n(n+1)\cdot\frac{1}{2}n(n+1)=\left\{\frac{1}{2}n(n+1)\right\}^2$$

$$\therefore 1^3+2^3+3^3+\ldots+n^3=(1+2+3+\ldots+n)^2.$$

Series such as those here considered are, however, usually summed by the Method of Differences, or as it is sometimes called,—

199. **The Differential Method.**—Let a, b, c, d, e, &c. be any series of numbers following a regular law of increase or decrease, then if each term be subtracted from the next following, the series of remainders is called *the first order of differences;* and if, in like manner, the successive differences of the terms of this new series be taken, the results will form *the second order of differences*, and so on.

Problem I. To find a general expression for the first term of the nth order of differences. Let a, b, c, d, e, &c. be the series, and let Δ_1, Δ_2, Δ_3, &c., stand for the first term of the first, second, third, &c., orders of differences respectively; then

$$\begin{array}{lllll}
\Delta_1=b-a, & c-b, & d-c, & e-d, \text{ &c.} \\
& b-a & c-b & d-c \\
\hline
\Delta_2=c-2b+a, & d-2c+b, & e-2d+c, \text{ &c.} \\
& & c-2b+a & d-2c+b \\
\hline
\Delta_3=d-3c+3b-a, & & e-3d+3c-b, \text{ &c.} \\
& & & d-3c+3b-a \\
\hline
& & \Delta_4=e-4d+6c-4b+a, \text{ &c.}
\end{array}$$

And it is easy to see that, by continuing the operation, the numerical coefficients in Δ_n are the same as those in the development of the binomial $(a-x)^n$ or $(1-1)^n$: hence, reversing the order of the terms,—

$$\left.\begin{array}{l}
\text{If } n \text{ is even, } \Delta_n=a-nb+\dfrac{n(n-1)}{2}c-\dfrac{n(n-1)(n-2)}{2.3}d+\ldots \\
\text{If } n \text{ is odd, } \Delta_n=-a+nb-\dfrac{n(n-1)}{2}c+\dfrac{n(n-1)(n-2)}{2.3}d-\ldots
\end{array}\right\}(1)$$

Problem II. To find the nth term of the series a, b, c, d, e, &c. In the general expressions (1), put 1, 2, 3, &c. for n in succession, then—

$$\begin{array}{lll}
b= & a+\Delta_1 & b=a+\Delta_1 \\
c= & -a+2b+\Delta_2. \text{ And by substitution} & c=a+2\Delta_1+\Delta_2 \\
[1]\ d= & a-3b+3c+\Delta_3 & d=a+3\Delta_1+3\Delta_2+\Delta_3 \qquad (2) \\
e= & -a+4b-6c+4d+\Delta_4 & e=a+4\Delta_1+6\Delta_2+4\Delta_3+\Delta_4 \\
f= & a-5b+10c-10d+5e+\Delta_5, & f=a+5\Delta_1+10\Delta_2+10\Delta_3+5\Delta_4+\Delta_5 \\
& \text{&c.} \qquad \text{&c.} & \text{&c.} \qquad \text{&c.}
\end{array}$$

Consequently the $(n+1)$th term of the series is

$$(n+1)\text{th term}=a+n\Delta_1+\frac{n(n-1)}{2}\Delta_2+\frac{n(n-1)(n-2)}{2.3}\Delta_3+\ldots\ldots[A]$$

$$\therefore n\text{th term}=a+(n-1)\Delta_1+\frac{(n-1)(n-2)}{2}\Delta_2+\frac{(n-1)(n-2)(n-3)}{2.3}\Delta_3+\ldots\ldots[B].$$

PROBLEM III. To find the sum of n terms of the series $a, b, c, d, e,$ &c. Of the series $0, a, a+b, a+b+c, a+b+c+d,$ &c. ...[1], the first order of differences is $a, b, c, d,$ &c. ...[2]; and the sum of n terms of this is the $(n+1)$th term of the preceding series; all we have to do, therefore, is to find the $(n+1)$th term of [1]. As this series commences with zero, the $(n+1)$th term will be found by putting in $[A]$, 0 for a, a for Δ_1, Δ_1 for Δ_2, &c., so that the sum S of the proposed series is

$$S=na+\frac{n(n-1)}{2}\Delta_1+\frac{n(n-1)(n-2)}{2.3}\Delta_2+\ldots.$$

(1) Required the sum of n terms of $1.2+2.3+3.4+4.5+\ldots\ldots$

Taking the differences as in the annexed operation, we see that $\Delta_1=4$, $\Delta_2=2$, and $\Delta_3=0$: hence

	a	b	c	d
	2,	6,	12,	20
Δ_1	4	6	8	
Δ_2	2	2		
Δ_3	0			

$$S=2n+2n(n-1)+\frac{1}{3}n(n-1)(n-2)=\frac{1}{3}n\{(n-1)(n-2)+6n\}=\frac{1}{3}n(n+1)(n+2).$$

(2) Find the sum of n terms of $1.3+2.4+3.5+4.6+\ldots\ldots$

Taking the differences, we see that $\Delta_1=5$, $\Delta_2=2$, and $\Delta_3=0$: hence

	3,	8,	15,	24
	5	7	9	
	2	2		
	0			

$$S=3n+\frac{5n(n-1)}{2}+\frac{n(n-1)(n-2)}{3}=\frac{(2n^2+9n+7)n}{6}.$$

These two examples have been worked by a different method at p. 158; the student is recommended to solve the following examples by both processes :—

EXAMPLES FOR EXERCISE.

(1) $1^2+2^2+3^2+\ldots+n^2$.
(2) $1^3+2^3+3^3+\ldots+n^3$.
(3) $1+3+6+10+\ldots+\frac{1}{2}n(n+1)$.
(4) $1+4+8+13+\ldots$ to 12 terms.
(5) $1+3+5+7+\ldots+2(n-1)$.
(6) $1.2.3+2.3.4+3.4.5+\ldots+n(n+1)(n+2)$.
(7) $1+4+10+20+35+\ldots$ to n terms.
(8) $1^3+3^3+5^3+\ldots+(2n-1)^3$.

200. **Construction of Tables.**—One of the most important purposes to which the Method of Differences is applied is the construction of Mathematical Tables. Suppose, for instance, it were required to construct tables of squares and cubes, we might proceed thus. By differencing a few of the leading squares 1, 4, 9, 16, &c. as in the margin, we find that the second differences are all constant. Adding, therefore, 2 to 7, a new term 9 will be introduced into the series of first differences, and adding this

1	4	9	16	
3	5	7		
2	2			
0				

9 to the 16, the next square 25 is obtained. The row of first differences, therefore, may be prolonged to any extent by simply continuing to add 2, and the row of squares, by adding to the last-found square the last-found first difference; and the arrangement may stand as below, where 16 is added to 9, 25 to 11, 36 to 13, and so on.

The second difference .	2								
Row of first differences.	7	9	11	13	15	17	19	21	23 &c.
Row of squares . . .	16	25	36	49	64	81	100	121	144 &c.

In like manner for the cubes. Differencing as in the margin, and computing the several rows as above the work stands thus :—

	1	8	27	64	125				
	7	19	37	61					
	12	18	24						
	6	6							
	0								

The third diff. . . .	6							
Row of second diffs. .	24	30	36	42	48	54	60	66 &c.
,, first diffs. . .	61	91	127	169	217	271	331	397 &c.
,, cubes . . .	125	216	343	512	729	1000	1331	1728 &c.

And it is plain that whenever the differences, found as above, at length become constant, the series of numbers may be extended to any limit by easy successive additions only.

But in the more advanced class of tables—Logarithmic and Trigonometrical Tables—where the numbers tabulated are given, not accurately, but only to a limited extent of decimals, the differences never become constant; yet as the successive differences continually diminish, we may carry on the operation till they at length become so small as to have no influence within the range of decimals to which the tabulated numbers are confined; the preceding differences may therefore be regarded as constant without practical error, and many additional numbers for tabulation may be computed as above. The small error, however, consequent upon supposing any order of differences to be constant, will by its repeated introduction have at length a sensible effect, so that the extent of numbers thus deduced must not exceed a certain limit; when this is reached, a new set of numbers, arrived at independently of differences, are to be differenced; and another series deduced from *them*, and so on. But for very ample information on this mode of constructing a table of logarithms the student is referred to the author's "Essay on the Computation of Logarithms."

201. **Interpolation.**—The process alluded to in last article is in strictness one of interpolation; its object being to insert a set of numbers, computed by differences, between two numbers computed independently, and separated by an interval more or less wide, and such that the entire series of numbers may follow a uniform law. The formulæ [*A*], [*B*] at p. 160, express how any remote term [the $(n+1)$th, or the nth] may be computed by help of only the first term (a), and the first of the several orders of differences. But when a single term only is to be interpolated among a set of previously-computed terms, one or other of the forms marked [1], at the preceding page are to be employed: for example :—

Given the logs of 101, 102, 104, and 105, to interpolate log 103. Here of the five terms a, b, c, d, e, all are known but the third c. Regarding then, as we may safely do, $\Delta_4=0$, we have by [1],

$$e=-a+4b-6c+4d \quad \therefore \quad c=\frac{4(b+d)-(a+e)}{6};$$ and

this we calculate as in the margin, and find that $c=\log 103=2{\cdot}0128372$. And in this way may the logs of prime numbers be found from those of composite numbers.

$a=\log 101=$ 2·0043214
$b=\log 102=$ 2·0086002
$d=\log 104=$ 2·0170333
$e=\log 105=$ 2·0211893
$\therefore 4(b+d)=$16·1025340
$(a+e)=$ 4·0255107
6) 12·0770233
$c=\log 103=$ 2·0128372

202. But if instead of it being required to supply an absent term, it is wanted to interpose a new term, between two consecutive terms of a regular series, we must work by the formula $[A]$ or $[B]$. If, for instance, it were required to find, not log 103, but log 103·55, from having the logs of 102, 103, 104, 105, given, then, regarding a as log 103=2·0128372, we must make $n=\cdot55$: and since by differencing the logs of 103, 104, 105, given above in the margin, we find (omitting leading zeros), $\Delta_1=41961$, and $\Delta_2=-401$, by neglecting Δ_3, on account of its smallness, we have

$$\log 103{\cdot}55=\log 103+n\Delta_1+\frac{n(n-1)}{2}\Delta_2=2{\cdot}0128372+{\cdot}55(4196+\frac{{\cdot}45}{2}401)=2{\cdot}0151500.$$

The small error arising from neglecting Δ_3, will be still further diminished, if instead of taking Δ_2 for the first term of the second order of differences, we take for it the arithmetical mean of the first *two* terms, as obtained from differencing the four given logs: this mean is -405, instead of -401, which will give 2·0151501 for log 103·55.

NOTE.—What is done in this last article is not a strictly-legitimate deduction from the foregoing theory, in which n is assumed to be, not a fraction, but a whole number. That the theorems $[A,]$ $[B]$ comprehend the case of n=a fraction, will, however, be satisfactorily shown hereafter. At present we can enter no further into the subject of Finite Differences than to prove the following interesting theorem:—

THEOREM. If in a rational and integral polynomial of degree n we substitute for x a series of numbers in arith. prog., the nth differences of the results will be constant.

In the polynomial $A_nx^n+A_{n-1}x^{n-1}+\ldots+A_2x^2+A_1x+A_0$, let $x+h$ be put for x: the difference will evidently be of the form $A'_{n-1}x^{n-1}+A'_{n-2}x^{n-2}+\ldots+A'_2x^2+A'_1x+A'_0$, a polynomial a unit lower in degree, from which the constant A_0 has disappeared. In like manner putting $x+h$ for x in this, and differencing, the result will be of the form $A''_{n-2}x^{n-2}+A''_{n-3}x^{n-3}+\ldots+A''_2x^2+A''_1x+A''_0$, and so on. Hence after thus differencing n times, the leading power of x will be $x^{n-n}=1$, and therefore the nth differences constant. As an example, take $3x^3+x^2+2$; and put 1, 2, 3, &c., successively for x: then differencing, as in the margin, we find that $\Delta_3=18$, and $\Delta_4=0$.

6 30 92 210 402
24 62 118 192
38 56 74
18 18
0

---

203. **Infinite Series.**—Imitating the method explained at (198), let the nth term of an infinite series be $\frac{q}{n'(n'+p)(n'+2p)\ldots(n'+mp)}$, where

n' differs from a multiple of n only by a constant number; then it is easily seen that this expression is equal to

$$\frac{1}{mp}\left\{\frac{q}{n'(n'+p)(n'+2p)\ldots n'+(m-1)p}-\frac{q}{(n'+p)(n'+2p)\ldots(n'+mp)}\right\}\ldots[D].$$

(1) Required the sum Σ of the infinite series $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots\ldots$ where $p=2$, and $m=1$.

$$\left.\begin{matrix}1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\\ -\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\right)\end{matrix}\right\}=1.\therefore\frac{1}{p}=\frac{1}{2}=\Sigma.$$

(2) Required the sum S of n terms of the above series:

$$\left.\begin{matrix}1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+\frac{1}{2n-1}\\ -\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\quad+\frac{1}{2n-1}+\frac{1}{2n+1}\right)\end{matrix}\right\}=1-\frac{1}{2n+1}=\frac{2n}{2n+1}.\therefore S=\frac{n}{2n+1}.$$

(3) Required the sum Σ of the infinite series $\frac{3}{5.8.11}+\frac{9}{8.11.14}+\frac{15}{11.14.17}+\ldots$ where $p=3$, and $m=2$.

$$\frac{1}{6}\left\{\begin{matrix}\frac{3}{5.8}+\frac{9}{8.11}+\frac{15}{11.14}\cdots\\ -\left(\frac{3}{8.11}+\frac{9}{11.14}\cdots\right)\end{matrix}\right\}=\frac{1}{2.5.8}+\frac{1}{8.11}+\frac{1}{11.14}+\ldots$$

$$\text{But }\frac{1}{8.11}+\frac{1}{11.14}+\ldots=\frac{1}{3}\left\{\begin{matrix}\frac{1}{8}+\frac{1}{11}+\ldots\\ -\left(\frac{1}{11}+\ldots\right)\end{matrix}\right\}=\frac{1}{24}.\therefore\Sigma=\frac{1}{80}+\frac{1}{24}=\frac{13}{240}.$$

(4) Required the sum Σ of $\frac{1}{1.3.5.7}+\frac{2}{3.5.7.9}+\frac{3}{5.7.9.11}+\ldots$ where $p=2$, and $m=3$.

$$\frac{1}{6}\left\{\begin{matrix}\frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+\ldots\\ -\left(\frac{1}{3.5.7}+\frac{2}{5.7.9}+\ldots\right)\end{matrix}\right\}=\frac{1}{6}\left(\frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+\ldots\right).$$

$$\text{But }\frac{1}{1.3.5}+\frac{1}{3.5.7}+\ldots=\frac{1}{4}\left\{\begin{matrix}\frac{1}{1.3}+\frac{1}{3.5}+\ldots\\ -\left(\frac{1}{3.5}+\ldots\right)\end{matrix}\right\}=\frac{1}{12}.\therefore\Sigma=\frac{1}{72}.$$

Examples for Exercise.

(1) $\Sigma=\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\ldots=$.

(2) $\Sigma=\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-\ldots=$.

(3) $\Sigma=\frac{1}{1.5}+\frac{1}{5.9}+\frac{1}{9.13}+\ldots=$.

(4) $S=\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\ldots$ to n terms.

(5) $S=\frac{1}{3.8}+\frac{1}{6.12}+\frac{1}{9.16}+\ldots$ to n terms.

(6) $\Sigma=\frac{4}{1.2.3}+\frac{5}{2.3.4}+\frac{6}{3.4.5}+\ldots=$.

(7) $\Sigma=\frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+\ldots=$.

(8) $\Sigma=\frac{1}{1.3.5}=\frac{4}{3.5.7}+\frac{7}{5.7.9}+\ldots=$.

(9) $\Sigma=\frac{6^2}{1.2.3.4}+\frac{7^2}{2.3.4.5}+\frac{8^2}{3.4.5.6}+\ldots=$.

(10) $S=\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}-\frac{5}{9.11}+\ldots$. to n terms.

Still following the same principle, since $\frac{a(a+b)(a+2b)\ldots(a+mb)}{n'(n'+b)(n'+2b)\ldots(n'+mb)}$ is equal to

$$\frac{1}{n'-a-b}\left\{\frac{a(a+b)(a+2b)\ldots(a+mb)}{n'(n'+b)(n'+2b)\ldots[n'+(m-1)b]}-\frac{a(a+b)(a+2b)\ldots[a+(m+1)b]}{n'(n'+b)(n'+2b)\ldots(n'+mb)}\right\}$$

we are enabled to sum such series as the following, namely:—

(1) Required the sum S of $\frac{1}{2}+\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6}+\frac{1.3.5.7}{2.4.6.8}+\ldots$ to n terms. Here $a=1$ $b=2$, and $n'=2$. Consequently,

$$S=\frac{1}{-1}\left\{\begin{array}{l}1+\frac{1.3}{2}+\frac{1.3.5}{2.4}+\ldots+\frac{1.3.5.7\ldots(2n-1)}{2.4.6\ldots(2n-2)}\\ \frac{1.3}{2}+\frac{1.3.5}{2.4}+\ldots\ldots\ldots\ldots\ldots\ldots+\frac{1.3.5.7.9\ldots(2n+1)}{2.4.6.8\ldots\ldots 2n}\end{array}\right\}=$$

$$\frac{1.3.5.7.9\ldots(2n+1)}{2.4.6.8\ldots\ldots 2n}-1,$$

which continually increases as n increases, so that the sum of the series to infinity is infinitely great.

(2) Required the sum Σ of the inf. series $\frac{2}{5.6}+\frac{2.3}{5.6.7}+\frac{2.3.4}{5.6.7.8}+\ldots$

(3) Required the sum S of $\frac{2}{3}+\frac{2.4}{3.5}+\frac{2.4.6}{3.5.7}+\frac{2.4.6.8}{3.5.7.9}+\ldots$ to n terms.

---

204. On account of the great variety of forms which infinite series may assume, the subject of their summation is an inexhaustible one. The formulæ given in the present section comprehend, as we have seen, some extensive and interesting classes of series: they were first published by the author of the present work in his "Treatise on Algebra" (1823): a further extension of them may be seen in his "Mathematical Dissertations," pp. 120–134. We shall now give a few examples of the summation of particular series by aid of known algebraic developments.

(1) Find Σ when $\Sigma=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots$, $\Sigma=\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots$, &c. By division,

$1\div(1-x)=1+x+x^2+x^3+$ &c.: hence putting $\frac{1}{2}$, $\frac{1}{3}$, &c., for x, we have

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots=1,\ \frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots=\frac{1}{2},\text{ and so on.}$$

It must be observed that the infinite series in x cannot have its sum expressed by $1\div(1-x)$ when $x>1$. The *remainder* arising from this division, and which is regarded as concealed under the "&c.," must not, in such cases, be neglected. This remainder when divided by $1-x$ must, in fact, be so great as to exceed the sum of the series, for the right-hand member of the identity must be negative, whereas the sum of the *series* is positive.

(2) Find the involvement of $F(x)=x+2^2x^2+3^2x^3+4^2x^4+$ &c.

By the Binomial Theorem, $(1-x)^{-3}=1+3x+6x^2+10x^3+$ &c.

$\therefore x(1-x)^{-3}=x+3x^2+6x^3+10x^4+$ &c.

Hence, by subtraction, $F(x)-x(1-x)^{-3}=x^2+3x^3+6x^4+$ &c. $=x^2(1-x)^{-3}$

$$\therefore F(x)=\frac{x(1+x)}{(1-x)^3}.$$

(3) Find $\Sigma=\frac{x^3}{2.3}+\frac{x^5}{2.3.4.5}+\frac{x^7}{2.3.4.5.6.7}+\ldots$ By the Exponential Theorem (page 80),

$$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\ldots$$

$$e^{-x}=1-x+\frac{x^2}{2}-\frac{x^3}{2.3}+\frac{x^4}{2.3.4}-\ldots$$

$$\therefore \frac{e^x-e^{-x}}{2}=x+\frac{x^3}{2.3}+\frac{x^5}{2.3.4.5}+\ldots \therefore \Sigma=\frac{e^x-e^{-x}}{2}-x.$$

(4) Find the involopment of $\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\frac{x^4}{4.5}+$ &c.

By (112), $-\log_e(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+$ &c. $=\log_e\frac{1}{1-x}$

$$\therefore x\log_e\frac{1}{1-x}=x^2+\frac{x^3}{2}+\frac{x^4}{3}+ \text{ &c.}$$

Subtracting, $(x-1)\log_e\frac{1}{1-x}=-x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{3.4}+$ &c.

$$\therefore 1+\frac{x-1}{x}\log_e\frac{1}{1-x}=\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\frac{x^4}{4.5}+ \text{ &c.}$$

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205. **Recurring Series.**—There is one class of series which has not as yet been adverted to—Recurring Series. These arise from developing rational fractions of the form $\frac{a+bx+cx^2+\ldots+px^m}{1+a'x+b'x^2+c'x^3+\ldots+q'x^{m+1}}$.

Thus $\frac{a+bx}{1+a'x+b'x^2}=A+Bx+Cx^2+Dx^3+Ex^4+$ &c., where, by the principle of indeterminate coefficients,* $A=a$, $B+a'A=b$, $C+a'B+b'A=0$, $D+a'C+b'B=0$, $E+a'D+b'C=0$, &c. In these equations of condition we see that the coefficients of A, B, C, of B, C, D, of C, D, E, &c., are those of the denominator of the generating fraction taken in reverse order. The values deduced from these conditions are $A=a$, $B=b-a'A$, $C=-b'A-a'B$, $D=-b'B-a'C$, $E=-b'C-a'D$, &c. Consequently each term of the development, after the first two terms $A+Bx$, is got by multiplying the two terms preceding it by $-b'x^2$, $-a'x$, and adding the results: $-b'x^2$, $-a'x$ is called the *scale of relation* of the recurring series. It consists of the terms of the denom. of the fraction (omitting the unit) taken in reverse order, and with changed signs. To take a particular case: $\frac{3-x-6x^2}{1-2x-x^2+2x^3}=3+5x+7x^2+13x^3+23x^4+$ &c., in which the scale of relation is $-2x^3$, x^2, $2x$.

(1) Required the involopment of the recurring series $1+2x+8x^2+28x^3+100x^4+$ &c. Assume the scale of relation to be px^2, qx; then $p+2q=8$, and $2p+8q=28$ $\therefore q=3$ $\therefore p=2$. Upon trial, these are found to be correct, so that the denom. of the involopment is $1-3x+2x^2$.

Assume then $\frac{A+Bx}{1-3x+2x^2}=1+2x+8x^2+$ &c. $\therefore A=1$, $B=2-3=-1$

$$\therefore \frac{1-x}{1-3x+2x^2}=1+2x+8x^2+28x^3+ \text{ &c.}$$

---

* It is plain that *here* the "&c." vanishes when $x=0$.

(2) Required the involvement of $1+3x+5x^2+7x^3+9x^4+$ &c. Assume the scale of relation to be px^2, qx; then $p+3q=5$, and $3p+5q=7$ $\therefore q=2 \therefore p=-1$, which, upon trial, are found to answer: hence the denom. of the involvement is $1-2x+x^2$.

Assume therefore $\dfrac{A+Bx}{1-2x+x^2}=1+3x+5x^2+$ &c. $\therefore A=1$, $B=3-2=1$

$$\therefore \frac{1+x}{(1-x)^2}=1+3x+5x^2+7x^3+9x^4+ \text{ &c.}$$

NOTE.—Should p and q, determined as in these examples, ever be found upon trial to fail, in reference to the advanced terms of the series, we should then assume the scale to be px^3, qx^2, rx, and determine p, q, r from three conditions: and so on. If the series be really recurring, the scale must in this way be at length determined. If the sum of n terms only of a recurring series be required, we must in general find the involvement, as above, of the entire series, then the involvement of the series which follows the first n terms—the scale of relation being uniform—and subtract the latter fraction from the former.

By *involvement* (a term very judiciously introduced by De Morgan) we mean the fraction or function which generates the series, and whatever besides may be included in the "&c." In all interminable algebraic series, the "&c." stands for the involvement, *minus* the series itself: when the involvement is, as above, an algebraic fraction, the "&c." represents the remainder with the divisor underneath. We have not, therefore, as is customary, called this involvement the *sum* of the series to infinity, inasmuch as it is this and something more.

EXAMPLES FOR EXERCISE.

Find the involvements of the following recurring series:—

(1) $1+3x+5x^2+7x^3+$ &c.
(2) $1+2x+3x^2+5x^3+$ &c.
(3) $1+4x+9x^2+16x^3+$ &c.
(4) Find the sum of n terms of $1+2x+3x^2+4x^3+$ &c.

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206. **Reversion of Series.**—To revert a series such as $y=ax+bx^2+cx^3+\ldots$ is to express x in a series proceeding according to the powers of y. The principle of indeterminate coefficients enables us to effect this. Assume $x=Ay+By^2+Cy^3+\ldots$, which call Y: then we have $aY+bY^2+cY^3+dY^4+\ldots-y=0$; that is, putting for Y its value,

$$\left.\begin{matrix} aA \\ -1 \end{matrix}\right\} y \quad \left.\begin{matrix} +aB \\ +bA^2 \end{matrix}\right\} y^2 \quad \left.\begin{matrix} +aC \\ +2bAB \\ +cA^3 \end{matrix}\right\} y^3 \quad \left.\begin{matrix} +aD \\ +2bAC \\ +bB^2 \\ +3cA^2B \\ +dA^4 \end{matrix}\right\} y^4+\ldots\ldots=0.$$

Hence the coefficients of y and its powers are each $=0$, so that $aA-1=0$, $aB+bA^2=0$, $aC+2bAB+cA^3=0$, $aD+2bAC+bB^2+3cA^2B+dA^4=0$, &c. And from these equations of condition we get

$$A=\frac{1}{a},\ B=-\frac{b}{a^3},\ C=\frac{2b^2-ac}{a^5},\ D=-\frac{5b^3-5abc+a^2d}{a^7},\ \text{&c.}$$

For ex. Let it be required to revert the series $y-1=x+\dfrac{x^2}{2}+\dfrac{x^3}{2.3}+\dfrac{x^4}{2.3.4}+\ldots$

Here $a=1$, $b=\frac{1}{2}, c=\frac{1}{2.3}$, &c. $\therefore \frac{1}{a}=1, -\frac{b}{a^3}=-\frac{1}{2}, \frac{2b^2-ac}{a^5}=\frac{1}{3}, -\frac{5b^3-5abc+a^2d}{a^7}=\frac{1}{4}$, &c.

$$\text{Hence } x=(y-1)-\frac{(y-1)^2}{2}+\frac{(y-1)^3}{3}-\frac{(y-1)^4}{4}+ \text{ &c.}$$

From (110), we know that this expression for x is the Napierian log of y; but if

$$x=\log y_e \therefore e^x=y \therefore e^x=1+x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\ldots$$

which is the exponential theorem, otherwise established at (110).

NOTE.—The series to be reverted is, of course, considered to be a *series*, and nothing more:—it may be finite or infinite. The reversion of it is, however, in all cases an *infinite* series, and will involve in general a supplementary &c.; we must again, therefore, caution the student against treating what may be implied in this vague symbol as of no moment, and thus deluding himself by the notion that it may be regarded as valueless. On the contrary, it may turn out to be the most important part of the entire expression, and serious error may therefore arise from neglecting it. This may be easily illustrated in the matter before us. Let us take $y=x+x^2+2$, or $y-2=x+x^2$: then by reverting, as above, we find for x,

$$x=(y-2)-(y-2)^2+2(y-2)^3-5(y-2)^4+ \text{ &c.}$$

or, for $y=1$, $x=-1-1-2-5-$&c. But the values of x, in $-1=x+x^2$, or in $x^2+x+1=0$, are both *imaginary;* the "&c," therefore conceals the most important part of the foregoing expression for x. Such expressions are of course practically useless; we can never estimate the value of a series, or of any expression, which contains in it a quantity about which we know nothing. Such series or expressions can be available in computation only in those cases in which we know that what is hidden under the "&c." must disappear.

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207. **Convergency of Infinite Series.**—An infinite series is convergent, as already observed at (111), when the more of its terms we add together, the nearer do we approach to some fixed numerical limit, the remote terms diminishing continually down to zero.

The fixed numerical limit here spoken of, could never be actually reached by adding term after term, on account of the terms being infinite in number, though the greater the number of terms we include in such an approximate summation, the more nearly does the rejected part of the series approach to zero. The limit, or *exact sum* of the series, is attained only when nothing is rejected; how this sum may be calculated with the most rigid accuracy—in a great variety of instances—has been sufficiently explained in the preceding sections.

A diverging series, on the contrary, is such that, take as many terms of it as we may, their sum does not approach to any fixed limit: there is no limit which, by including more and more of the terms, the sum may not exceed. That the series is diverging when the terms continually increase is obvious; but there are many series which have one of the two distinguishing features of converging series, namely, that have their terms continually diminishing down to zero, and are nevertheless diverging; the other peculiarity, namely, the existence of a numerical limit, which the sum of the terms, however numerous, cannot exceed, being absent. Such a series, for instance, is $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\ldots+\frac{1}{n}$,

for however remote the nth term may be, n more terms of the series would be $\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n}$, the sum of which evidently exceeds $\frac{1}{2n}\times n$, or $\frac{1}{2}$, however great n may be. It is of importance, therefore, to have some means of ascertaining whether a series, such as this, is convergent, and therefore summable—at least approximately—or whether it is divergent, and therefore not summable. A series such as $1-1+1-1+1-1+\ldots$, in which the terms neither increase nor diminish, has been called a *neutral* series. Its sum, if we take an even number of terms, is 0; if we take an odd number it is 1; its sum to *infinity*, since ∞ is as much even as odd, is ambiguously 0 or 1. *Some* writers say it is $\frac{1}{2}$, but that is impossible, for no *fractions* enter the terms. The test of convergency for a series, whose terms are all positive, is the following:—

THEOREM I. An infinite series, of which the terms are all positive is convergent, when, commencing at any term however remote, the quotient, arising from dividing that term by the immediately preceding one, is less than unit.

Let the series be $\ldots+e+f+g+h+\ldots$, and such that $f\div e, g\div f, h\div g$, &c. are each less than a certain number r, then $f<er$, $g<fr$, $h<gr$, &c., $\therefore f<er$, $g<er^2$, $h<er^3$, and so on, $\therefore e+f+g+h+\ldots<e+er+er^2+er^3+\ldots$ Let now r be less than 1, then (85) $e+f+g+h+\ldots<\frac{e}{1-r}$, so that the sum of the series on the left is a number less than that on the right; the series is therefore convergent, the terms, commencing at e, arising from continually multiplying by fractions less than unit, being, in the aggregate, less than a given finite number.

This conclusion remains undisturbed, though the multiplying fractions continually increase, provided that the nth fraction, however great n may be, even $n=\infty$, is *less* than 1, but not otherwise. Thus, in the harmonic series instanced above, since $\frac{1}{n}\div\frac{1}{n-1}=1-\frac{1}{n}$ is not less than 1, when $n=\infty$, the series is divergent.

Again: take the series $x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\ldots\frac{x^n}{2.3.4\ldots n}$.

Here $\frac{x^n}{2.3.4\ldots n}\div\frac{x^{n-1}}{2.3\ldots(n-1)}=\frac{x}{n}$.

Now, however great be the value given to x, a value still greater may be given to n: hence the proposed series will at length become convergent. The series for e^x is thus always convergent, whatever real value be given to x.

THEOREM II. If the terms of a continually-decreasing series are alternately $+$ and $-$, the series is a converging series.

$$\text{For } a-b+c-d+\ldots=\left\{\begin{matrix}(a-b)+(c-d)+(e-f)+\ldots\\ \text{or, } a-(b-c)-(d-e)-(f-g)-\ldots\end{matrix}\right\}\ [1],$$

where if a, b, c, d, &c. continually decrease, the terms within the brackets are all *positive:* hence the sum Σ of the series is such that $\Sigma>(a-b)$, and $<a$; Σ, therefore, has a determinate numerical value, so that the series

is convergent. In like manner, from [1], $\Sigma > (a-b)+(c-d)+(e-f)$, and $< a-(b-c)-(d-e)$, and so on: hence, in approximating to Σ, by summing term after term, we see that the error committed is always numerically less than the term immediately following that at which we stop: the error is in excess if we stop at a positive term (as e), and in defect if we stop at a negative term (as f): we have just seen that by stopping at e, Σ is made too great, yet if f be subtracted, Σ, as the first inequality shows, will be made too small: by stopping at e, therefore, no error so great as f is made, and so on.

But if the terms of the converging series are all united by the sign *plus*, a limit to the amount of error arising from taking the sum of a finite number of its terms for Σ is found thus. Take the term immediately following that at which the summation is stopped, and form a geometrical series, having this for its first term, and for the common ratio, the *greatest* ratio that can be furnished by two consecutive terms of the neglected portion of the series; it is plain that the sum of this geom. prog. continued to infinity must exceed the sum of the neglected terms: we shall therefore thus have a limit which the error committed cannot be large enough to reach. For example:—

Required the limit of error committed in summing ten terms of the series $e=2+\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}\ldots$, as at p. 80. Here the general ratio of two consecutive terms is $\frac{1}{2.3.4\ldots n(n+1)} \div \frac{1}{2.3.4\ldots n} = \frac{1}{n+1}$, which continually diminishes as n increases: hence the ratio of the first two terms of the neglected portion of the series is greater than the ratio of any two consecutive terms more remote. The limit of error, therefore, in summing only n terms, is the sum of a geom. prog., commencing with the $(n+1)$th term, and in which the common ratio is $1 \div (n+1)$. This sum, by (85), is

$$\frac{a}{1-r} = \frac{1}{2.3.4\ldots n(n+1)} \div \left(1-\frac{1}{n+1}\right) = \frac{1}{n} \cdot \frac{1}{2.3.4\ldots n} \therefore \text{Error} < \frac{1}{n} \cdot \frac{1}{2.3.4\ldots n},$$

so that when $n=10$, we have Error $< \frac{\cdot 1}{2.3.4\ldots 10}$, or $< \cdot 00000003$.

Examples for Exercise.

(1) Prove that the series $1+2x+3x^2+4x^3+\ldots$ is always convergent for $x<1$.

(2) Prove that the series $1+2x+2.3x^2+2.3.4x^3+\ldots$ is always divergent.

(3) Is the series $\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+\frac{4}{7.9}\ldots$ convergent or divergent?

(4) Between what numerical limits for x is the series $x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\frac{x^4}{4^2}+\ldots$ always convergent?

(5) What is the limit of error in summing five terms of the series

$$\text{Nap. log } \frac{1}{2} = -\left(\frac{1}{2}+\frac{1}{8}+\frac{1}{24}+\frac{1}{64}+\frac{1}{160}+\frac{1}{384}+\ldots\right)?$$

Show that the error is always double of the first of the neglected terms.

Scholium.—In the foregoing discussions on infinite series, whenever the series considered is to be taken as complete in itself, without any

supplemental correction, we have uniformly written a row of dots after a few of its leading terms; but when the series is such that, in particular cases of it, it may become divergent, and therefore equal to its involopment only when a supplementary correction is appended to it, we have, in general, instead of dots, introduced the "&c." And it would be well if this distinction were always observed by writers on Series. (See the Note, p. 88.)

[The treatise on Algebra which we here conclude—on account of the multiplicity of topics comprehended under that name—has occupied a considerable portion of our space, but not more, we submit, than the extent and importance of the subject justifies. It is, indeed, difficult to determine, at the present day, where Algebra ends: in recent times, the researches of Professor Sir W. R. Hamilton have opened up entirely new fields of algebraic investigation—and so have those of Mr. Cayley, Professor Sylvester, and Mr. Spottiswoode. For an account of these the student is referred to Hamilton's "Lectures on Quaternions," and to Salmon's "Lessons on the Higher Modern Algebra."]

II. PLANE TRIGONOMETRY.

208. **Trigonometry** signifies the measurement of *Triangles*. It is divided into two parts:—Plane Trigonometry, and Spherical Trigonometry; the former being devoted exclusively to triangles traced upon a plane surface: the latter to triangles described upon the surface of a sphere.

But plane trigonometry is not, at present, thus limited in meaning: it is now understood to comprehend the general theory of plane angles, whether they belong to a triangle or not; it therefore divides itself into two distinct branches:—the doctrine of the plane triangle, and the theory of angular magnitude in general. It appears to us that the distinction here adverted to should be strictly preserved in an elementary treatise, and that what suffices for the plane triangle only—the simplest of all rectilinear figures—should be taught first; and, with this preparation, that the Trigonometrical Analysis, in general, should then be introduced to the student. This is analogous to what is done in Algebra: nobody would think of commencing with the general theory of equations, and of then deducing, from results of the widest application, special rules for a common quadratic. In the following treatise, after the definitions and necessary first principles, are laid down, the subject will be delivered under three heads, and in the order following:—

First. Plane Trigonometry properly so called: that is, the calculation of the sides and angles of plane triangles; as also of certain figures compounded of these.

Secondly. The theory of Angular Magnitude in general, or as it is more commonly called—The Trigonometrical Analysis.

And thirdly. The extension of that analysis to the developments of certain trigonometrical functions in the form of infinite series.

An *angle* is a magnitude of a peculiar kind: we cannot speak of the length or breadth of an angle: we call it a magnitude, or a measurable

quantity, only in reference to its amount of opening: this is numerically expressed as follows:—

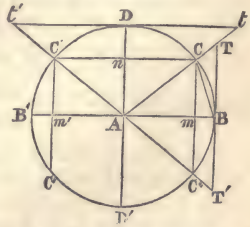
209. **Measurement of Angles.**—If the circumference of any circle be divided into 360 equal parts, each part is called a *degree* of that circle. If the degree be divided into 60 equal parts, each part is called a *minute* of that circle. And if the minute be, in like manner, divided into 60 equal parts, each part is called a *second* of that circle. The subdivision might, in this way, be carried on to *thirds*, *fourths*, &c.; but portions of an arc, smaller than the second, are usually expressed as fractions or decimals of the *second*. In speaking of an arc of a circle of so many degrees, minutes, &c., no information is afforded as to the linear *magnitude* of that arc:—we merely refer to the part which it is of the whole circumference: thus, if an arc of 30 degrees is referred to, we know that the twelfth part of the circumference of a circle is meant, but of the actual length of this twelfth part we can conclude nothing, unless we know the magnitude of the circle. It is otherwise, however, with regard to the angle at the centre of the circle which that arc subtends:—the mere degrees, &c., in the arc imply the *magnitude* of the angle. Arcs very different in magnitude may contain the same number of degrees; but the subtended angle is invariably the same, as to amount of opening, for all. If the subtending arc be one of 30 degrees, the subtended angle at the centre will be just the third part of a right angle; for the fourth part of a circumference or 90 degrees subtends a whole right angle at the centre, and (Euc. Prop. 33, B. VI.) the angles vary as the arcs subtending them. Arcs of circles, thus viewed in reference, not to their lengths, but to the number of degrees they contain, are therefore said to *measure* the angles they subtend at the centre: an angle of 1 degree is the 90th part of a right angle, and an angle of n degrees is the $\frac{n}{90}$ of a right angle.

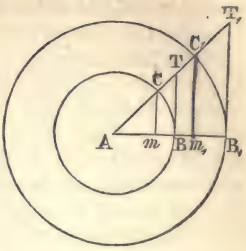
For the words *degree*, *minute*, *second*, a particular notation is used: thus 28° 43′ 12″ means 28 degrees, 43 minutes, 12 seconds. An angle having this measure would be the same part of a right angle that 28° 43′ 12″ is of 90°.

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210. *Definitions.*—Although, as its name implies, trigonometry has for its main object the calculation of the angles and sides of a triangle, yet an angle itself never directly enters into the requisite computations. Certain dependent quantities, now to be considered, enter instead.

211. **Sine.**—The sine of an *arc*, BC, is the straight line Cm, drawn from its extremity, C, at right angles to the diameter BB' from the commencement, or *origin* B, of that arc. If BC is an arc of 40°, then is Cm the sine of 40° of *that* circle. But, as remarked above, 40° of one circle may imply a very different length of arc from 40° of another, so that the sine of 40 —or as it is more briefly written, sin 40°—is not a line of invariable length. Yet the ratio of sin 40° to the radius of the circle to which the arc belongs—be this circle great or small—*is* of

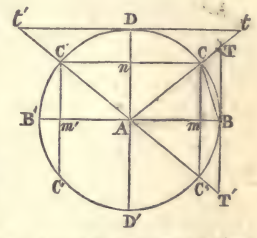




invariable amount: a glance at the annexed diagram is sufficient to justify this assertion; for however the two arcs BC, B_1C_1, may differ in length, provided only that each is the same part of the whole circumference to which it belongs, the ratio $\frac{Cm}{AC}$, or $\frac{C_1m_1}{AC_1}$, of the sine to the radius, is invariable (Euc. Pr. 4, B. VI.), like the angle A itself, which either arc, indifferently, measures. And it is this invariable ratio that is called the *sine of the angle*. The sine of an *angle* is therefore an *abstract number*, while the sine of the arc which measures it is a straight line.

From what is here said, the student will at once perceive that if we always regard 1 to be the numerical representative of the radius of the circle, the sine of any arc will have for *its* numerical representation the number which is strictly the sine of the angle that arc subtends at the centre; and thus it is common to say that *the sine of an angle is the sine of its subtending arc, to radius* 1.

212. The four equal arcs into which the circumference is divided by the two perpendicular diameters BB', DD', are called *quadrants: BD* is the first quadrant, DB' the second, $B'D'$ the third, and $D'B$ the fourth. The sine DA, of the first quadrant, is = radius; therefore the sine of a right angle, or sin $90°=1$: it is plain that no sine can exceed this limit; the sines increase from sin 0°, which is 0, up to sin 90°, which is 1; and they then as gradually diminish down to sin 180°, which is also 0; so that the sine of any angle—as the angle BAC'—greater than 90°, is a proper fraction, or a decimal: whatever part the *linear* sine $C'm'$ is of the radius DA, the same part, of course, is the *numerical* sine, that is, the *trigonometrical* sine of the angle BAC', of 1.



213. **Cosine.**—The cosine of an arc, BC, is the part Am, of the diameter BB', which is intercepted between the centre A, and m, the foot of the sine. The numerical value of this intercept, conformably to the scale radius=1, is the trigonometrical cosine of the angle BAC. We use the prefix *trigonometrical*, merely to impress upon the mind that we are here speaking of the sines and cosines exclusively employed in trigonometry: we shall dispense with it in future, taking for granted that the student will recollect that the sine or cosine of an *angle* is always an abstract number. He may satisfy himself—by a reference to Euclid as above—that the ratio $\frac{Am}{AC}$ is invariable, whatever be the length of the radius AC: it is this constant ratio that is the cosine of the angle CAB. The sines and cosines of *arcs*, that is, the *linear* sines and cosines, do not enter into the investigations of trigonometry.

If the angle be a right angle, its cosine is 0; that is, cos $90°=0$; for if C coincide with D, the intercept Am vanishes. The cosine increases as the angle diminishes; for the nearer C is to B, the greater does the

intercept Am become, till C coincides with B, when the angle CAB vanishes, and the cosine becomes 1; that is, cos 0°=1. The cosine, Am', of an angle BAC' greater than 90°, increases—in an opposite direction to Am—till the obtuse angle reaches its utmost limit 180°, or till C' coincides with B', in which case m' also coincides with B', so that cos 180°=1, its linear representation lying in the opposite direction. To mark this opposition of direction, the *minus sign* is always prefixed to the cosine of an obtuse angle, so that we write cos 180°=−1. In like manner, if the angle BAC' be 150°, and if the numerical value of the cosine of it be p, we should write cos 150°=$-p$. This distinction of algebraic sign is important:—precision requires it. In the absence of it, we could not discover, upon being told that the cosine of an angle A is $\frac{3}{4}$, whether that angle was less or greater than 90°; as it is, however, we know that the statements cos $A=\frac{3}{4}$, and cos $A'=-\frac{3}{4}$, imply two different angles, the latter, A', being as much greater than 90°, as the former, A, is less than 90°.

214. **Tangent, Secant.**—The tangent of an arc BC is the straight line BT drawn from B, the origin of the arc, touching the arc at that point, and terminating in AT, the line from the centre through the end C of the arc. And this line, AT, is the *secant* of the arc BC. If the arc exceed 90°, as the arc BC', the tangent of it BT' takes the opposite direction, which opposition is marked, as in the case of the cosine, by the minus sign. It is easy to see that the secant continually varies its direction as the arc varies in length:—the secant of the arc BC', conformably to the above general definition, is AT'.

The ratio of the linear tangent or secant to the radius, is the tangent or secant of the *angle* BAC, or BAC': each ratio, as it is easy to see, is invariable, whatever be the length of the radius: taking, as before, 1 for the numerical representation of the radius, the numerical values of the lines just defined *are* these ratios.

If C coincide with B, that is, if the arc be 0, the tangent BT vanishes, and the secant AT becomes AB, the radius: $\therefore$ tan 0°=0, sec 0°=1. But if C coincide with D, it is plain that AT, BT never meet, so that tan 90°=∞, and sec 90°=∞. For an obtuse angle the tangent like the cosine is *negative*.

215. **Cotangent, Cosecant.**—If a line Dt be drawn from D, the extremity of the first quadrant, touching the arc at that point, and continued till it meet the secant of BC, or that secant prolonged, in t, the touching line Dt is called the *cotangent*, and At, the cosecant of that arc. In like manner, Dt' is the cotangent and At' the cosecant of the arc BC'. The ratio of each of these lines to the radius, or the numerical values of the lines themselves, to the scale radius=1, are the cotangent and cosecant of the angle BAC, or BAC'.

216. The ratios or abstract numbers to which the above *six* names are given, comprehend all that are peculiar to trigonometry. There is, however, another term sometimes used: the term *versed sine*; it is employed to denote the excess of unity above the cosine of an angle; thus vers A means 1−cos A. The versed sine of an arc BC is the line $Bm=AB-$

Am. Also covers A means $1-\sin A$. In reference to the arc BC, it is the line Dn.

217. **Complement.**—Whatever must be added (*algebraically* added) to an arc or angle to make up 90°, is called the *complement* of that arc or angle: thus, DC is the comp. of BC, and DC' (taken negatively) is the comp. of BC'. Also 50° is the comp. of 40°, and −50° is the comp. of 140°. The *co*sine, *co*tangent, &c., of an arc or angle, are no other than the sine, tangent, &c. of the *complement* of that arc or angle, regarding the complements of arcs as measured from D.

It must be remembered then that if two arcs or angles p, q, together make up 90°, each is the complement of the other, and that therefore $\sin p=\cos q$, $\cos p=\sin q$, $\tan p=\cot q$, $\cot p=\sin q$, &c.

218. **Supplement.**—Whatever must be added to an arc or angle to make up 180°, is called the *supplement* of that arc or angle: thus CB' is the supplement of the arc BC, and $C'B'$ the supplement of BC'. Also 140° is the supplement of 40°, 60° is the supplement of 120°, and so on. And it may here be noticed, in connection with the supplement, that mB' is called the *suversed sine* of the arc BC:—it is the diameter *minus* the versed sine mB. Of an angle A, the suversed sine, or as it is more frequently called, the suversine, is $2-\cos A$. It will have been seen, from what has preceded, that the *directive* signs (+ or −) prefixed to a cosine, tangent, or cotangent, have been suggested by the position of the geometrical lines in the diagram. To avoid all confusion and ambiguity about direction, the arc whose sine, cosine, tangent, &c., is to be determined, is always considered to have its origin or commencement at one and the same point B, and to be measured from that origin in one and the same direction $BCDC'$, &c. For instance, if we had to discuss these lines in reference to the arc CC', we should conceive the point C applied to B, or which is the same thing, should replace CC' by an arc, equal to it, measured from B, in the direction BCD. And this is the way supplemental arcs are considered: the supplement of BC is CB', as to length, or number of degrees; but to determine its sine, cosine, &c., we conceive C to be brought down to B, and the arc, thus displaced, to terminate at C', so that $C'm'$ is the sine of that supplement, and Am' its cosine. As a supplement is what the arc wants of 180°, or of a semicircumference, it is plain that if BC' is equal to CB, we must have $BC=B'C'$, and $Cm=C'm'$, as also $Am=Am'$: hence an arc or angle has the same sine as the supplement of that arc or angle: the cosine too is the same in magnitude for both, but the signs of direction (+ or −) are opposite: thus, if $\sin 40°=p$, then also, $\sin 140°=p$; but if $\cos 40°=q$, then $\cos 140°=-q$. The signs of direction prefixed to the trigonometrical ratios, thus have their origin in the linear representation of those ratios:—in a diagram, fixing direction as well as length: we could not with propriety speak of a ratio, or abstract number, taking a directive sign, unless that ratio or number were derived from geometrical considerations. The lines in the diagram at (213), and which have been defined above, may always be regarded as linear representations of the trigonometrical ratios bearing the same names, the radius representing 1; so that many properties and relations of these ratios may be established by help of their representative lines, as in the following article.

219. **Fundamental Trigonometrical Relations.**—Referring to the diagram at art. (213), and remembering that the trigonometrical

radius is 1, the 47th prop. of Euclid's first book furnishes the following relations, where, for simplicity of notation, $(\sin A)^2$, $(\cos A)^2$, &c. are written $\sin^2 A$, $\cos^2 A$, &c.

$$\left.\begin{array}{ll} \sin^2 A+\cos^2 A=1^2 & \therefore \sin A=\sqrt{(1-\cos^2 A)},\ \cos A=\sqrt{(1-\sin^2 A)} \\ 1^2+\tan^2 A=\sec^2 A & \therefore \sec A=\sqrt{(1+\tan^2 A)},\ \tan A=\sqrt{(\sec^2 A-1)} \\ 1^2+\cot^2 A=\operatorname{cosec}^2 A & \therefore \operatorname{cosec} A=\sqrt{(1+\cot^2 A)},\ \cot A=\sqrt{(\operatorname{cosec}^2 A-1)} \end{array}\right\} \quad [1].$$

Again: since the sides about the equal angles of equiangular triangles are proportional (Euc. Prop. 4, B. VI.), by comparing together the equiangular triangles TAB, CAm, and the equiangular triangles $t'AD$, $C'Am'$, where $Am'=Am$, and $Cm'=Cm$, we have the relations

$$\left.\begin{array}{l} \dfrac{\tan A}{1}=\dfrac{\sin A}{\cos A},\ \dfrac{\cot A}{1}=\dfrac{\cos A}{\sin A},\ \dfrac{\cos A}{1}=\dfrac{1}{\sec A},\ \dfrac{\sin A}{1}=\dfrac{1}{\operatorname{cosec} A}, \\ \text{that is, } \tan A=\dfrac{\sin A}{\cos A},\ \cot A=\dfrac{\cos A}{\sin A},\ \sec A=\dfrac{1}{\cos A},\ \operatorname{cosec} A=\dfrac{1}{\sin A} \end{array}\right\} \quad [2].$$

The following pairs of quantities therefore, namely, tan A, cot A; cos A, sec A; sin A, cosec A, are the reciprocals of each other; and this it will be necessary to recollect.

The foregoing fundamental relations show that from the two trigonometrical quantities sin A, cos A, all the others may be derived, as well as regards sign, as numerical value: we thus see, for instance, that the sign of the secant of an angle is the same as that of the cosine, and that the sign of the cosecant, is the same as that of the sine.

220. All the quantities in the preceding equations are *numbers*:—the trigonometrical ratios: they apply exclusively to *angles*. But they may easily be converted into geometrical relations, applicable to the sines, cosines, &c., of arcs belonging to any circle. If we write the terms SINE, COSINE, &c., in capitals, when we refer to arcs instead of angles, we have only to replace sin A, cos A, &c., above, by $\dfrac{\text{SIN } A}{R}$, $\dfrac{\cos A}{R}$, &c., where R represents the radius of the circle to which the arc A belongs. And to this R we may, of course, afterwards attribute any numerical value we please. We shall now give an example or two in reference to the relations marked [1] and [2] above.

(1) Required the sine of 45°. Here $A=45°=90°-45°$: hence (217)

$$\sin A=\cos A \therefore [1],\ \sin^2 A+\cos^2 A=2\sin^2 A=1$$

$$\therefore \sin A=\sqrt{\frac{1}{2}}=\frac{1}{2}\sqrt{2}=\cos A.$$

(2) Required the tangent, cotangent, &c., of 45°. Dividing the sine by the cosine, we have [2], $\tan 45°=\frac{1}{2}\sqrt{2}\div\frac{1}{2}\sqrt{2}=1$, also $\cot 45°=1$; and since the secant is the reciprocal of the cosine, and the cosecant of the sine, we have $\sec 45°=1\div\frac{1}{2}\sqrt{2}=\sqrt{2}$, and $\operatorname{cosec} 45°=\sqrt{2}$.

(3) Required the sine, cosine, &c., of 60°. This is the measure of each angle of an equilateral triangle, since the three equal angles amount together to two right angles, or 180°. If from the vertex of such a triangle a perp. be drawn, it will bisect both the base and the vertical angle, dividing the equil. triangle into two equal right-angled triangles.

Let CAC'', in the diagram at p. 171, be an equil. triangle, CC'' being equal to the radius of the circle passing through C, C'': then $BAC=30°$, therefore to radius $AB=1$, $Am=\frac{1}{2}=\sin 30°$, $\therefore$ [1],

$$\cos 30°=\sqrt{\left(1-\frac{1}{4}\right)}=\frac{1}{2}\sqrt{3}.$$

Also [2], $\tan 30°=\frac{\sin 30°}{\cos 30°}=\frac{1}{\sqrt{3}}=\frac{1}{3}\sqrt{3}$, $\cot 30°=\sqrt{3}$,

$\sec 30°=\frac{1}{\cos 30°}=\frac{2}{\sqrt{3}}=\frac{2}{3}\sqrt{3}$, $\operatorname{cosec} 30°=\frac{1}{\sin 30°}=2$.

Since the complement of 30° is 60°, we infer from these values (217) that $\sin 30°=\cos 60°=\frac{1}{2}$, $\cos 30°=\sin 60°=\frac{1}{2}\sqrt{3}$, $\tan 30°=\cot 60°=\frac{1}{3}\sqrt{3}$, $\cot 30°=\tan 60°=\sqrt{3}$, $\sec 30°=\operatorname{cosec} 60°=\frac{2}{3}\sqrt{3}$, $\operatorname{cosec} 30°=\sec 60°=2$.

221. **Tables.**—But it would not be possible, in this way, to determine the trigonometrical ratios for all values of the angle A. For this general purpose, certain series have been investigated, and certain algebraic expedients devised, which will be explained hereafter. With such aid, the sines, cosines, &c., for all values of A, from $A=0°$, up to $A=90°$, have been computed to several places of decimals, and arranged in *Tables*. For values of A above 90°, such computations would be superfluous, for the sine, cosine, &c., of an angle above 90°, are the same, in numerical value, as the sine, cosine, &c., of an angle as much below 90°. A mere inspection of the diagram at p. 171 is sufficient to show this.

There are two kinds of trigonometrical tables. The first contains the several trigonometrical ratios, computed as above hinted. This is called a table of *natural sines, cosines, &c.* The second is a table of *logarithmic sines, cosines, &c.* These are not strictly the logs of the natural sines, cosines, &c.:—they are these logs each increased by the number 10, which addition to them is made solely to avoid the inconvenience of negative logs. The trigonometrical sines and cosines, as we have already seen, are all less than 1; so that the logs of these are all negative: by increasing each log by 10, we in effect compute the table to the radius $R=10^{10}$, instead of to $R=1$. And we have seen (220) that the trigonometrical ratios remain the same whatever be R.

222. Before concluding these introductory observations, there is one other inference, from the diagram at p. 171, to which the attention of the student must be directed. The *chord* CC'' of an arc CBC'' is obviously double the sine Cm of half that arc; and still regarding the circle to which the arc belongs as the *trigonometrical circle*—or the circle whose radius is the linear representation of the numerical unit—we infer that the chord of any angle is twice the sine of half that angle, or that $\text{chd } A=2\sin\frac{1}{2}A$. Now by Euc. 8, VI., the chord CB is a mean proportional between BB', Bm; that is, $CB^2=BB'\times Bm$, giving to the lines their numerical values, but $BB'=2$,

$$\therefore 2(1-\cos A)=2\text{ vers }A=(2\sin\tfrac{1}{2}A)^2\ \therefore\ \text{vers }A=2\sin^2\tfrac{1}{2}A,$$

a property that will be called in request in investigating the third case of oblique-angled triangles.

Having thus disposed of the necessary preliminaries, we may now proceed to the business of Trigonometry Proper.

Part 1. Solution of Plane Triangles.

223. **Right-angled Triangles.**—There are two cases to be considered:—1. *When an acute angle and a side are given.*—Rule 1. With the vertex of the given angle as centre, and the given side, terminating in that vertex, as radius, imagine an arc to be described, and notice what *trigonometrical name*, in reference to that arc—whether sine, or cosine, or tangent, &c., the sought side takes.

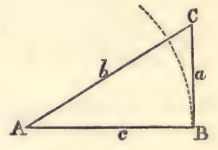
2. Enter the table of natural sines, &c., with the given angle, and take out the number under that *name*. Multiply the given side by that number: the product will be the sought side.

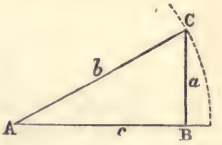
[Note.—When one of the acute angles of a right-angled triangle is given, the other is virtually given: for if A be one, $90°-A$ is the other].

2. *When two sides are given to find an angle.*—Rule 1. With the vertex of the required angle as centre, and a given side, terminating in that vertex, as radius, imagine an arc to be described, and notice what trigonometrical name the other given side takes.

2. Divide this latter by the former: the quotient will be the tabular number of the *same name*, over which, in the table, the value of the required angle will be found.

For, in the right-angled triangle ABC, let, first, the side $AB=c$, and the angle A be given: then if with centre A, and radius AB, we imagine an arc to be described, as in the diagram, we see that BC or a becomes the *tangent* of that arc, and AC or b, the *secant*; and we know, whatever be the length of our radius, that we have the constant ratio $\frac{a}{c}=\tan A$, and $\frac{b}{c}=\sec A$. Next, let the hypotenuse $AC=b$, and the angle A, be given: then if with centre A, and radius AC, we imagine an arc to be described, it is plain that BC or a becomes the sine of that arc, and AB or c the cosine; and we know, whatever be the length of our radius, that we have the constant ratio $\frac{a}{b}=\sin A$, and $\frac{c}{b}=\cos A$, $\therefore a=c\tan A$, $b=c\sec A$, $a=b\sin A$, $c=b\cos A$, and





$$\tan A=\frac{a}{c},\ \sec A=\frac{b}{c},\ \sin A=\frac{a}{b},\ \cos A=\frac{c}{b}.$$

results which sufficiently establish the foregoing rules. From the expressions for $\tan A$, and $\sin A$, and from (219), it is plain that $\cot A=\frac{c}{a}$, and $\operatorname{cosec} A=\frac{b}{a}$; and therefore that $c=a\cot A$, and $b=a\operatorname{cosec} A$.

Note 1.—The following examples, on the solution of right-angled triangles, are all worked by the table of natural sines, cosines, &c. In the

case of right-angled triangles, the operations are, in general, easier by the natural numbers than by logarithms. Prefixed to the tables will be found all the necessary directions for using them. In the common applications of plane trigonometry it is but seldom requisite to compute the angles nearer than to degrees and minutes.

2. In the table of natural sines, &c., the secants and cosecants are not inserted: for since $\sec A=\frac{1}{\cos A}$, and $\operatorname{cosec} A=\frac{1}{\sin A}$, we may change multiplication by sec A, or by cosec A, for division by cos A, or by sin A; and division by sec A, or by cosec A, for multiplication by cos A, or by sin A. In all multiplications and divisions by large numbers, where several decimals are concerned, it is better to use the contracted methods, as explained in the Arithmetic. (See Weale's Rudimentary Treatise.)

When any two sides of a right-angled triangle are given to find the third side, the solution is effected by the 47th of Euclid's first book; and no aid from trigonometry is required. Putting for the sides opposite to the angles A, B, C, the corresponding small letters a, b, c, if B be the right angle, we have, by the prop. referred to,

$b^2=a^2+c^2 \therefore b=\sqrt{(a^2+c^2)}$, $a=\sqrt{(b^2-c^2)}=\sqrt{\{(b+c)(b-c)\}}$, $c=\sqrt{\{(b+a)(b-a)\}}$.

(1) In the right-angled triangle ABC, are given $c=174$ feet, and $A=18°\ 19'$, to find the sides a and b.

Here $a=c \tan A$, and $b=c \sec A=c\div\cos A$ (art. 219)

tan 18° 19′ = ·33104	cos 18° 19′ = ·9,4,9,3,3)	174	(183·29 = b
c, reversed, = 471		94933	
33104		79067	
23173		75946	
1324		3121	
a=57·601		2848	
		273	
		190	
		83	

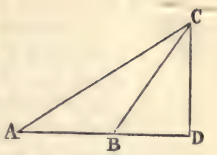
Hence the side BC=57·6 feet, and the side AC=183·29 feet.

(2) From the edge of a ditch 18 feet wide, which surrounded a fort, the angle of elevation of the top of the wall was taken, and found to be 62° 40′: required the height of the wall, and the length of a ladder necessary to scale it. Here c=18 feet, and A=62° 40′. And tan 62° 40′ × 18 = 1·93470 × 18 = 34·82 feet = a, the height of the wall, 18 ÷ cos 62° 40′ = 18 ÷ ·45917 = 39·2 feet = b, the length of the ladder.

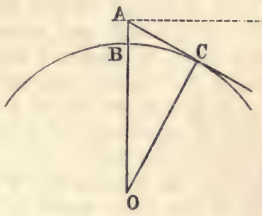
(3) The hypotenuse AC is 643·7 feet, and the base AB, 473·8 feet: required the angles A, C, and the perpendicular, BC. Here $\frac{AB}{AC}=\cos A$, that is, 473·8 ÷ 643·7 = ·73606 = cos 42° 36′ $\therefore A$=42° 36′, and C=90° − A=47° 24′. Also (Euc. 47 of I.), $\sqrt{(643·7^2-473·8^2)}$=435·87 feet=BC.

(4) The tower DC of an enemy's fortress cannot be safely approached nearer than B: at this point the angle of elevation CBD of the top is found to be 55° 54′. The observer, upon going back a further distance BA of 100 feet, finds the angle of elevation A to be 33° 20′: required the height of the tower and its distance from B.

$DA=CD$ tan ACD, $DB=CD$ tan BCD, that is, $DA=CD$ cot A, $DB=CD$ cot CBD $\therefore$ $AB=DA-DB=CD(\cot 33^\circ 20' - \cot 55^\circ 54')=CD(1{\cdot}52043-{\cdot}67705)=100$ $\therefore$ $CD=100\div{\cdot}84338=118{\cdot}57$ ft., the height of the tower. And $\therefore$ CD cot $CBD=118{\cdot}57\times{\cdot}67705=80{\cdot}28$ ft., the distance BD.



(5) From the top of a mountain BA, m miles high, the angle of depression EAC, below the horizontal line AE, of the remotest visible point of the surface of the sea, is taken, and found to be θ: it is required to determine the radius OC of the earth.



Since $OAE=90^\circ$, EAC is the complement of OAC, but O is also the complement of OAC $\therefore$ $O=EAC=\theta$.

Now $OA=OC$ sec $\theta=OB+BA$

$\therefore$ $BA=m=OC$ sec $\theta-OC=OC(\sec\theta-1)$ $\therefore$ $OC=\dfrac{m}{\sec\theta-1}$, or, since

$$\sec\theta=1\div\cos\theta,\ OC=\frac{m\cos\theta}{1-\cos\theta}\text{ miles.}$$

NOTE.—This would be a very expeditious method of determining the earth's diameter, if the refraction of the atmosphere, near the horizon, could be accurately estimated. But owing to the fluctuations of this horizontal refraction, it is of uncertain amount; so that the foregoing method cannot be regarded as giving more than a somewhat rough approximation to the truth.

The radius of the earth being known, however, from other investigations, it is easy to find—sufficiently near for the purpose, and without the aid of trigonometry—at what distance at sea the top of a mountain of known height just emerges above the distant horizon. For, calling the height h, the distance d, and the radius of the earth R, we know by Euclid, Prop. 36, B. III., that $(2R+h)h=d^2$. As h^2 is comparatively insignificant, we may neglect it: we shall then have $d^2=2Rh$. Thus, R being about 4000 miles, if h be 1 mile, $d=\sqrt{8000}=89$ miles. Also, the distance being known, we may compute the height of the mountain from $h=\dfrac{d^2}{2R}$; thus, if a ship just loses sight of the top of a mountain, when at the distance of 30 miles from it, its height will be

$$\frac{900}{8000}=\frac{9}{80}\text{ miles }=594\text{ feet.}$$

EXAMPLES FOR EXERCISE.

(1) The base of a right-angled triangle is 246 feet, and the angle at the base 33° 45′: required the perpendicular and hypotenuse.

(2) The hypotenuse is 430 yards, the perpendicular 214 yards: required the angle between them.

(3) Given $c=73$, $A=52^\circ 34'$ to find a, b.

(4) Given $c=288$, $A=53^\circ 8'$ to find a, b.

(5) Given $a=85$, $b=111{\cdot}4$ to find A, c.

(6) Given $a=75{\cdot}18$, $c=53{\cdot}42$ find b, A, C.

(7) Given $c=138{\cdot}24$, $C=52^\circ 57'$ find A.

(8) The perp. is double the base: required the angles.

(9) A tower $53\frac{1}{2}$ feet high, is surrounded by a ditch 40 feet broad: the angle of the elevation of the top of the tower from the opposite bank is $53° 13'$: required the length of a ladder sufficient to scale the tower.

(10) If the top of a mountain, known to be 6600 feet high, can just be seen at sea 100 miles off, what must be the diameter of the earth?

(11) From the top of a ship's mast, 80 feet above the water, the angle of depression of another ship's hull was found to be $20°$: required the distance between the two ships.

(12) A ladder, 40 feet long, is so placed as to reach a window 33 feet from the ground; and upon being turned over, the foot remaining in the same place, it reaches a window 21 feet from the ground, on the opposite side of the street: required the breadth of the street.

(13) From the top of a castle standing on a hill by the sea-side, the angle of depression of a ship at anchor was observed to be $4° 52'$; at the bottom of the castle, or top of the hill, the angle of depression was $4° 2'$: required the horizontal distance of the ship, as also the height of the hill, that of the castle itself being 60 feet.

(14) The height of the mountain called the Peak of Teneriffe is about $2\frac{1}{5}$ miles; the angle of depression of the remotest visible point of the surface of the sea is found to be $1° 58'$: required the diameter of the earth, and the utmost distance at which an object can be seen from the top of the mountain—that is, the length of the line from the eye to the remotest visible point.

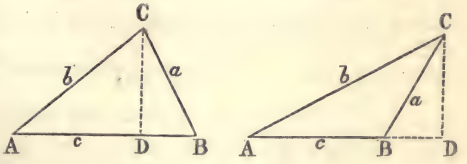
(15) The angles of elevation of a balloon were taken by two observers at the same time; both were in the same vertical plane as the balloon, and on the same side of it; the angles were $35°$ and $64°$, and the observers were 880 yards apart: required the height of the balloon.

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224. **Oblique-angled Triangles.**—Every triangle has *six parts* as they are called—three sides and three angles. If any three of the six be given, except they be the three *angles*, the remaining three may be determined by computation. It is plain that the three angles would not suffice for the determination of the triangle, because there may be an infinite variety of *equiangular* triangles all different in size. There are three cases to be considered.

225. Case I. *When two of the three given parts are opposite parts—an angle and a side.*

Let ABC be any triangle, the sides being denoted by the small letters a, b, c, and from either vertex, as C, let a perpendicular CD be drawn to the opposite side c. If the angles A, B, be both acute, CD will fall within the triangle: if one be obtuse, as in the second diagram, it will fall without the triangle, but in each case ADC, BDC, will be right-angled triangles. From the first of these we have $CD = b \sin A$: from the second, $CD = a \sin B$, the sine of an angle being the same as the sine of its supplement.



$$\therefore a \sin B = b \sin A \therefore \frac{a}{b} = \frac{\sin A}{\sin B}, \text{ or } a : b :: \sin A : \sin B,$$

that is, in any plane triangle *the sides are as the* SINES *of the angles opposite to them:* hence the following rule, which is applicable alike to oblique-angled and right-angled triangles.

RULE. As one side of a triangle is to another, so is the sine of the angle opposite the former, to the sine of the angle opposite the latter.

And this proportion it is usual, and in general, for oblique-angled triangles, more easy, to work by logarithms; that is, to subtract the log of the first term from the sum of the logs of the second and third terms: the remainder is the log of the fourth term.

NOTE.—If a side be required, the first term of the proportion is, of course, the *sine* of a given angle; but if an angle be required, the first term is a given side. As the fourth term is then a *sine:*—the sine of the sought angle, and as to every sine there corresponds *two* angles, the acute angle given in the tables, and the *supplement* of that angle (218), a doubt may sometimes exist as to which of these two—the acute angle or the obtuse one—really belongs to the triangle in question. Under the following circumstances, however, there can never be any ambiguity:—

1. When the given angle is *obtuse;* for a triangle cannot have *two* obtuse angles: the sought angle must, therefore, be *acute.*

2. When the given angle is *acute,* and it so happens that the side opposite to this is *greater* than the given side opposite to the sought angle; for, by Euc. Prop. 18, B. I., this latter angle must then be *less* than the former, and therefore acute also.

Moreover, in actual practice, the shape of the triangle will, in general, be sufficiently marked to enable us to pronounce at once whether any particular angle is acute or obtuse: but when mere numerical measurements are given, unaccompanied with a geometrical representation, or when the triangle is not actually figured before us, and the given angle is acute, and the side opposite to it happens to be *less* than the other given side, the example is said to belong to the *ambiguous case;* for whether we take, as belonging to our computed sine, the acute angle furnished by the tables, or the obtuse angle which is its supplement, *either* will subsist with the given conditions; and two distinct triangles, each satisfying those conditions, will be determinable. Example 3, following, is an instance of the ambiguous case of plane triangles.

(1) In a plane triangle ABC, are given the base AB or c=408, and the angles A, B,=58° 7′, and 22° 37′, respectively: required the other two sides. The angle C opposite to the side c, being the supplement of $A+B$=80° 44′, sin C=sin 80° 44′ (218). Hence by the rule,

	Logs.		Logs.
As sin C= sin 80° 44′	9·994295	As sin C= sin 80° 44′	9·994295
: sin A= sin 58° 7′	9·928972	: sin B=sin 22° 37′	9·584968
: : AB=408	2·610660	: : AB=408	2·610660
	12·539632		12·195628
: BC=351	2·545337	: AC=159	2·201333

NOTE.—In these operations the rule has been strictly followed: the first log in each proportion has been substracted from the sum of the other two logs. But a subtractive log may always be replaced by an additive one, and the work thereby shortened, thus: Instead of writing down from the table, the subtractive log, write down what that log wants

of 10, in this way: Looking at the leading figure, write what it wants of 9, then what the next figure wants of 9, and so on, till the last significant figure is reached, when there must be put down what *it* wants of 10. It is plain that in this way we shall get what the entire log wants of 10. This defect from 10 is called the *arithmetical complement* of the log, or its *co-log*. Now, if this colog be *added*, instead of the log itself being subtracted, the result will evidently exceed the true result by 10, which excess is allowed for without any trouble—we have only to dismiss 10 from the erroneous result. With this modification, the work of the above example will be as follows:—

As sin C = sin 80° 44′		0·005705 = colog	As sin C = sin 80° 44′		0·005705 = colog
: sin A = sin 58° 7′		9·928972	: sin B = sin 22° 37′		9·584968
:: AB = 408		2·610660	:: AB = 408		2·610660
: BC = 351		2·545337	:: AC = 159		2·201333

With a very little practice, a colog may be written down as quickly as the log itself can be transcribed.

226. We shall now exhibit the work of an example in which *seconds* are concerned. In the Table of log sines, cosines, &c., there will be found several columns headed "D:" these are columns of *differences*, each difference being placed between two consecutive numbers in the table. In some tables, these differences are those due to 1 minute, or 60 seconds, and are got by simply subtracting the greater of the two numbers from the less. The difference d, due to any smaller number (a) of seconds is found, from such tables, by the proportion $60 : a :: D : d$, so that $d = \frac{Da}{60}$. But in the tables intended to accompany the present work, the differences are those due not to 60 but to 100 seconds; so that in in these tables, $d = \frac{Da}{100}$; and thus d is found, somewhat more readily.

The most convenient way of correcting for the odd seconds is this: Take from the table the number answering to the degrees and minutes, and against it write the tabular D. Observe whether this D is additive or subtractive, and put against it the + or − accordingly.* When all the logs, with the corresponding tabular differences, have been transcribed, then compute for the seconds thus: multiply each D by the number of seconds in the angle against which that D stands; add up all the products, reject the last two figures of the sum, for the division by 100, and the remaining figures will furnish the proper correction of the result, as in the following example.

Note.—Observe that for an angle greater than 90°, the sign of D must be changed, and that, for a colog, D takes a sign opposite to that for the log. For the method of determining a *required angle* to seconds, see ex. 3.

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* As regards *acute* angles, D is always additive except for a *co*-quantity, when it is always subtractive. In the case of an *obtuse* angle, we enter the Table with the supplement of the degrees and minutes, and multiply the corresponding D by the given seconds, changing the sign as directed in the Note. If we replace the obtuse by the acute angle which is its complete supplement, the D will of course preserve its proper sign.

(2) Given $A=41°\,13'\,22''$, $B=71°\,19'\,5''$, $a=55$, to find b.

		Logs.	D	(*Parts for secs.*)
As sin A = sin $41°\,13'\,22''$		$0{\cdot}181175$=colog	-240	-5288
: sin B = sin $71°\,19'\,5''$		$9{\cdot}976489$	$+\ 71$	$+\ 355$
: : a =	55	$1{\cdot}740363$		
				$-49{,}33$
		$1{\cdot}898027$		
		-49=Correction for the seconds.		
: b =	$79{\cdot}063$	$1{\cdot}897978$		

If the correction for seconds had been additive, we should have placed it under the three tabular logs, and have added all up together. In using tables where the differences are to 60″, instead of 100″, in place of cutting off two figures from the sum of the "parts for seconds," we should divide by 60.

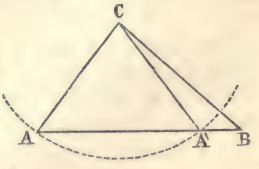
(3) Given $a=178{\cdot}3$, $b=145$, and $B=41°\,10'$, to find A, C, and c.

As $b=145$	$7{\cdot}838632$=colog
: $a=178{\cdot}3$	$2{\cdot}251151$
: : sin B = sin $41°\,10'$	$9{\cdot}818392$
: sin A = sin $54°\ 2'\,22''$	$9{\cdot}908175$
	$9{\cdot}908141$=sin $54°\,2'$
$\therefore A+B=$ $95°\,12'\,22''$	
$180°$	$D=153)\ 3400(22''$
$\therefore C=$ $84°\,47'\,38''$	

As sin B = sin $41°\,10'$	$0{\cdot}181608$= colog	(*Parts.*)	
: sin C = sin $84°\,47'\,38''$	$9{\cdot}998197$	$D=20$	$7{,}60$
: : $b=145$	$2{\cdot}161368$		
	8=Cor. for the secs.		
: $c=219{\cdot}37$	$2{\cdot}341181$		

$\therefore A=54°\,2'\,22''$, $C=84°\,47'\,38''$, $c=219{\cdot}37$

But the present example is in the *ambiguous case*. The log sine $9{\cdot}908175$ belongs equally to the angle $54°\,2'\,22''$, and to its supplement $125°\,57'\,38''$; and we have nothing here to guide us as to which of these is the angle A. We have proceeded above to calculate c on the supposition that A is *acute*. If, however, it be obtuse, then $C=180°-(125°\,57'\,38''+41°\,10')=12°\,52'\,22''$, and sin $C=9{\cdot}347890$. Substituting this, therefore, for sin C above, the sum of the three logs will be $1{\cdot}690866$, which answers to the number $c=49{\cdot}076$. Hence the triangle in question may be either of the two, ABC, $A'BC$, figured in the margin, where $CB=a$, and $CA=CA'=b$, the angle B being common to both triangles. In the larger triangle, the angle A is $54°\,2'\,22''$, and the side $AB=219{\cdot}37$, the opposite angle C being $84°\,47'\,38''$. In the smaller triangle, the angle A' is $125°\,57'\,38''$, and the side $A'B=49{\cdot}076$, the opposite angle C being $12°\,52'\,22''$.



Examples for Exercise.

(1) Given $A=22°\ 37'$, $B=134°\ 46'$, $c=351$, to find C, a and b.

(2) Given $A=44°\ 13'\ 24''$, $B=79°\ 46'\ 38''$, $c=368$, to find a and b.

(3) Given $b=153$, $c=137$, $B=78°\ 13'$, to find C and a, the former to seconds.

(4) Given $a=117$, $b=216$, $A=22°\ 37'$ to find B to seconds. Is the case ambiguous?

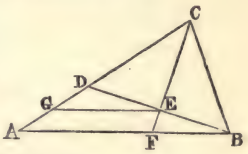
(5) Given two sides of a plane triangle 50 and 40 respectively, and the angle opposite the latter 32°; to determine the *two* triangles, to each of which these parts belong.

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227. Case II. *When the three given parts are two sides and the included angle.*

Let ABC be any plane triangle, in which the two sides CA, CB, and the included angle C, are given. If the given sides are *equal*, the solution comes under the case of right-angled triangles, because a perp. from C, bisecting both the angle C and the opposite side, we shall have the hypothenuse AC, and the vertical angle $\left(\frac{1}{2}C\right)$ given to find the base $\left(\frac{1}{2}AB\right)$. Each of the base-angles A, B, will be half the supplement of C.

Let, then, CA be the greater of the two given sides, and from it cut off $CD=CB$: join BD, and draw the perp. CE, which produce to F: lastly, draw EG parallel to AB. The perp. CE bisects the angle C, and also the line DB: hence (Euc. 2 of VI.),



$$DG=GA\ \therefore\ CG=\frac{1}{2}(BC+CD+DA)=\frac{1}{2}(BC+CA)=\textit{half the sum of the sides.}$$

$$\text{Also } AG=\frac{1}{2}(CA-CB)=\textit{half the difference of the sides.}$$

Again $(A+B+C)=(CBD+CDB+C)\ \therefore\ CBD=$*half the sum of A and B*, and ABD is *half the difference*, because, being added to the half sum, it gives the greater, B.

Now, by right-angled triangles, $CE=BE$ tan CBD, and $EF=BE$ tan ABD

$$\therefore\ CE : EF :: \tan CBD : \tan ABD.$$

But (Euc. 2 of VI.), $CG : GA :: CE : EF\ \therefore\ 2CG : 2GA :: \tan CBD : \tan ABD$, which proportion, in words, gives the following rule:—

Rule. As the sum of the two given sides is to their difference, so is the tangent of half the sum of the opposite angles, to the tangent of half their difference. [Note. $\text{Tan}\ \frac{1}{2}(A+B)=\cot\frac{1}{2}C.$]

The sum of A and B is got from subtracting C from 180°; and thus, having the half sum and the half difference, the angles themselves are obtained by addition and subtraction; and thence the third side AB, by last case. The above, as a formula, is

$$\frac{a+b}{a\sim b}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A\sim B)}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A\sim B)}\ \therefore\ \tan\frac{1}{2}(A\sim B)=\frac{a\sim b}{a+b}\cot\frac{1}{2}C.$$

(1) Given $a=137$, $b=153$, $C=40°\ 33'\ 12''$, to find the other parts.

$$A+B=180°-C=139°\ 26'\ 48''\ \therefore \tfrac{1}{2}(A+B)=69°\ 43'\ 24''.$$

As $a+b=290$	7·537602=colog.		
: $a\sim b=16$	1·204120	D	(*Parts.*)
: : tan 69° 43′ 24″	10·432291	+ 648	155,52
		156 Cor. for secs.	
: tan 8° 29′ 37″	9·174169		
Adding, $\therefore B=78°\ 13'\ 1''$	9·173634=tan 8° 29′		
Subtracting, $A=61°\ 13'\ 47''$	D=1442) 53500(37″		

		D	(*Parts.*)
As sin B, 78° 13′ 1″ (colog)	0·009250	−45	−45
: sin C, 40° 33′ 12″	9·812988	246	2952
: : $b=153$	2·184691		29,07
		29 Cor. for secs.	
: $c=101·615$	2·006958		

Examples for Exercise.

(1) Given $a=526$, $b=378$, $C=32°\ 18'\ 26''$, to find the other parts.

(2) Given $a=17·802$, $b=21·704$, $C=26°\ 12'\ 16''$, to find the remaining parts.

(3) Given $b=2065$, $c=1637$, $A=132°\ 7'\ 12''$, to find the remaining parts.

(4) Given $a=500$, $b=400$, $C=88°\ 30'$, to find the remaining parts.

(5) Given $a=960$, $c=1686$, $B=128°\ 4'$, to find A, C, and b.

(6) Given $a=1672$, $b=1014$, $C=45°\ 1'\ 3''$, to find A, B, and c.

Note.—A method of computing the side c, without first finding the angles A, B, will be given hereafter. (See p. 204.)

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228. Case III. *When the given parts are the three sides.*

By Euclid (12 and 13 of II.), we have the following values of AC^2, in the triangles at page 180, namely,

When B is acute, $AC^2=BC^2+BA^2-2BA\,.\,BD$.
When B is obtuse, $AC^2=BC^2+BA^2+2BA\,.\,BD$.

Now $BD=BC\cos CBD$, and this angle being the supplement of B, when obtuse, its cosine is *minus* cos B. Hence, introducing the minus in this case, we have, whether B be acute or obtuse, the single formula $b^2=a^2+c^2-2ac\cos B$, which is a general expression for one side of a triangle in terms of the other two and the included angle, but it is not so well adapted to computation by logs as the method already given. As b is *any* side of the triangle, expressions similar to this have place for a^2, and c^2; so that, for the cosines of the angles, in terms of the sides, we have

$$\cos A=\frac{b^2+c^2-a^2}{2bc},\ \cos B=\frac{a^2+c^2-b^2}{2ac},\ \cos C=\frac{a^2+b^2-c^2}{2ab}\ldots[1].$$

expressions which are very easily remembered: they may be employed with advantage whenever the three given sides are small numbers. The student is, of course, aware that, to adapt an expression to logarithmic

computation, it should consist entirely of factors and divisors: no additive or subtractive operations should be implied in it. In each of the above, the denominator is in a suitable form already; we have, therefore, now to convert the numerator into factors; and this we may do by simply subtracting each side of the equation from unit:

$$\text{Thus, } 1-\cos A=\frac{a^2-(b^2-2bc+c^2)}{2bc}=\frac{a^2-(b-c)^2}{2bc}=\frac{(a-b+c)(a+b-c)}{2bc},$$

$$\text{that is, (216), versin } A=\frac{(a-b+c)(a+b-c)}{2bc}. \quad \text{But (222) vers } A=2\sin^2\tfrac{1}{2}A,$$

$$\therefore \sin^2\tfrac{1}{2}A=\frac{\frac{1}{2}(a-b+c).\frac{1}{2}(a+b-c)}{bc};$$

$$\text{or, putting } s=\tfrac{1}{2}(a+b+c) \therefore s-b=\tfrac{1}{2}(a-b+c), \text{ and } s-c=\tfrac{1}{2}(a+b-c)$$

$$\therefore \sin\tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{bc}}\ldots\ldots[2],$$

where b, c, are the sides including the angle A, and s the half sum of all the three sides. Similar formulæ may, of course, be written down, from this, for the sines of the other half angles. Formulæ for the cosines of the half angles may also be readily deduced: thus,

$$\text{Since } \tfrac{1}{2}(1-\cos A)=\sin^2\tfrac{1}{2}A \therefore 1-\tfrac{1}{2}(1-\cos A)=\tfrac{1}{2}(1+\cos A)=1-\sin^2\tfrac{1}{2}A=\cos^2\tfrac{1}{2}A.$$

Hence, adding 1 to each side of the first equation [1], we have

$$1+\cos A=\frac{b^2+2bc+c^2-a^2}{2bc}=\frac{(b+c)^2-a^2}{2bc}=\frac{(a+b+c)(b+c-a)}{2bc}=2\cos^2\tfrac{1}{2}A$$

$$\therefore \cos\tfrac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}}\ldots[3]. \quad \text{And, } [2]\div[3], \tan\tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\ldots[4].$$

Instead of the half angles, we may now find at once the whole angles, thus:

$$(1-\cos A)(1+\cos A)=\sin^2 A=4\sin^2\tfrac{1}{2}A\cos^2\tfrac{1}{2}A \therefore \sin A=2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A$$

$$\therefore [2]\times[3], \sin A=\frac{2}{bc}\sqrt{\{s(s-a)(s-b)(s-c)\}}\ldots[5].$$

As sin A belongs equally to the angle A in the tables, and to its supplement, when a is the greatest of the three sides, there may be some doubt as to which of these angles really belongs to the triangle, for there can be only *one* triangle having the three given sides (Euc. 8. I.). But half an angle of a triangle is always acute, and, therefore, one or other of the three formulæ before given is, in general, to be preferred. These are easily retained in the memory: the numerator of [2] consists of two factors: s minus one side, and s minus another. If we were to write s minus the *opposite* side for one factor, then the other factor would be ambiguous: it would be s minus *either* of the adjacent sides: this is sufficient to apprise us that the two subtractive sides in the num. are the *adjacent* sides, and these same sides form the factors of the denom. In the numerator of [3], there is only one subtractive side: it cannot be an adjacent side, because the *adjacent* sides have each an equal claim: it must, therefore, be the opposite side, which cannot be one of the two sides in the denom., because the other side would then be ambiguous. To preclude

mistake, as to which of these two forms belongs to $\sin\frac{1}{2}A$ and which to $\cos\frac{1}{2}A$, we have only to observe that the expression for $\cos\frac{1}{2}A$ is shorter than that for $\sin\frac{1}{2}A$, and to connect in the memory the *o* in *cos*, with the *o* in *shorter*. In every practical example, the student should write down from memory the formula he intends to employ, and work from *it*, instead of from a verbal rule. The formula [4], arising from dividing [2] by [3], can at any time be recalled from these.

229. It may be noticed here, that it is not always matter of indifference which of the forms [2], [3], be employed. If the sought angle A is foreseen to be very near to 180°, then $\frac{1}{2}A$ will be very near to 90°: and the *sines* of angles very near to 90° differ so little from one another that, for five or six places of decimals, the sines of several such angles all agree; so that the tables would give a choice of angles increasing in magnitude, second by second, through a range of some minutes. In the case, therefore, of an *ill-conditioned* triangle of this kind, [3] is the preferable formula. On the contrary, when A is a very small angle, [2], or [4] should be employed.

230. If it be required to determine *all* the angles, we may find two only, by one or other of the above formulæ, and then subtract the sum of these from 180°. But this course is not recommended. It is better to compute all three, in order that the accuracy of the work may be tested by trying whether or not their sum amounts to 180°, as it ought—at least, within a second or two—if our results are correct. In such a case, [4] is the most convenient formula to work from; for, since

$$\tan\tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}},\ \tan\tfrac{1}{2}B=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}},\ \tan\tfrac{1}{2}C=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$\therefore \tan\tfrac{1}{2}B=\frac{s-a}{s-b}\tan\tfrac{1}{2}A,\ \tan\tfrac{1}{2}C=\frac{s-a}{s-c}\tan\tfrac{1}{2}A.\quad \text{Or in logarithms,}$$

$$\log\tan\tfrac{1}{2}B=\log\tan\tfrac{1}{2}A+\log(s-a)-\log(s-b)=\log\tan\tfrac{1}{2}A-\operatorname{colog}(s-a)-\log(s-b)+10$$

$$\log\tan\tfrac{1}{2}C=\log\tan\tfrac{1}{2}A+\log(s-a)-\log(s-c)=\log\tan\tfrac{1}{2}A-\operatorname{colog}(s-a)-\log(s-c)+10.$$

Since colog $(s-a)$, log $(s-b)$, and log $(s-c)$, have all been previously employed in computing log tan $\frac{1}{2}A$, the additional work for finding log tan $\frac{1}{2}B$, and log tan $\frac{1}{2}C$, is very trifling. This work, arranged in the most convenient manner, is exhibited in the solution of example 2, next page, where the colog, and the two succeeding logs are marked $[p]$, $[q]$, $[r]$, respectively, and log tan $\frac{1}{2}A$ is marked $[T]$.

(1) Given $a=1{\cdot}372$, $b=6$, $c=5{\cdot}523$, to find A.

$$\sin\tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{bc}}$$

$a=$	$1{\cdot}372$	
$b=$	6	colog $9{\cdot}221849$
$c=$	$5{\cdot}523$	colog $9{\cdot}257825$
	$2)12{\cdot}895$	
$s=$	$6{\cdot}4475$	
$s-b=$	$\cdot4475$	$\bar{1}{\cdot}650793$
$s-c=$	$\cdot9245$	$\bar{1}{\cdot}965907$
		$2)18{\cdot}096374$
$\sin\frac{1}{2}A$,	$6°\ 24'\ 55''$	$9{\cdot}048187$
	$\sin 6°\ 24'\ =$	$9{\cdot}047154$

$D=1875)103300(55''$

$\therefore A=12°\ 49'\ 50''$

(2) Given $a=95{\cdot}12$, $b=162{\cdot}34$, $c=98$: find A, B, C.

$$\tan\tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$a=$	$95{\cdot}12$	
$b=$	$162{\cdot}34$	
$c=$	98	
	$2)355{\cdot}46$	
$s=$	$177{\cdot}73$	colog $7{\cdot}750239$
$s-a=$	$82{\cdot}61$	colog $8{\cdot}082967[p]$
$s-b=$	$15{\cdot}39$	$1{\cdot}187239[q]$
$s-c=$	$79{\cdot}73$	$1{\cdot}901622[r]$
		$2)18{\cdot}922067$
$\tan\frac{1}{2}A$,	$16°\ 7'\ 27''$	$9{\cdot}461033[T]$
	$\tan 16°\ 7'\ =$	$9{\cdot}460823$

$D=789)21000(27''$

$1{\cdot}378066=[T-p]$

$\tan\frac{1}{2}B$, $57°\ 12'\ 4''$ $10{\cdot}190827=[T-p]-q$

$\tan\frac{1}{2}C$, $16°\ 40'\ 29''$ $9{\cdot}476444=[T-p]-r$

Verification $90°\ 0'\ 0''$

NOTE.—In determining $\frac{1}{2}A$, in each of these examples, two cologs, or arithmetical complements, have been employed; it might seem, therefore, that two *tens*, or 20, should be deducted from the sum of the four logs. But it must be remembered that, in the logarithmic tables, each log sin, log cos, &c., is increased by 10 (221), so that, in logarithms, the above formulæ are

$$\log\sin\tfrac{1}{2}A=\tfrac{1}{2}\{\log(s-b)+\log(s-c)-\log b-\log c\}+10$$

$$\log\tan\tfrac{1}{2}A=\tfrac{1}{2}\{\log(s-b)+\log(s-c)-\log s-\log(s-a)\}+10.$$

After the division by 2, in the foregoing operations, the results, in consequence of two cologs being incorporated in the sum, each exceed the true $\log\sin\frac{1}{2}A$, and $\log\tan\frac{1}{2}A$, by 10, and it is by this quantity that the log sin and log tan, in the table, are each actually increased, so that no correction of our results is necessary. In deducing $\tan\frac{1}{2}B$, $\tan\frac{1}{2}C$, in the second example, 10 is added to the remainders $[T-p]-q$, $[T-p]-r$, agreeably to the logarithmic forms in last page.

EXAMPLES FOR EXERCISE.

(1) Given $a=15{\cdot}236$, $b=12{\cdot}414$, $c=9{\cdot}018$, to find B.

(2) Given the three sides 4, 5, 6 respectively: determine the angles.

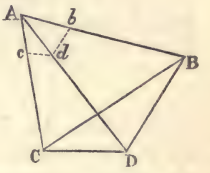
(3) Given $a=101{\cdot}616$, $b=153$, $c=137$, to find A, B, C.

(4) Given $a=33$, $b=42{\cdot}6$, $c=53{\cdot}6$ to find A, B, C.

---

231. Miscellaneous Problems involving Plane Triangles.

(1) In order to find the distance between two inaccessible objects A, B, a base line CD was measured = 536 yards, and at C, D, the following angles were taken, namely, $ACB=57°\,40'$, $BCD=40°\ 16'$, $ADC=42°\ 22'$, $ADB=71°\ 7'$: required the distance AB.



1. In the triangle ACD are given CD and all the angles, to find DA.

2. In the triangle BCD are given CD and all the angles, to find DB.

3. In the triangle ADB are given DA, DB, and the included angle, to find AB.

These computations being performed, we find $AB=939\frac{1}{2}$ yards.

(2) Two eminences C, D, can be easily reached, but from the nature of the intervening ground, the distance between them cannot be conveniently measured. For the purpose of finding it, two spires, known to be 6594 yards apart, were observed, and the following angles taken, namely:—

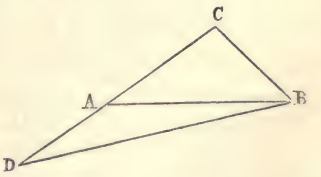
$$ACB=85°\ 46',\ BCD=23°\ 56',\ ADC=31°\ 48',\ ADB=68°\ 2';$$

required the distance CD.

Assume any point c in AC, and regard cd, parallel to CD, as unit; then, drawing db parallel to DB, the two quadrilaterals $ABCD$, $abcd$, will be similar. Compute Ab by last problem, from the given angles equal to those observed; then since the sides about the equal angles of similar figures are proportional, $Ab : cd :: AB : CD=\dfrac{AB}{Ab}$, since $cd=1$.

CD is thus found = 4694 yards.

(3) The distances of three objects A, B, C, from each other are $AB=462$ yards, $AC=328$ yards, and $BC=297$ yards. A person at D, in the continuation of CA, wishing to know his distance from each object, takes the angle ADB, and finds it to be $24°\ 16'\ 21''$; required the distances DA, DC, DB.



(4) The distance AB (fig. to Prob. I.) is one mile, the angle $ACD=76°\ 30'$, $BCD=44°\ 10'$, $BDC=81°\ 12'$, $ADC=46°\ 5'$: required the distance CD.

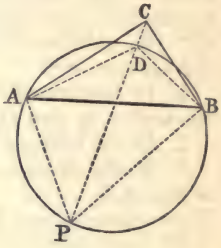
(5) The angle of elevation of the top of a steeple is 40°, when the observer's eye is on a level with the bottom; and from a window 18 feet directly above the first station, the angle of elevation is $37°\ 30'$: required the height and distance of the steeple.

(6) Three objects, A, B, C, whose distances apart are $AB=12$ miles, $AC=8$ miles, $BC=7\frac{1}{5}$ miles, are visible from a station D, in the line joining A, B: the angle at D subtended by AC is 107° 56′ 13″: required the distances of the objects from D.

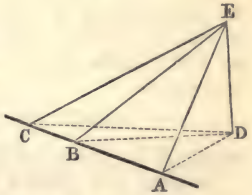
(7) From the top of a hill 360 feet high, I observed a column of infantry halting in a straight road directly before me: the angle of depression of the foremost rank was 57°, that of the most remote 25° 30′: required the length of the column.

(8) Wanting to know the height of a castle on the top of an inaccessible hill, I took the angle of elevation of the top of the hill, 40°, and of the top of the castle 51°; then retiring in a direct line a distance of 180 feet, I again took the angle of elevation of the top of the castle, and found it to be 33° 45′: required the height of the castle.

(9) Given the distances $AB=800$, $AC=600$, $BC=400$, between the three objects A, B, C, and the angles subtended by the two latter distances at a point P in the same plane as the objects, namely, $APC=33° 45′$, $BPC=22° 30′$, to determine the respective distances of P from each.



(10) Given the angles of elevation of an object E, taken at three stations A, B, C, in a horizontal line, equal to α, α', α'', respectively, the distances between the stations being $AB=a$, $BC=b$: prove that the height h of the object E is



$$h=\sqrt{\frac{ab(a+b)}{a\cot^2\alpha''+b\cot^2\alpha-(a+b)\cot^2\alpha'}}.$$

(11) In the last problem, if the measured distances AB, BC, are equal, the expression for the height h is $h=a\div\sqrt{\left(\frac{1}{2}\cot^2\alpha+\frac{1}{2}\cot^2\alpha''-\cot^2\alpha'\right)}$. It is required to calculate h, when $\alpha=36° 50′$, $\alpha'=21° 24′$, $\alpha''=14°$, and the two equal distances AB, BC, 84 feet.

(12) Suppose the distances are $AB=100$ yards, $BC=400$ yards, and the angles at A, B, C, 5° 24′, 6° 27′·5, 8° 36′, respectively: required h.

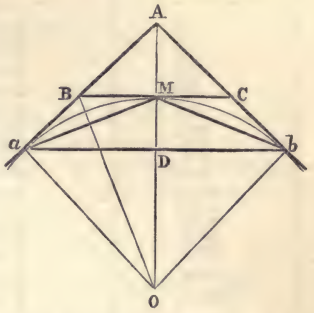
NOTE.—The foregoing problems are all solvable by aid of the principles established respecting the plane triangle in the preceding articles. But, with the additional assistance derived from a more comprehensive knowledge of trigonometrical properties, some of them may be solved with less numerical labour, as we shall show hereafter.

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232. **Approximate Quadrature of the Circle.**—By the *geometrical* quadrature of the circle, is meant the finding of a rectilinear figure, or of a *square*, which shall be equal to the circle in surface, or area. The attempt to solve this problem has long been abandoned, but it would be wrong to pronounce the solution as impossible, since there is no

doubt that a rectilinear area equal to the circular one *exists*. The *numerical* quadrature of the circle, means the determination of its area in *finite numbers*, when the numerical value of the radius is given. This problem *is* impossible, since no finite expression for the area, in numbers, exists. But the approximate quadrature may be obtained, to as many places of decimals as we please, as follows.

Imagine a regular polygon of n sides to be inscribed in the circle. Let ab be one of those sides, and let the surface of this inscribed polygon be p_n. Then the tangents aA, bA, will be each half a side of a similar circumscribed polygon. Call the surface of this P_n. Let M be the middle of the arc aMb; then will the chords aM, bM, be two sides of an inscribed polygon p_{2n}, of $2n$ sides, and the tangent BMC will be a side of a similar circumscribed polygon P_{2n}. All this is proved in Euclid, Book IV.



Now O being the centre of the circle, in and about which the regular polygons are described, it is plain that the triangle

ODa is the $2n$th part of the polygon			p_n	
that OaM	,,	,,	,, p_{2n}	[1]
that OaA	,,	,,	,, P_n	
and that $OaBM$	,,	,,	,, P_{2n}	

By similar right-angled triangles, ODa, OaA,
$OD : Oa :: Oa : OA$
or, $OD : OM :: OM : OA$
and since triangles of the same altitude are to one another as their bases, $ODa : OaM :: OaM : OaA$.

Consequently the numerical measure, or area, of the triangle OaM, is a mean proportional between the areas of ODa, OaA. And since the wholes are as their like parts, it follows that—

The area of p_{2n} *is a mean between the areas of* p_n *and* P_n, *that is,* $p_{2n}=\sqrt{P_n p_n}$...[2].

Again: since the right-angled triangles ODa, BMA, are similar, the angle BAM being $=OaD$, since AOa is the complement of each, we have

$$OD : Oa :: BM : BA, \text{ or } OD : OM :: aB : BA$$

and since triangles of the same altitude are as their bases, it follows that

$$ODa : OMa :: OaB : OBA \therefore ODa : ODa+OMa :: OaB : OaB+OBA$$

$$\therefore [1],\ p_n : p_n+p_{2n} :: \frac{1}{2}P_{2n} : P_n \therefore P_{2n}=\frac{2P_n p_n}{p_n+p_{2n}}...[3].$$

The expressions [2], [3], enable us to compute the areas of inscribed and circumscribed polygons of $2n$ sides, when those of polygons of n sides are given. Let 1 be the numerical value of the radius of the circle; and, to begin with, let the inscribed and circumscribed polygons be *squares*, that is, let $n=4$: then we have given $p_4=2$, and $P_4=4$; and therefore from [2], [3], above,

$$p_8=\sqrt{8}=2{\cdot}8284271,\ P_8=\frac{2\times 8}{2+\sqrt{8}}=\frac{16}{4{\cdot}8284271}=3{\cdot}3137085$$

$$\therefore p_{16}=\sqrt{(3{\cdot}3137085\sqrt{8})}=3{\cdot}0614674,\ P_{16}=\frac{2\times 3{\cdot}137085\sqrt{8}}{\sqrt{8}+3{\cdot}0614674}=3{\cdot}1825979.$$

And in a similar manner may p_{32}, P_{32}; p_{64}, P_{64}, &c., be computed in succession, and thus the following table formed :—

Surfaces of Inscribed and Circumscribed Regular Polygons. Radius =1.

No. of sides.	Surf. of ins. Polygons.	Surf. of circ. Polygons.
4	2·0000000	4·0000000
8	2·8284271	3·3137085
16	3·0614674	3·1825979
32	3·1214451	3·1517249
64	3·1365485	3·1441184
128	3·1403311	3·1422236
256	3·1412772	3·1417504
512	3·1415138	3·1416321
1024	3·1415729	3·1416025
2048	3·1415877	3·1415951
4096	3·1415914	3·1415933
8192	3·1415923	3·1415928
16384	3·1415925	3·1415927
32768	3·1415926	3·1415926
&c.	&c.	&c.

233. It thus appears that the numerical expressions for the surfaces of inscribed and circumscribed regular polygons of 32768 sides each, agree as far as seven places of decimals ; and as the surface of the circle itself is intermediate between these two polygons, it follows that the area of a circle of radius 1, as far as seven places of decimals, must be 3·1415926.

Now the area of any regular polygon of n sides, whether inscribed or circumscribed, is the sum of all the n equal triangles formed by lines from the centre of the circle to the extremities of the sides : the area of each triangle is half the product of a side of the polygon by the perpendicular upon it from the centre. In the case of a *circumscribed* polygon, this perp. is r, the radius of the circle : hence, calling the semi-perimeter of the polygon π, the area is $r\pi$, which, when $r=1$, is simply π. Now when n is indefinitely great, we have seen that the area π is 3·1415926... But in this case the perimeter 2π of the polygon, and the circumference of the circle, become confounded ; therefore π=3·1415926...is the numerical expression for the semicircumference of a circle of radius 1 ; and as we have just seen, this number, as so many *square* units, instead of linear units, as here, expresses the area of the same circle. It will be remembered therefore that π units expresses the *length* of 180° of the trigonometrical circle, or of the circle whose radius is one unit.

Cor. It follows from the above that the area of a *sector aMbO* is equal to the radius multiplied by half the length of the arc.

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234. **Unit of Circular Measure.**—We have just seen that the numerical measure of 180° of the trigonometrical circle is the number π. Hence the length of one degree of the trigonometrical circle is 3·1416÷ 180, and consequently the length of $A°$ is $\frac{3\cdot1416}{180}A$, therefore the length of $A°$ of a circle whose radius is R times 1, or R, is

$$\textit{Length of } A° \textit{ to rad. } R, = \frac{3\cdot1416}{180}A\times R = \frac{\cdot5236}{30}A\times R.$$

For ex., the length of 15° of a circle whose radius is 10 feet, is $\frac{\cdot 5236}{30} \times 150 = 2\cdot 618$ feet. If the length of an arc of given radius R be known, we may also readily find the number of degrees in it; for calling that number A, we have

$$A = \frac{\text{Length of arc}}{R} \times \frac{30}{\cdot 5236} = \frac{\text{Length of arc}}{R} \times 57\cdot 295\ldots$$

If the arc be equal in length to R, then there must be $57\cdot 295\ldots$ degrees in it, or more correctly $57°\cdot 2957795$. And this arc is called the *unit of circular measure:* in any circle it is the radius, bent round that circle.

The angle subtended by this unit-arc is taken as the *unit of angular measure:* the *number* of such units in any angle, subtended by an arc of any radius R, is therefore expressed by $\frac{\text{Length of arc}}{R}$. If, then, we drop the angular unit, which is the angle subtended by $57°\cdot 2957795$, we may express every angle by the abstract number which is the above ratio. When $R=1$, this ratio is simply "Length of arc," which for $A°$ is $3\cdot 1416\ \frac{A}{180}$, as already noticed above: suppose, for ex., that $A=60$, then we should have angle $60° = \pi\ \frac{60}{180} = 1\cdot 04719\ldots$, which is the *length* of 60° of the trigonometrical circle; and the *number* thus expressing the length of any arc, of this circle (the circle whose semicircumference is π) is always what is here called the *measure* of the subtended angle.

It would be well to call this the *arcual* measure of the angle, and its measure in degrees, the *grādual* measure, as proposed by De Morgan. Perhaps, to preclude mispronunciation, the latter term might be written *gradeal.*

It is plain that although the arcual measure is not the same thing as the gradeal measure, yet that the sine, cosine, &c., of the one is the same as the sine, cosine, &c., of the other; so that $\sin \frac{180°}{n}$, $\cos \frac{180°}{n}$, &c., is the same as $\sin \frac{\pi}{n}$, $\cos \frac{\pi}{n}$, &c.; or if θ be the arcual measure of $A°$, then $\sin \theta = \sin A°$, $\cos \theta = \cos A°$, &c. But in expressing angles in gradeal measure, the symbol for degrees is usually suppressed, $A°$ being written simply A. The following is an example of the application of arcual measure. Ex. It is found that the Earth's radius, 4000 miles, subtends at the centre of the sun an angle of 8·57...seconds: required the sun's distance from the earth. We may proceed in either one or other of these ways:—

Let R be the distance; then 4000 miles of arc, of rad $=R$, is $8''\cdot 57\ldots$ of that arc; but R miles of arc is $57°\cdot 295\ldots$

$$\therefore 8''\cdot 57\ldots : 57°\cdot 295\ldots :: 4000 : R = 4000\ \frac{57\cdot 295\ldots \times 60 \times 60}{8\cdot 57\ldots} = 96260000 \text{ miles.}$$

And this operation is obviously the same as the following, namely:—

The arcual measure θ, of the angle, being $\frac{\text{Length of arc}}{R}=\frac{4000}{R}=\frac{8\cdot57\ldots}{57\cdot295\ldots\times60\times60}$

$$\therefore R=4000\frac{57\cdot295\ldots\times60\times60}{8\cdot57\ldots}=96260000 \text{ miles.}$$

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235. **Comparison of French and English Degrees.**—Instead of dividing the right angle into 90 degrees, certain French analysts of great eminence, towards the close of the last century, proposed to divide it into 100 degrees, and each degree into 100 minutes, and so on. But although adopted by some distinguished writers of that period, the proposal, so far from having met with general acceptance on the Continent, is at present entirely discarded, in consideration of the valuable records in which the sexagesimal division is used. For distinction, we call the French degree a *grade*, and employ the notation $32^g\ 24'\ 48''$ to mark grades, minutes, and seconds in this centesimal division of the right angle. But the object of the proposed change was to do away with the necessity for any such notation for the subdivisions of a grade:—it is plain that the angle here written is better expressed thus $32^g\cdot2448$. We shall briefly show how to convert the French measure into English, and *vice versâ*.

Let E°, F^g, each denote the same angle: then since

$$100 : 90 :: F : E \therefore E=\frac{9}{10}F=F-\frac{F}{10}. \quad \text{And } F=\frac{10}{9}E=E+\frac{E}{9}.$$

(1) How many degrees, minutes and seconds are there in the angle $32^g\ 2448$?

(2) How many grades are there in $24^\circ\ 18'\ 24''$?

1.

$F=32\cdot2448$

Subtract $\frac{1}{10}F=\ 3\cdot22448$

$\therefore E=29\cdot02032$

60

$1\cdot2192$

60

$13\cdot152$

$\therefore E^\circ=F^g=29^\circ\ 1'\ 13''\cdot152.$

2.

60)24

60)18·4

$E=24\cdot30666\ldots=$No. of degrees in the angle

Add $\frac{1}{9}E=\ 2\cdot70074$

$\therefore F=27\cdot00741$

$\therefore F^g=E^\circ=27^g\cdot00741.$

Part II. Theory of Angular Magnitude in General.

236. As introductory to the more general theory, which, dismissing the triangle, treats of angles without restriction, we shall return for a moment to the triangle itself, and deal with it in a manner more strictly algebraical. Referring to the diagrams at page 180, we see that $c=a\cos B+b\cos A$, and similarly for the other sides a, and b; that is,

$$a=b\cos C+c\cos B,\ b=c\cos A+a\cos B,\ c=a\cos B+b\cos A\ldots[1]$$

three algebraic equations from which all that concerns the plane triangle may be deduced. This is obvious, for any three of the six quantities entering these equations being given, the other three (except the given quantities be the three *angles*) can always be found by the rules of algebra. The exception here stated is suggested by the equations themselves; for writing ma, mb, mc, for a, b, c, amounts to the same as merely multiplying each equation by m, so that $\cos A$, $\cos B$, $\cos C$, would be the same for any equimultiples of a, b, c, as for these quantities themselves; therefore the angles being given or fixed, does not determine fixed lengths for the sides.

In order to find the cosines of the angles in terms of the sides, multiply the equations, in order, by a, b, c; we then have

$$a^2=ab\cos C+ac\cos B,\ b^2=bc\cos A+ab\cos C,\ c^2=ac\cos B+bc\cos A.$$

And subtracting each of these from the sum of the other two,

$$b^2+c^2-a^2=2bc\cos A\ \therefore\ \cos A=\frac{b^2+c^2-a^2}{2bc}$$

$$a^2+c^2-b^2=2ac\cos B\ \therefore\ \cos B=\frac{a^2+c^2-b^2}{2ac}$$

$$a^2+b^2-c^2=2ab\cos C\ \therefore\ \cos C=\frac{a^2+b^2-c^2}{2ab}$$

If the three sides are equal, we see that the angles must be equal: the cosine of each being $\frac{1}{2}$; that is, $\cos 60^\circ=\sin 30^\circ=\frac{1}{2}$. Though the three equations [1] involve the whole theory of the plane triangle, yet we shall not further develope that theory. Like the formulæ for the cosines, just deduced, all our results would present themselves in a form ill-adapted to logarithmic computation. The practical facilities offered by logarithms have caused additional labour to devolve on the theorist, inasmuch as his algebraical results must appear in a form fitted, as much as possible, for logarithmic computation.

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237. **Extension of the Trigonometrical Definitions.**—An angle of a triangle can never become so great as 180°; and, in the popular sense, the term *angle* is always understood to be an opening, within this limit. The meaning of the word is now to be extended: we are henceforth to regard any multiple whatever of an angle, as also *an angle*: and this extension is countenanced by Euclid, in Prop. 33 of Book VI., where he speaks of "any multiple of the angle BGC" as also an angle. In like manner, if angles be combined by addition or subtraction, that is in any way united by the signs $+$, $-$, the result of the combination is an angle. Now for the symbols of algebra to be generally applicable to any species of magnitude, those symbols should always be interpretable in reference to that magnitude, and since the result of such a combination of angles as that just referred to may be *negative*, it is necessary for us, by appropriate conventions at the outset, to establish the meaning of a *negative angle*, consistently with the meaning given to a positive angle. What is laid down at page 173, respecting negative cosines, &c., will in some measure prepare the student for the extension of the term *negative* to angles.

Employing again the diagram at page 171, the conventions necessary to bring the general theory of angular magnitude under the dominion of algebra are as follows:—

1. An angle is regarded as generated by the revolution of a straight line AC, starting from a fixed position AB: it is a positive angle if the line revolve in the direction BCD, &c., and a negative angle if it revolve in the contrary direction $BC''''C'$, &c.: for it is agreed, in the former case, to mark the algebraic symbol for the angle, *plus*, and therefore, consistently with the nature of the algebraic opposites, +, —, the symbol for the angle in the latter case is marked *minus*.

2. The line AB, the initial position of the revolving line, is always marked +; the directly opposite line AB' is therefore marked —. In like manner it being agreed to call AD, +, the opposite line AD', is marked —.

3. The line DD' is regarded as a fixed boundary from which all lines parallel to AB, or AB' are measured; those parallels which like AB all lie to the right, are marked +, those which like AB' all lie to the left, are marked —. The line BB' is another fixed boundary from which all lines parallel to AD or AD' are measured; those which like AD are above this boundary are marked +, those which like AD' are below it are marked —.

238. These are the only conventions which are necessary. Being adopted by universal consent they render the theory of analytical trigonometry universally intelligible: the algebraic symbols of quantity serve to denote the magnitudes of the lines and angles employed, and the algebraic signs +, —, mark the directions or positions of these in reference to fixed boundaries, while they at the same time discharge their office of signs of addition and subtraction. The *lines* to which we have just referred are replaced in trigonometry by their numerical values, conformably to the hypothesis that AB is *unit*, and these numerical values take of course the same signs as their linear representatives. Thus:

If the revolving line AC stop at C in the diagram, mC is *plus*, and Am is also *plus*; therefore both the sine and cosine of the angle CAB is +. If the revolving line stop at C', $m'C'$ is *plus*, and Am' is minus; therefore sin $C'AB$ is +, and cos $C'AB$ is —. If the revolving line stop at C'', $m'C''$ is *minus*, and Am' is also *minus*, therefore the sine of the angle generated is —, and so is the cosine. In like manner, if the revolving line continue onward to C'''', the sine of the angle generated is —, and its cosine +. And from sine and cosine all the other trigonometrical ratios are derivable (219).

Again, if the revolving line starting from AB proceed in the opposite direction, generating the negative angle $C''''AB$, the sine of this angle is —, and its cosine +. Of the negative angle $C''AB$ both sine and cosine are —. And it is plain that we have only to notice in which of the four quadrants BC, DB', $B'D'$, $D'B$, the revolving line stops to determine the proper algebraic sign for the sine or cosine of the angle generated, whether that angle be positive or negative:—the sine and cosine, both as to numerical value and algebraic prefix, being respectively the same, whether the revolving line has terminated a positive angle or a negative angle. It is also obvious, that when the revolving line has performed more than a complete revolution, the sine and cosine of the angle generated is also the sine and cosine of the angle which remains after subtracting

from the former as many times four right angles as there have been revolutions: thus $\sin 583°=\sin(583°-360°)=\sin 43°$, and $\cos 583°=\cos 43°$; $\sin 762°=\sin(762°-720°)=\sin 42°$, and $\cos 762°=\cos 42°$. In like manner $\sin 486°=\sin 126°$, and $\cos 486°=\cos 126°$. The sine in this case is +, and the cosine −. A glance at the diagram shows,

1. That if the revolving line stop in the first or second quadrant, the sine of the generated angle is + ; if it stop in the third or fourth the sine is −.

2. If the revolving line stop in the first or fourth quadrant, the cosine of the angle is + : if it stop in the second or third, it is −. Every angle expressed in arcual (or circular) measure must be comprehended in the general form $2n\pi\pm\theta$ where θ measures less than two right angles: if the revolving line terminating the angle lie in the upper semicircle, the upper sign applies, if in the lower semicircle, the lower sign applies. And since in determining the trigonometrical ratios for an angle the multiple $2n\pi$ may be rejected, as mentioned above, and as the diagram sufficiently shows, it follows that, however large the angle, the sine, cosine, &c., of it are equally the sine, cosine, &c., of an angle below 180°, either positive or negative: thus $\sin(720°\pm 124°)=\sin(\pm 124°)$, and $\cos(720°\pm 124°)=\cos(\pm 124°)=\cos 124°$. And generally, θ measuring less than two right angles,

$$\sin(2n\pi+\theta)=\sin\theta,\ \sin(2n\pi-\theta)=\sin(-\theta)=-\sin\theta;$$
$$\cos(2n\pi\pm\theta)=\cos\theta=\cos(-\theta)\ldots[1].$$

But $\sin\theta=\sin(\pi-\theta)$, and $\cos\theta=-\cos(\pi-\theta)$, where θ or $\pi-\theta$ is less than *one* right angle: consequently the trigonometrical ratios for any angle, however large, are equally the ratios for an angle less than a right angle.

If in [1], we substitute $\pi-\theta$ for θ, we have

$$\sin\{2n\pi+(\pi-\theta)\}=\sin(\pi-\theta)=\sin\theta,\ \sin\{2n\pi-(\pi-\theta)\}=-\sin(\pi-\theta)=-\sin\theta;$$
$$\cos\{2n\pi\pm(\pi-\theta)\}=\cos(\pi-\theta)=-\cos\theta,$$

and remembering that to any angle we may always add 2π, without altering the trigonometrical ratios, these equations are the same as the following:—

$$\sin\{(2n+1)\pi-\theta\}=\sin\theta,\ \sin\{(2n+1)\pi+\theta\}=-\sin\theta;$$
$$\cos\{(2n+1)\pi\mp\theta\}=-\cos\theta\ldots[2].$$

The first and third of the results [2] suggest an extension of the definition of *Supplement* (218). And if instead of putting $\pi-\theta$ for θ in [1], we had put $\frac{\pi}{2}-\theta$, a like extension of the definition of *Complement* would have been suggested.

The principles and conventions now established enable us freely to apply algebra to investigate the general theory of angular magnitude, and are such as to render the symbolical results arrived at clearly interpretable in consistency with them. As a simple illustration, we may refer to the relations $\cos A=\sqrt{(1-\sin^2 A)}$, and $\sin A=\sqrt{(1-\cos^2 A)}$. Here $\cos A$, and $\sin A$, ought each to have *two* values, numerically equal, but opposite in sign. The double value for $\cos A$ is justified by what is shown in Part I.; for as an angle and its supplement have the same sine, the angle A, under the radical, may be either an acute angle or its supplement, while in the former case, the cosine is +, and in the latter − : the double

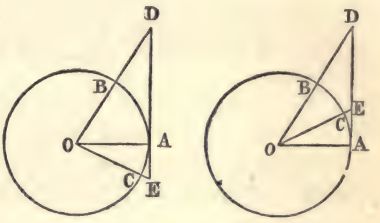
sign is thus accounted for. But the double sign for sin A is not intelligible upon any principle delivered in Part I.: *now*, however, we know that the cosine of A is the same whether A be positive or negative, but that the *sine* is in the one case $+$, and in the other $-$: the double sign for sin A is thus accounted for by a reference to the foregoing conventions.

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239. What has been said above, in reference to angles and the trigonometrical ratios, equally applies to the arcs which subtend the angles, and to the linear sines, cosines, &c., of those arcs. By the definition (214), tangents of arcs all originate at B, and are measured along the touching line in one or other of the two directions BT, BT'. The tangent limits the secant, so that wherever the arc originating at B may terminate, the tangent and secant of that arc always form, with the radius AB, a right-angled triangle situated to the right of the centre A. Moreover, the sine and cosine of the arc are always the same, *in magnitude*, as the sine and cosine of the arc, less than a quadrant, which terminates in the secant: thus the arc of which AT is the secant (p. 171), terminates either in C, or C'', and it is plain that the sine and cosine Cm, Am are respectively equal in length to $C''m'$ and Am'. Similarly the arc of which AT' is the secant, terminates either in C''', or C', and the sine and cosine of one of these arcs are respectively the same in length as the sine and cosine of the other. And from such geometrical considerations, the trigonometrical truths, stated in the last article, might have been deduced, like as the relations at (219) were deduced from the trigonometrical circle, which relations we now know to have place, whatever be the magnitude of the angle A.

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240. **Sine and Cosine of** $(A \pm B)$.—Let AD, AE, in the annexed diagrams, be the tangents of any two arcs A, B, and OD, OE, their secants. Wherever the two arcs may terminate, their tangents and secants cannot fall otherwise than are here represented,* and they are equally the tangents and secants of the arcs AB, AC, each less than a quadrant: so that the one pair of arcs may replace the other as regards their tangents and secants. Now (228) and Euclid, Props. 4 and 7, B. II, we have



$$DE^2 = OD^2 + OE^2 - 2OD \cdot OE \cos DOE \ldots [1]$$
$$DE^2 = AD^2 + AE^2 \pm 2AD \cdot AE \ldots\ldots\ldots\ldots [2]$$

Let the radius $OA = 1$, then $OD^2 - AD^2 = 1$ and $OE^2 - AE^2 = 1$. Hence subtracting [2] from [1], $0 = 1 + 1 \mp 2 \tan A \tan B - 2 \sec A \sec B \cos (A \pm B)$,

$$\therefore \cos (A \pm B) = \frac{1 \mp \tan A \tan B}{\sec A \sec B} = \cos A \cos B \mp \sin A \sin B \ldots [1'].$$

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* Of course, in the second diagram, there may be *below* OA, a similar construction to that above it; but it is unnecessary to encumber the figure with it.

Again: by Euc. Prop. 35, B. I., the surface of a triangle is equal to half the rectangle of its base and altitude; and any side of a triangle may be considered as the base. Taking DE for the base, OA is the alt., and taking OD for the base, OE sin DOE is the alt.: hence

$$OD \,.\, OE \sin DOE = DE \,.\, OA \;\therefore \text{ for } OA = 1,$$

$$\sin (A \pm B) = \frac{\tan A \pm \tan B}{\sec A \sec B} = \sin A \cos B \pm \cos A \sin B \ldots [2'.]$$

241. In the preceding investigation, the tangents and secants of the unrestricted arcs A, B, as noticed at the outset, are correctly represented as in the diagrams; but in deducing the expressions for sin $(A \pm B)$, cos $(A \pm B)$, we have regarded these sines and cosines to be the same as those of the arcs $(AB \pm AC)$, where each component arc is less than a quadrant. To show that this was justifiable, put θ, θ' for these component arcs, regarding the circle as the trigonometrical circle. Then the arcs $(A \pm B)$, of which the components have the same tangents and secants as those of $(\theta \pm \theta')$, will all be comprehended in one or other of the forms $2n\pi + (\theta \pm \theta')$, or $(2n+1)\pi + (\theta \pm \theta')$. Now it has been seen (p. 197) that sin $\{2n\pi + (\theta \pm \theta'\} = \sin (\theta \pm \theta')$, and that cos $\{2n\pi + (\theta \pm \theta')\} = \cos (\theta \pm \theta')$. Also that sin $\{(2n+1)\pi + (\theta \pm \theta')\} = -$ sin $\{(\theta \pm \theta')\}$ and cos $\{(2n+1)\pi + (\theta \pm \theta')\} = -$ cos $(\theta \pm \theta')$. This latter form applies to the case where only *one* of the arcs, as AB, is increased by π or by an odd multiple of it, and it is proper that the results should take the minus sign, for sec A will then be negative in [1'] and [2'], notwithstanding its geometrical position in the diagrams, because sec $A = \frac{1}{\cos A}$. And here we may remark that, as concerns *negative* arcs or angles, it is not true that sec $A = \frac{1}{\cos A}$: the 1 in the numerator is the numerical representation of that linear radius, of which the prolongation is the secant (see page 196); and in the case here spoken of it should be -1.

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242. **Expressions involving two Angles.**—The expressions [1'], [2'], to which the foregoing remarks apply, form the basis of the entire theory of angular magnitude; we shall here write them separately for future reference.

$$\left.\begin{aligned} \sin (A+B) &= \sin A \cos B + \cos A \sin B \\ \sin (A-B) &= \sin A \cos B - \cos A \sin B \\ \cos (A+B) &= \cos A \cos B - \sin A \sin B \\ \cos (A-B) &= \cos A \cos B + \sin A \sin B \end{aligned}\right\} \ldots [\text{I.}]$$

And from the same general expressions [1'], [2'], we further see that

$$\tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}, \quad \tan (A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \ldots [\text{II.}]$$

and dividing num. and den. of each fraction by tan A tan B, and taking the reciprocals, we have

$$\cot (A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}, \quad \cot (A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A} \ldots [\text{III.}]$$

If $A+B+C=180°$, then $\tan(A+B)=-\tan C$,

$\therefore$ [II.], $\tan A \tan B \tan C = \tan A + \tan B + \tan C$.

If $A+B+C=90°$, then $\tan(A+B)=\cot C=\dfrac{1}{\cot(A+B)}$

$\therefore$ equating this with the reciprocal of the fraction for $\cot(A+B)$ in III., we have

$$\cot A \cot B \cot C = \cot A + \cot B + \cot C.$$

The first of these deductions shows that the product of the tangents of the three angles of a triangle (or of any three angles into which 180° may be divided), is equal to their sum; and the second, that the product of the cotangents of any three angles into which 90° may be divided is also equal to their sum. And it is readily seen that the former property applies whenever the three angles amount to any even multiple of 90°, and the latter, when they amount to any odd multiple of 90°. If A, B, C, be any three angles, unrestricted, we leave the student to prove, from [II.], that

$$\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan A\tan C-\tan B\tan C}.$$

Other useful deductions from the fundamental formulæ [I.] are the following:—

1. By adding and subtracting

$$\left.\begin{aligned}
\sin(A+B)+\sin(A-B)&=2\sin A\cos B\\
\sin(A+B)-\sin(A-B)&=2\cos A\sin B\\
\cos(A+B)+\cos(A-B)&=2\cos A\cos B\\
\cos(A+B)-\cos(A-B)&=-2\sin A\sin B
\end{aligned}\right\}\ldots[\text{IV.}]$$

2. By multiplying, $\sin(A+B)\sin(A-B)=\sin^2 A\cos^2 B-\cos^2 A\sin^2 B$

$$\cos(A+B)\cos(A-B)=\cos^2 A\cos^2 B-\sin^2 A\sin^2 B$$

and since $\sin^2 A+\cos^2 A=1$, and $\sin^2 B+\cos^2 B=1$, these, by substitution, become

$$\left.\begin{aligned}
\sin(A+B)\sin(A-B)&=\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A\\
\cos(A+B)\cos(A-B)&=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A
\end{aligned}\right\}\ldots[\text{V.}]$$

so that $\sin(A+B)\sin(A-B)=(\sin A+\sin B)(\sin A-\sin B)$.

Again: since any angles may be put for A and B, substitute $\frac{1}{2}(A+B)$ for the former, and $\frac{1}{2}(A-B)$ for the latter; then [IV.] becomes

$$\left.\begin{aligned}
\sin A+\sin B&=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)\\
\sin A-\sin B&=2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)\\
\cos A+\cos B&=2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)\\
\cos B-\cos A&=2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)
\end{aligned}\right\}\ldots[\text{VI.}],$$

whence, by division, $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\tan\tfrac{1}{2}(A+B)$, $\dfrac{\sin A-\sin B}{\cos A+\cos B}=\tan\tfrac{1}{2}(A-B)$,

and dividing the first of these by the second, $\dfrac{\sin A+\sin B}{\sin A-\sin B}=\dfrac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$.

Now in the *triangle*, $\frac{a}{b}=\frac{\sin A}{\sin B}$ $\therefore$ (38) $\frac{a+b}{a-b}=\frac{\sin A+\sin B}{\sin A-\sin B}$,

$$\text{that is, } \frac{a+b}{a-b}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$$

which establishes the rule, otherwise arrived at (227), for solving the second case of oblique triangles.

In like manner, by division, $\frac{\cos B-\cos A}{\cos A+\cos B}=\tan\frac{1}{2}(A+B)\tan\frac{1}{2}(A-B)$,

$$\frac{\cos B-\cos A}{\sin A\pm\sin B}=\tan\frac{1}{2}(A\mp B),$$

so that $\frac{\sin A+\sin B}{\cos A+\cos B}=\frac{\cos B-\cos A}{\sin A-\sin B}$, and $\frac{\sin A-\sin B}{\cos A+\cos B}=\frac{\cos B-\cos A}{\sin A+\sin B}$.

But these may be got from principles still simpler, thus

since $\sin^2 A+\cos^2 A=\sin^2 B+\cos^2 B$ $\therefore$ $\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A$

$\therefore$ $\frac{\sin A+\sin B}{\cos B+\cos A}=\frac{\cos B-\cos A}{\sin A-\sin B}$, from which $\frac{\sin A-\sin B}{\cos B+\cos A}=\frac{\cos B-\cos A}{\sin A+\sin B}$.

The above are the most useful deductions, involving two angles, from the fundamental formulæ [I.]. But as any of the operations of algebra may be applied to these formulæ, it is obvious that a variety of other results may be deduced. Instead of further multiplying these we shall proceed to derive some important expressions involving only single angles, their halves, and their multiples.

243. **Expressions involving Single Angles and their Halves.**—Putting A for B in [I.], and remembering that $\cos^2 A=1-\sin^2 A$ and $\sin^2 A=1-\cos^2 A$, we have

$$\sin 2A=2\sin A\cos A,\ \cos 2A=\cos^2 A-\sin^2 A=1-2\sin^2 A=2\cos^2 A-1\ldots[\text{VII.}].$$

Putting A for B in [II.], $\tan 2A=\frac{2\tan A}{1-\tan^2 A}$...[VIII.]

These expressions for sine, cosine, and tangent of $2A$ may be changed for those of $\frac{1}{2}A$ by putting this last for A, we thus get

$$\sin A=2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A,\ \cos A=\cos^2\tfrac{1}{2}A-\sin^2\tfrac{1}{2}A=1-2\sin^2\tfrac{1}{2}A=2\cos^2\tfrac{1}{2}A-1\ldots[\text{IX.}]$$

$$\text{And } \tan A=\frac{2\tan\frac{1}{2}A}{1-\tan^2\frac{1}{2}A}\ldots[\text{X.}]$$

If the first of [IX.] be added to and subtracted from $1=\sin^2\frac{1}{2}A+\cos^2\frac{1}{2}A$,

we shall have $1+\sin A=\left(\sin\frac{1}{2}A+\cos\frac{1}{2}A\right)^2$, $1-\sin A=\left(\sin\frac{1}{2}A-\cos\frac{1}{2}A\right)^2$...[XI.]

And proceding in like manner with the second of [IX.], we have

$$1+\cos A=2\cos^2\tfrac{1}{2}A,\ 1-\cos A=2\sin^2\tfrac{1}{2}A\ldots[\text{XII.}]$$

These last expressions, for angles in general, were found useful at page 186, where A is an angle of a triangle. From [IX.] and [XII.], by division, we have

$$\frac{\sin A}{1+\cos A}=\tan\frac{1}{2}A=\frac{1-\cos A}{\sin A}=\sqrt{\frac{1-\cos A}{1+\cos A}}\ldots[\text{XIII.}]$$

It is worthy of notice that since $\sin 45°=\cos 45°=\sqrt{\frac{1}{2}}$, therefore

$$\left.\begin{array}{l}\sin(A\pm45°)=(\sin A\pm\cos A)\sqrt{\frac{1}{2}}\ \therefore\ \sin A\pm\cos A=\sin(A\pm45°)\,.\,\sqrt{2}\\ \text{In like manner, since}\\ \cos(A\pm45°)=(\cos A\mp\sin A)\sqrt{\frac{1}{2}}\ \therefore\ \cos A\mp\sin A=\cos(A\pm45°)\sqrt{2}.\end{array}\right\}\ldots[\text{XIV.}]$$

Also, since

$$\left.\begin{array}{l}\tan 45°=1,\ \therefore[\text{II.}],\ \tan(45°+A)=\frac{1+\tan A}{1-\tan A},\ \tan(45°-A)=\frac{1-\tan A}{1+\tan A}\\ \therefore\ \tan(45°+A)-\tan(45°-A)=\frac{4\tan A}{1-\tan^2 A}=2\tan 2A\end{array}\right\}\ldots[\text{XV.}]$$

244. **Expressions involving Multiple Angles.**—Referring to the formulæ [IV.], put A for B, and $(n-1)A$ for A, and we then have

$$\left.\begin{array}{l}\sin nA+\sin(n-2)A=2\sin(n-1)A\cos A\\ \cos nA+\cos(n-2)A=2\cos(n-1)A\cos A\end{array}\right\}\ldots[\text{XVI.}]$$

And writing 2, 3, &c. for n, we have for the sines and cosines of multiple angles,

$$\sin 2A=2\sin A\cos A,\text{ as found above. Also, since }\cos^2 A=1-\sin^2 A,$$

$$\sin 3A=2\sin 2A\cos A-\sin A=4\sin A\cos^2 A-\sin A=3\sin A-4\sin^3 A\ldots[\text{XVII.}]$$

&c. &c.

$$\cos 2A=2\cos^2 A-1,\text{ as found above,}$$

$$\cos 3A=2\cos 2A\cos A-\cos A=4\cos^3 A-3\cos A\ldots[\text{XVIII.}]$$

&c. &c.

Making the same substitutions in [II.], we have

$$\tan nA=\frac{\tan(n-1)A+\tan A}{1-\tan A\tan(n-1)A}\ldots[\text{XIX.}]$$

which for $n=2$, 3, &c. gives $\tan 2A=\frac{2\tan A}{1-\tan^2 A}$, as before found,

$$\tan 3A=\frac{\tan 2A+\tan A}{1-\tan A\tan 2A}=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A},\ \&c.\ldots[\text{XX.}]$$

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245. **Trigonometrical Ambiguities.**—In the trigonometrical expressions established in the foregoing articles it has been seen, conformably to what was stated at page 171, that the trigonometrical ratios, and not the angles to which they belong, alone enter the various formulæ; these angles A, B, &c., occur in the notation, but it is sin A, cos A, tan A, &c., which furnish the numerical values concerned. Now as these values equally belong to a variety of different angles, ambiguities may present themselves unless a knowledge of the angle itself, or at least of the limits within which it lies, precludes such ambiguities. Thus taking the expressions [XII.],

$$\sin\frac{1}{2}A=\sqrt{\frac{1-\cos A}{2}},\ \cos\frac{1}{2}A=\sqrt{\frac{1+\cos A}{2}}.$$

Here $\sin\frac{1}{2}A$, $\cos\frac{1}{2}A$, are each ambiguous, as to sign, in the absence of all knowledge respecting the magnitude of the angle A, to which the pro-

posed cosine belongs: for it *may* belong to either of several angles. If A be known to lie between the limits 360°, 2.360°; or 3.360°, 4.360°; or 5.360°, 6.360°, &c., then $\sin \frac{1}{2}A$ must be negative: for other limits it is positive. Again, if A lie between the limits 180°, 3.180°; or 5.180°, 7.180°; or 9.180°, 11.180°, &c., then $\cos \frac{1}{2}A$ will be negative, and for other limits positive: the radical sign very properly provides for these different cases. And in all such instances of ambiguity the proper sign for the trigonometrical ratio will always be readily found by noticing in what quadrant the angle to which it belongs terminates. In the following formulæ [XXI.] derived from [XI.], putting $2A$ for A, there is a double ambiguity,

$$\sin A + \cos A = \sqrt{(1+\sin 2A)},\quad \sin A - \cos A = \sqrt{(1-\sin 2A)}$$

$$\therefore \sin A = \frac{1}{2}\left\{\sqrt{(1+\sin 2A)} + \sqrt{(1-\sin 2A)}\right.$$

$$\left.\cos A = \frac{1}{2}\left\{\sqrt{(1+\sin 2A)} - \sqrt{(1-\sin 2A)}\right.\right\} \ldots [XXI.]$$

By giving the double sign to each radical *four* values would be implied, and we might assign, as above, between what limits the angle must lie for any one of these four to be the true value; but as already observed, by knowing the limits beforehand, we can always readily affix the proper sign; we may just remark, however, that restricting the signs to those actually exhibited above, the forms will be correct for all angles A between the limits 180°, 3.180°; and if the signs of the second radical in each be interchanged, they will be correct for all angles between 0, 180°; or 3.180°, 4.180°. The student will remember that the ambiguities here commented upon all arise from this source, namely, that

$$\sin \theta = \sin (2n\pi + \theta) = \sin \{(2n+1)\pi - \theta\},\ \cos \theta = \cos (2n\pi \pm \theta).$$

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246. We shall now give a few applications of the preceding formulæ.

(1) Required the sine and cosine of 15°. By [IV.], we have

$$\cos (30°+15°) + \cos (30°-15°) = 2 \cos 30° \cos 15° \therefore \cos 45° = (2 \cos 30° - 1) \cos 15°.$$

But (220), $\cos 45° = \frac{1}{2}\sqrt{2}$, and $\cos 30° = \frac{1}{2}\sqrt{3}$ $\therefore \cos 15° = \frac{\sqrt{2}}{2(\sqrt{3}-1)} = \frac{1}{4}\sqrt{2}(\sqrt{3}+1)$.

Again, by [IX.], $\sin 15° \cos 15° = \frac{1}{2} \sin 30° = \frac{1}{4}$ (page 195),

$$\therefore \sin 15° = \frac{1}{4 \cos 15°} = \frac{1}{\sqrt{2}(\sqrt{3}+1)} = \frac{1}{4}\sqrt{2}(\sqrt{3}-1),$$

(2) Required the sine and cosine of 18°. By [IX.], $\sin 36° = 2 \sin 18° \cos 18°$, but $\sin 36° = \cos 54° = \cos 3.18° = 4\cos^3 18° - 3 \cos 18°$ by [XVIII.]: hence, dividing by $\cos 18°$, $2 \sin 18° = 4 \cos^2 18° - 3 = 4\ (1 - \sin^2 18°) - 3 = 1 - 4 \sin^2 18°$. Transposing, $4 \sin^2 18° + 2 \sin 18° = 1$, and the roots of this quadratic are $\frac{1}{4}(-1 \pm \sqrt{5})$, of which the positive root only is admissible,

$\therefore \sin 18° = \frac{1}{4}(-1+\sqrt{5})$. And since, as just shown, $2 \sin 18° = 4 \cos^2 18° - 3$,

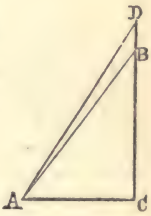
$$\therefore \tfrac{1}{2}(-1+\sqrt{5})+3 = 4\cos^2 18° \therefore \cos 18° = \tfrac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}} = \tfrac{1}{4}\sqrt{(10+2\sqrt{5})}.$$

(3) Given $\operatorname{cosec} A = \sin A + \tan \frac{1}{2}A$, to find $\sin A$. Transpose, then, by [XIII.],

$$\sin A = \frac{1}{\sin A} - \frac{1-\cos A}{\sin A} = \frac{\cos A}{\sin A} \therefore \cos A = \sin^2 A = 1 - \cos^2 A \therefore \cos^2 A + \cos A = 1$$

$$\therefore \cos A = \tfrac{1}{2}(-1 \pm \sqrt{5}) \therefore \sin A = \sqrt{\cos A} = \tfrac{1}{2}\sqrt{(-2 \pm 2\sqrt{5})}.$$

(4) An object 6 feet high placed on the top of a tower subtends an angle, of which the tangent is ·015, at a place in the same horizontal plane as the bottom of the tower, and 100 feet from it: required the height of the tower.



Put θ for the angle DAC, and θ' for BAC, then $\tan \theta = \frac{CD}{100}$, and $\tan \theta' = \frac{BC}{100} \therefore \tan \theta - \tan \theta' = \frac{6}{100} = ·06$. Now [II.], $\tan(\theta - \theta') =$

$$\frac{\tan\theta - \tan\theta'}{1+\tan\theta\tan\theta'} = \frac{·06}{1+\tan\theta(\tan\theta - ·06)} = ·015,$$

$$\text{or } \frac{4}{\tan^2\theta - ·06\tan\theta + 1} = 1 \therefore \tan^2\theta - ·06\tan\theta = 3.$$

Solving this quadratic, $\tan\theta = 1·7623\ldots \therefore \tan\theta' = \tan\theta - ·06 = 1·7023\ldots$ and $100\tan\theta' = 170·23\ldots$ feet, the height of the tower.

(5) In a plane triangle ABC are given two sides a, b, and the included angle C to find the third side c, without first finding the angles A, B. By art (236), and formula [XII.],

$$c^2 = a^2+b^2-2ab\cos C = a^2+b^2-2ab\left(1-2\sin^2\tfrac{1}{2}C\right) = (a-b)^2+4ab\sin^2\tfrac{1}{2}C =$$

$$(a-b)^2+\left(2\sqrt{ab}\sin\tfrac{1}{2}C\right)^2 = (a-b)^2\left\{1+\left(\frac{2\sqrt{ab}}{a-b}\sin\tfrac{1}{2}C\right)^2\right\}$$

Put now the second term within the brackets, (the square), $= \tan^2\varphi$, φ being an angle determinable from that square of its tan being known; then $c^2 = (a-b)^2\sec^2\varphi \therefore c = (a-b)\sec\varphi$. Hence c may be computed by logarithms; thus

1st, for φ, $\log\tan 2 + \frac{1}{2}\log a + \frac{1}{2}\log b - \log(a-b) + \log\sin\frac{1}{2}C = \log\tan\varphi$

From this value of $\log\tan\varphi$, φ is found from the tables, and thence $\sec\varphi$.

2nd, for c, $\log(a-b)+\log\sec\varphi - 10 = \log c$, and thence c from the tables.

This expedient of introducing a *subsidiary angle* φ, to bring an expression into a form calculable by logs, is frequently resorted to. The following is another way of effecting the object:—

$$c^2 = a^2+b^2-2ab\cos C = a^2+b^2-2ab\left(2\cos^2\tfrac{1}{2}C - 1\right), \text{ by [XII.]}$$

$$=(a+b)^2-\left(2\sqrt{ab}\cos\frac{1}{2}C\right)^2=(a+b)^2\left\{1-\left(\frac{2\sqrt{ab}}{a+b}\cos\frac{1}{2}C\right)^2\right\}.$$

Now $2\sqrt{ab}$ can never be greater than $a+b$, or than $(\sqrt{a}\sim\sqrt{b})^2+2\sqrt{ab}$, otherwise $(\sqrt{a}\sim\sqrt{b})^2$ would be negative, so that the square within the brackets can never be greater than 1; we may therefore put it $=\sin^2\varphi$, $\therefore c^2=(a+b)^2(1-\sin^2\varphi)$ $\therefore c=(a+b)\cos\varphi$. Hence c may be computed by logs as before.

(6) As another instance of the convenient introduction of a subsidiary angle, suppose, as sometimes happens, that instead of the sides a, b, being immediately given, the logs of them are given, and that from these and the angle C it be required to find the angles A, B.

$$\text{By (227) } \tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\frac{1}{2}C.$$

$$\text{Put } \frac{a-b}{a+b} \text{ or } \frac{\frac{a}{b}-1}{\frac{a}{b}+1}=\frac{\tan\varphi-1}{\tan\varphi+1}, \text{ that is, assume } \tan\varphi=\frac{a}{b};$$

then [XV.], $\tan\frac{1}{2}(A-B)=\tan(\varphi-45^\circ)\cot\frac{1}{2}C$. Hence, by logs,

1st, for φ, $\log a-\log b+10=\log\tan\varphi$ $\therefore$ φ is found from the tables,

2nd, for $\frac{1}{2}(A-B)$, $\log\tan(\varphi-45^\circ)+\log\cot\frac{1}{2}C=\log\tan\frac{1}{2}(A-B)$ $\therefore$ $\frac{1}{2}(A-B)$ is found from the tables, and $\frac{1}{2}(A+B)$ being known from $\frac{1}{2}C$, A and B become known.

(7) To find the distance AB between two inaccessible objects A, B, from the measurements given in ex. 1, page 189,

Put $CD=a$, $BCD=\alpha_1$, $CDB=\beta_1$, $ACD=\alpha_2$, $ADC=\beta_2$.

$$\text{From the triangles } CBD,\ CDA,\ DB=\frac{a\sin\alpha_1}{\sin(\alpha_1+\beta_1)},\ DA=\frac{a\sin\alpha_2}{\sin(\alpha_2+\beta_2)}.$$

Therefore two sides DA, DB, and the included angle of the triangle ADB, are given to find AB: and this we may do by logs, as explained in example 5.

(8) Give a general formula suited to computation by logs of problem 4, page 178.

Put $CBD=\alpha$, $A=\alpha_1$, and $BA=a$, then $DB=CD\cot\alpha$, $DA=CD\cot\alpha_1$

$$\therefore DA-DB=a=CD(\cot\alpha_1-\cot\alpha)\ \therefore CD=\frac{a}{\cot\alpha_1-\cot\alpha}=a\frac{\sin\alpha\sin\alpha_1}{\sin\alpha\cos\alpha_1-\cos\alpha\sin\alpha_1}$$

$$\therefore DB=CD\cot\alpha=CD\frac{\cos\alpha}{\sin\alpha}=a\frac{\cos\alpha\sin\alpha_1}{\sin(\alpha-\alpha_1)} \text{ by [I.]}$$

(9) Convert the expression $\cos\alpha\cos\beta+\sin\alpha\sin\beta\cos\gamma$ into a form suited to logs. Since $\frac{\sin}{\cos}=\tan$ $\therefore$ sin=cos . tan; hence the expression is

$$\cos\alpha\cos\beta(1+\tan\alpha\tan\beta\cos\gamma).$$

Put the second term in the brackets $=\tan\theta$, and the expression is

$$\cos\alpha\cos\beta(1+\tan\theta)=\cos\alpha\cos\beta\frac{\cos\theta+\sin\theta}{\cos\theta}=\cos\alpha\cos\beta\frac{\sin(\theta+45^\circ)}{\cos\theta}\sqrt{2},$$

by [XIV.], which is the form required.

(10) Eliminate θ from the equations $\sec\theta-\cos\theta=m$, $\operatorname{cosec}\theta-\sin\theta=n$; that is, find the relation between m and n.

Here $m=\dfrac{1}{\cos\theta}-\cos\theta=\dfrac{1-\cos^2\theta}{\cos\theta}=\dfrac{\sin^2\theta}{\cos\theta}$, $n=\dfrac{1}{\sin\theta}-\sin\theta=\dfrac{\cos^2\theta}{\sin\theta}$

$\therefore m.n^2=\cos^3\theta$, and $n.m^2=\sin^3\theta$ $\therefore (m.n^2)^{\frac{2}{3}}+(n.m^2)^{\frac{2}{3}}=\cos^2\theta+\sin^2\theta=1$.

(11) Prove that $2\cos(A+45°)\cos(A-45°)=\cos 2A$. By [XIV.]

$\cos(A+45°)=\dfrac{1}{\sqrt{2}}(\cos A-\sin A)$, $\cos(A-45°)=\dfrac{1}{\sqrt{2}}(\cos A+\sin A)$

$\therefore 2\cos(A+45°)\cos(A-45°)=\cos^2A-\sin^2A=\cos 2A$, by [VII.]

Examples for Exercise.

Prove the truth of the following expressions: namely,

(1) $\cos^4A-\sin^4A=\sin 2A$.

(2) $\tan\frac{1}{2}A+\cot\frac{1}{2}A=2\operatorname{cosec}A$.

(3) $\tan A+\tan B=\dfrac{\sin(A+B)}{\cos A\cos B}$.

(4) $\tan A-\tan B=\dfrac{\sin(A-B)}{\cos A\cos B}$.

(5) $\dfrac{\tan A+\tan B}{\tan A-\tan B}=\dfrac{\sin(A+B)}{\sin(A-B)}$.

(6) $\dfrac{\sin A\pm\sin B}{\cos A+\cos B}=\tan\frac{1}{2}(A\pm B)$.

(7) $\operatorname{cosec}A=1+\tan A\tan\frac{1}{2}A$.

(8) $\cos 2A+\cos 2B=2\cos(A+B)\cos(A-B)$.

(9) $\operatorname{cosec}2A=\frac{1}{2}\sec A\operatorname{cosec}A$.

(10) $\sin(A+B+C)=\sin A(\cos B\cos C-\sin B\sin C)$
$+\cos A(\sin B\cos C+\cos B\sin C)$.

(11) If $A+B+C=180°$, then $\cos^2A+\cos^2B+\cos^2C+2\cos A\cos B\cos C=1$.

(12) $\tan 3A=\dfrac{\sin A+\sin 3A+\sin 5A}{\cos A+\cos 3A+\cos 5A}$.

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(13) Given $\sec A=4\sin A$, to find A.

(14) Given $\operatorname{cosec}A=\sin A+\tan\frac{1}{2}A$, find A.

(15) Given $\sin 2A=\sin^2 3A$, find A.

(16) Given $4\sin x\sin 3x=1$, find x.

(17) $\tan(x+\alpha)+\cos(x+\alpha)=\sin(x-\alpha)+\cos(x-\alpha)$, find x.

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(18) Given $\left.\begin{aligned}\tan A+\sin A&=m\\ \tan A-\sin A&=n\end{aligned}\right\}$ find the relation between m and n.

(19) $\left.\begin{aligned}a\sin^2A+b\cos^2A&=m\\ b\sin^2B+a\cos^2B&=n\\ a\tan A&=b\tan b\end{aligned}\right\}$ show that $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{1}{m}+\dfrac{1}{n}$.

(20) Show that by means of a subsidiary angle ϕ, the expression

$$\sqrt{\{a+\sqrt{(a^2-b^2)}\}}+\sqrt{\{a-\sqrt{(a^2-b^2)}\}},$$

where $a>b$, may be put in the form following, adapted to logs, namely,

$$2\sin\left(\tfrac{1}{2}\phi+45°\right)\sqrt{a}.$$

247. **Inverse Trigonometrical Functions.**—If we had $\sin A=p$, and wished to express the angle A itself, we might do so thus: "A = the angle whose sine is p." But it has been agreed that the notation $\sin^{-1}p$ shall stand for the same thing, namely, "the angle whose sine is p." In like manner $\cos^{-1}q$ means "the angle, whose cosine is q." Similarly, $\log^{-1}n$ means the number whose log is n, and so on. These are called *inverse functions.**

(1) Prove that $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{4}{5}=90°$. Since $A+B=\sin^{-1}(\sin A\cos B+\cos A\sin B)$, by putting $\frac{3}{5}$ for $\sin A$, $\frac{4}{5}$ for $\sin B$, and consequently $\frac{4}{5}$ for $\cos A$, and $\frac{3}{5}$ for $\cos B$, we have

$$\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{4}{5}=\sin^{-1}\left(\frac{3}{5}\cdot\frac{3}{5}+\frac{4}{5}\cdot\frac{4}{5}\right)=\sin^{-1}1=90°.$$

(2) Prove that $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=45°$. Since $A+B=\tan^{-1}\frac{\tan A+\tan B}{1-\tan A\tan B}$, by putting $\frac{1}{2}$ for $\tan A$, and $\frac{1}{3}$ for $\tan B$, we have

$$\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=\tan^{-1}1=45°.$$

From the above general expression for $A+B$, an interesting theorem, due to Euler, may be deduced. Putting in that expression t_1 for $\tan A$, and t_2 for $\tan B$, it becomes

$$\tan^{-1}t_1-\tan^{-1}t_2=\tan^{-1}\frac{t_1-t_2}{1+t_1t_2}\ldots[1].$$

Now the following is obviously an identical equation, namely,

$$A_1=(A_1-A_2)+(A_2-A_3)+(A_3-A_4)+\ldots+(A_{n-1}-A_n)+A_n\ldots[2];$$

that is, it holds, whatever be put for A_1, A_2, &c. Put, therefore, $A_1=\tan^{-1}t_1$, $A_2=\tan^{-1}t_2$, $A_3=\tan^{-1}t_3$, &c., then by [1],

$$\tan^{-1}t_1=\tan^{-1}\frac{t_1-t_2}{1+t_1t_2}+\tan^{-1}\frac{t_2-t_3}{1+t_2t_3}+\tan^{-1}\frac{t_3-t_4}{1+t_3t_4}+\ldots+\tan^{-1}t_n\ldots[3].$$

[The identity [2], above, has enabled us to arrive at this formula of Euler in a manner much more simple than is usually done].

Let $n=2$, and take $t_1=1$, $t_2=t_n=\frac{1}{2}$;

$$\text{then } \tan^{-1}1=45°=\tan^{-1}\frac{\frac{1}{2}}{1+\frac{1}{2}}+\tan^{-1}\frac{1}{2}=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}.$$

Again: let $n=3$, taking $t_1=1$, $t_2=\frac{1}{2}$, $t_3=t_n=\frac{1}{3}$;

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* The above notation, which is now pretty generally adopted by British analysts, was first introduced by Sir John Herschel. It is certainly open to the objection of using a symbol hitherto employed as an exponent in a totally different sense; yet by regarding it for the moment *as* an exponent, the notation enables us very readily to pass from the inverse to the direct function, and the contrary.

$$\text{then } 45°=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}.$$

In like manner it will be found, by putting $n=4$, $t_1=1$, $t_2=\frac{1}{2}$, $t_3=\frac{3}{11}$, $t_4=\frac{1}{8}$,

$$\text{that } 45°=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8};$$

and so on to any extent. And in this way may any given angle be split up into any number of component angles, each of which has a determinable tangent. And it is to be observed that every tangent except one (namely, t_n) may be fixed beforehand: thus, in the instance just adduced, assuming the first three fractions of [3] equal to $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, respectively, we find $t_2=\frac{1}{2}$, $t_3=\frac{3}{11}$, and thence $t_4=\frac{1}{8}$.

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248. **Solution of a Quadratic by Trigonometrical Tables.** — Every quadratic equation is of one or other of the four following forms, p, q, being written with their proper signs:—

1. $x^2-px+q=0$. 2. $x^2-px-q=0$. 3. $x^2+px+q=0$. 4. $x^2+px-q=0$.

1. The roots of the first are $x=\frac{p}{2}\left\{1\pm\sqrt{\left(1-\frac{4q}{p^2}\right)}\right\}$. Let q be *less* than $\frac{1}{4}p^2$, then we may assume $\frac{4q}{p^2}=\sin^2\phi$ $\therefore\sqrt{\left(1-\frac{4q}{p^2}\right)}=\cos\phi$. Hence the roots are

$$x=\frac{1}{2}p\{1\pm\cos\phi\}=p\cos^2\frac{1}{2}\phi, \text{ or } p\sin^2\frac{1}{2}\phi \text{ by formulæ [XII.].}$$

Therefore, for the computation of the two values a_1, a_2, of x, we have

$$\left.\begin{aligned}\log a_1&=\log p+2\log\cos\tfrac{1}{2}\phi-20\\ \log a_2&=\log p+2\log\sin\tfrac{1}{2}\phi-20\end{aligned}\right\}, \text{ where } \log\sin\phi=\log 2+\frac{1}{2}\log q-\log p+10.$$

If q be *greater* than $\frac{1}{4}p^2$, we may assume $\frac{4q}{p^2}=\sec^2\phi$, and we shall then have

$$x=\frac{1}{2}p\{1\pm\sqrt{(1-\sec^2\phi)}\}=\frac{1}{2}p\{1\pm\tan\phi\sqrt{-1}\}: \textit{imaginary.}$$

2. The roots of the second equation are $x=\frac{p}{2}\left\{1\pm\sqrt{\left(1+\frac{4q}{p^2}\right)}\right\}$. Assume

$\frac{4q}{p^2}=\tan^2\phi$ $\therefore\sqrt{\left(1+\frac{4q}{p^2}\right)}=\sec\phi$. Hence the roots are $x=\frac{1}{2}p\{1\pm\sec\phi\}$,

but $\frac{1}{2}p=\frac{\sqrt{q}}{\tan\phi}$ $\therefore x=\sqrt{q}\cdot\frac{1\pm\sec\phi}{\tan\phi}=\sqrt{q}\cdot\frac{\cos\phi\pm1}{\sin\phi}=\sqrt{q}\cdot\frac{1}{\tan\frac{1}{2}\phi}$, or $-\sqrt{q}\cdot\tan\frac{1}{2}\phi$.

Therefore, for the computation of the two values a_1, a_2, of x, we have

$$\left.\begin{aligned}\log a_1&=\frac{1}{2}\log q-\log\tan\tfrac{1}{2}\phi+10\\ \log(-a_2)&=\frac{1}{2}\log q+\log\tan\tfrac{1}{2}\phi-10\end{aligned}\right\}, \text{ where } \log\tan\phi=\log 2+\frac{1}{2}\log q-\log p+10.$$

The last two forms of the equation become the same as the first two by changing the sign of p: hence the roots of the third and fourth equations are the same as the roots of the first and second, respectively, with changed signs (141).

NOTE.—Having determined one root of either equation as above, the other root may be got by subtracting the found root from the coef. of x with a changed sign (68). We see that the determination of a root, by this method, requires five references to the tables: it would be found more readily by Horner's method, however large the coefficients p, q, may be.

249. **Solution of a Cubic by Trigonometrical Tables.**—Let the equation, deprived of its second term, be $x^3-qx-r=0$, and put $x=\frac{y}{n}$, then $y^3-qn^2y-rn^3=0$. But [XVIII.], $\cos^3\varphi-\frac{3}{4}\cos\varphi-\frac{1}{4}\cos 3\varphi=0$.

To render these two equations identical, put

$$\cos\varphi=y,\ \frac{3}{4}=qn^2,\ \frac{1}{4}\cos 3\varphi=rn^3,\ \therefore n=\left(\frac{3}{4q}\right)^{\frac{1}{2}},\ \cos 3\varphi=4r\left(\frac{3}{4q}\right)^{\frac{3}{2}}.$$

Let the least value of φ which satisfies this condition be found from the tables; then one value of y, namely, $y=\cos\varphi$, will be determined, $\therefore$ one value of x is $x=\frac{\cos\varphi}{n}=\left(\frac{4q}{3}\right)^{\frac{1}{2}}\cos\varphi$. But $\cos 3\varphi$ is also $=\cos(2\pi\pm 3\varphi)$, that is, either $\frac{2\pi+3\varphi}{3}$, or $\frac{2\pi-3\varphi}{3}$, may be written for φ: Hence the three roots of the proposed cubic are

$$a_1=2\left(\frac{q}{3}\right)^{\frac{1}{2}}\cos\varphi,\ a_2=2\left(\frac{q}{3}\right)^{\frac{1}{2}}\cos\left(\frac{2\pi}{3}+\varphi\right),\ a_3=2\left(\frac{q}{3}\right)^{\frac{1}{2}}\cos\left(\frac{2\pi}{3}-\varphi\right).$$

This method of solution applies exclusively to the case in which all the roots are real—Cardan's irreducible case: for since $\cos 3\varphi$ is less than 1,

$$\therefore 4r\left(\frac{3}{4q}\right)^{\frac{3}{2}}<1,\ \therefore \frac{r^2}{4}<\left(\frac{q}{3}\right)^3.$$ See art. (183).

EXAMPLE. Let the equation be $x^3-147x-285\cdot5=0$. Here

$$\cos 3\varphi=1142\left(\frac{1}{196}\right)^{\frac{3}{2}}=\frac{1142}{2744}=\cdot4162=\cos 65^\circ\ 24',\ \therefore \varphi=21^\circ\ 48',\ \therefore \cos\varphi=\cdot92849,$$

$$\cos(120^\circ+21^\circ\ 48')=\cos 141^\circ\ 48'=-\cos 38^\circ\ 12'=-\cdot78586,$$
$$\cos(120^\circ-21^\circ\ 48')=\cos\ 98^\circ\ 12'=-\cos 81^\circ\ 48'=-\cdot14263.$$

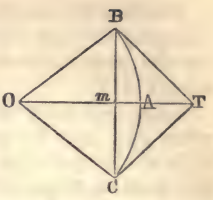
And multiplying each of these by $2(49)^{\frac{1}{2}}=14$, we have

$$a_1=12\cdot9988\ldots,\ a_2=-11\cdot0020\ldots,\ a_3=-1\cdot9968.$$

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250. **Construction of Trigonometrical Tables.**—Although the existing tables of the trigonometrical ratios supply every practical want, and render the labour of constructing them anew superfluous, yet it is well that the student should have some idea of the facilities which analysis affords for accomplishing such a work. We shall, therefore, give here a brief sketch of the mode of proceeding, first establishing one or two necessary preliminaries.

] An arc of a circle, less than a quadrant, is greater than the sine of that arc, and less than its tangent.



Let $AB=AC$, be an arc less than a quadrant; then $Bm=Cm$, is the sine of that arc: and since a straight line is the shortest distance between two points, $BmC<BAC$, $\therefore$ $Bm<$ arc BA: hence the arc is greater than its sine. Again: double the surface of the sector OBA is arc $AB\times OB$ (233), and double the surface of the triangle OBT, is $BT\times OB$; but this surface is greater than the former, $\therefore$ $BT>$arc BA: hence the arc is less than its tangent. We may regard the arc here referred to as belonging to the trigonometrical circle, and to be the arcual measure of the angle θ, and may therefore conclude that, of the three values $\sin\theta$, θ, $\tan\theta$, the middle value always lies between the other two values; therefore, of the three values 1, $\frac{\theta}{\sin\theta}$, $\frac{1}{\cos\theta}$, the middle value always lies between the other two values, however small θ be taken. But the more θ is diminished, the nearer does $\frac{1}{\cos\theta}$ approach in value to 1, which it ultimately becomes when θ is reduced to zero: hence $\frac{\theta}{\sin\theta}$, by diminishing θ, may be made to approach as near to 1 as we please, and it actually becomes 1 when θ is reduced to zero.

2. Although when θ is of any value, $\sin\theta<\theta$, yet if $\theta<\frac{1}{2}\pi$, $\sin\theta>\theta-\frac{\theta^3}{4}$.

For [IX.] $\sin\theta=2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$...[1], and as shown above, $\tan\frac{1}{2}\theta>\frac{1}{2}\theta$,

or $\sin\frac{1}{2}\theta>\frac{1}{2}\theta\cos\frac{1}{2}\theta$ $\therefore$ [1], $\sin\theta>\theta\cos^2\frac{1}{2}\theta$, or $>\theta\left(1-\sin^2\frac{1}{2}\theta\right)$.

But $\sin^2\frac{1}{2}\theta<\left(\frac{1}{2}\theta\right)^2$ $\therefore$ $\sin\theta>\theta-\frac{\theta^3}{4}$.

251. **To compute sin 10″ and cos 10″.**—The arcual measure of 10″ is

$$\frac{1}{180.60.6}\pi\text{ (art. 234)}=\frac{\pi}{64800}=\cdot000048481368110\ldots>\sin 10''.$$

$$\text{Also }\frac{\pi}{64800}-\frac{1}{4}\left(\frac{\pi}{64800}\right)^3=\cdot000048481368078\ldots<\sin 10''.$$

As these numerical values, between which sin 10″ lies, agree as far as twelve decimals, it follows that, to that extent,

$$\sin 10''=\cdot000048481368,\ \therefore\ \cos 10''=\sqrt{(1-\sin^2 10'')}=\cdot999999988248.$$

Having thus got the sine and cosine of 10″, we may compute the sine and cosine of 20″, of 30″, &c., from the formulæ [IV.], which give

$$\left.\begin{aligned}\sin(A+B)&=2\cos B\sin A-\sin(A-B)\\ \cos(A+B)&=2\cos B\cos A-\cos(A-B)\end{aligned}\right\}\ldots[1]$$

thus:

$\sin 20''=2\cos 10''\sin 10''$
$\sin 30''=2\cos 10''\sin 20''-\sin 10''$
$\sin 40''=2\cos 10''\sin 30''-\sin 20''$
&c.

$\cos 20''=2\cos 10''\cos 10''-1$
$\cos 30''=2\cos 10''\cos 20''-\cos 10''$
$\cos 40''=2\cos 10''\cos 30''-\cos 20''$
&c.

But after computing, as above, cos 10″, there is no occasion to compute the following cosines on the right, because the series of angles from 0° up to 90°, supply also the cosines. The labour of computing the successive sines is rendered considerable on account of the repeated multiplications by 2 cos 10″: but this may be diminished, thus:—Writing 10″ for B in [1], and $n.10''$ for A, the first equation becomes

$$\sin(n+1)10''=2\sin n\,10''\cos 10''-\sin(n-1)10'';$$

which, because $2\cos 10''=1{\cdot}9999999976496=2-{\cdot}0000000023504=2-k$, may be written $\sin(n+1)10''=2\sin n\,10''-k\sin n\,10''-\sin(n-1)10''$, or

$$\sin(n+1)10''=\sin n\,10''-\sin(n-1)10''+\sin n\,10''-k\sin n\,10''\ldots[2];$$

so that, making $n=1$, 2, 3, &c., we have

$$\begin{aligned}
\sin 20''&=\sin 10''-\sin 0''+\sin 10''-k\sin 10''\\
\sin 30''&=\sin 20''-\sin 10''+\sin 20''-k\sin 20''\\
\sin 40''&=\sin 30''-\sin 20''+\sin 30''-k\sin 30''\\
&\text{\&c.}\qquad\qquad\text{\&c.}
\end{aligned}$$

where the multiplier k is the comparatively small number ·0000000023504.

When the sines are calculated in this way up to sin 60°, the process may be replaced by a much more simple one. For [IV.]

$$\sin(60°+B)-\sin(60°-B)=2\cos 60°\sin B=\sin B\;\therefore\;\sin(60°+B)=\sin B+\sin(60°-B);$$

so that, making $B=10''$, 20″, &c., successively, as before, the sines above sin 60° will all be got from those below sin 60° by simple addition.

As to the tangents, they are found by dividing each of the several sines by the corresponding cosines: but this need be done only so far as tan 45°; because, since [XV.],

$$\tan(45°+B)-\tan(45°-B)=2\tan 2B,\;\therefore\;\tan(45°+B)=\tan(45°-B)+2\tan 2B.$$

The tangents up to 90° supply also the cotangents; and for the secants and cosecants, we have the formula

$$\operatorname{cosec}A=\frac{1}{2}\left(\tan\frac{1}{2}A+\cot\frac{1}{2}A\right)=\sec(90°-A).$$

The values of sin 10″, cos 10″, which form the basis of all the foregoing computations, it is necessary to calculate, as above, to a much greater number of decimals than are required to be inserted against each sine, cosine, &c., in the tables, in order that the original errors in sin 10″, cos 10″, may be so small that, even by accumulating, in consequence of their repeated introduction into the subsequent calculations, they may not amount to a value sufficiently great to affect the decimals intended to be preserved.

In the manner now explained, were the great tables computed by the French academicians, at the expense of the government, when the new division of the right angle into 100 parts, or *grades*, was resolved upon. As already noticed (p. 194), the innovation did not succeed, and the more recent French analysts, like those of Europe generally, employ the sexagesimal division.

In order to test the accuracy of the figures, as the work proceeded, they were compared, at intervals, with the rigorously exact results furnished by independent formulæ:—these were called *formulæ of verification*: the following are some of them, which the student may prove:—

$$\sin A=\cos(30°-A)-\cos(30°+A),\;\cos A=(\sin 30°+A)+\sin(30°-A),$$
$$\sin A+\sin(36°-A)+\sin(72°+A)=\sin(36°+A)+\sin(72°-A),$$

and [2], $\sin(n°+1°)=\sin n°-\sin(n°-1°)+\sin n°-k_1\sin n°$, where $k_1=2-2\cos 1°$.

The natural sines, cosines, &c., being computed and verified in this manner, their logarithms will form a table of log sines, log cosines, &c., but there are methods of constructing the logarithmic tables independently: they will be adverted to at the end of the PART next following.

PART III. GENERAL THEOREMS AND DEVELOPMENTS.

252. **Development of sin θ, cos θ.**—In the present article we propose to develope the functions sin θ and cos θ in a series, proceeding according to the ascending powers of the number θ which is the arcual measure of the angle.

1. Since $\sin(-\theta)=-\sin\theta$, it follows that the series for sin θ must be such that, when $-\theta$ is put for θ in the terms, all the signs of those terms must change: the series, therefore, cannot involve any even powers of θ, that is, it must be of the form

$$\sin\theta=A_1\theta+A_3\theta^3+A_5\theta^5+\ldots;\text{ so that }\frac{\sin\theta}{\theta}=A_1+A_3\theta^2+\ldots.$$

Let θ be diminished down to $\theta=0$; then (250) $1=A_1$; hence

$$\sin\theta=\theta+A_3\theta^3+A_5\theta^5+A_7\theta^7+\ldots[1].$$

2. Again: since $\cos(-\theta)=\cos\theta$, the substitution of $-\theta$ for θ, in the series for cos θ, cannot produce any change of sign in the terms: the series therefore cannot involve any odd powers of θ, that is, it must be of the form

$$\cos\theta=1+A_2\theta^2+A_4\theta^4+A_6\theta^6+\ldots[2].$$

The leading term must be 1, because, when $\theta=0$, $\cos\theta=1$.

$$\text{Put now } \theta=x+y, \text{ then } \left\{\begin{matrix}\sin\theta=\sin x\cos y+\cos x\sin y\\ \cos\theta=\cos x\cos y-\sin x\sin y\end{matrix}\right\}\ldots[3].$$

$$\text{Also [1], [2], } \sin\theta=(x+y)+A_3(x+y)^3+A_5(x+y)^5+A_7(x+y)^7+\ldots[4].$$
$$\cos\theta=\quad 1+A_2(x+y)^2+A_4(x+y)^4+A_6(x+y)^6+\ldots[5].$$

Substitute the developments of sin x, cos x, as given by [1] and [2], in [3], then the developments of sin θ, cos θ become

$$\sin\theta=(x+A_3x^3+A_5x^5+A_7x^7+\ldots)(1+A_2y^2+A_4y^4+A_6y^6+\ldots)+$$
$$(y+A_3y^3+A_5y^5+\ldots)(1+A_2x^2+A_4x^4+\ldots)\ldots[6]$$
$$\cos\theta=(1+A_2x^2+A_4x^4+A_6x^6+\ldots)(1+A_2y^2+A_4y^4+A_6y^6+\ldots)-$$
$$(x+A_3x^3+A_5x^5+\ldots)(y+A_3y^3+A_4y^4+\ldots)\ldots[7].$$

Now, by the Binomial Theorem, the coef. of y in [4] is the series

$$1+3A_3x^2+5A_5x^4+7A_7x^6+\ldots[8];$$

and from the last two factors in [6], we see that the coef. of y is

$$1+A_2x^2+A_4x^4+A_6x^6+\ldots[9].$$

The series [8], [9], being each the coef. of y, in the two identical series [4], [6], are equal, whatever be the value of x: hence the coefficients of the like powers of x in [8], [9], are equal.

In a similar manner, by equating the coefficients of y in the two identical series [5], [7], we have the two equal series

$$2A_2x+4A_4x^3+6A_6x^5+\ldots[10]$$
$$=-x-A_3x^3-A_5x^5-\ldots[11].$$

Equating now the coefficients of the like powers of x, in [10], [11], and in [8], [9], we have

$$2A_2=-1,\ 3A_3=A_2,\ 4A_4=-A_3,\ 5A_5=A_4,\ 6A_6=-A_5,\ \&c.$$

$$\therefore A_2=-\frac{1}{2},\ A_3=-\frac{1}{2.3},\ A_4=\frac{1}{2.3.4},\ A_5=\frac{1}{2.3.4.5},\ A_6=-\frac{1}{2.3.4.5.6},\ \&c.$$

$$\left.\begin{aligned}\therefore \sin\theta&=\theta-\frac{\theta^3}{2.3}+\frac{\theta^5}{2.3.4.5}-\frac{\theta^7}{2.3.4.5.6.7}+\ldots\\ \cos\theta&=1-\frac{\theta^2}{2}+\frac{\theta^4}{2.3.4}-\frac{\theta^6}{2.3.4.5.6}+\ldots\end{aligned}\right\}\ldots[\mathrm{A}].$$

253. This latter series, in addition to its application in higher inquiries, is of use in solving Case II. of plane triangles, whenever the given angle is so large as to render the Tables inefficient in giving its cosine with accuracy. Thus, C being the given angle, and $\pi-\theta$ its arcual measure, θ will be very small; so that in the series for $\cos\theta$, the third term, and the terms following, may be neglected. We may, therefore, write

$$c^2=a^2+b^2-2ab\cos C=a^2+b^2+2ab\cos\theta=a^2+b^2+2ab\left(1-\frac{\theta^2}{2}\right)=(a+b)^2-ab\theta^2=$$

$$(a+b)^2\left\{1-\frac{ab\theta^2}{(a+b)^2}\right\}.\quad\text{Extracting the square root, } c=(a+b)\left\{1-\frac{ab\theta^2}{2(a+b)^2}+\ldots\right\},$$

that is, still rejecting terms involving the fourth and higher powers of θ,

$$c=(a+b)\left\{1-\frac{ab\theta^2}{2(a+b)^2}\right\}=a+b-\frac{ab\theta^2}{2(a+b)}.$$

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254. **Euler's Expressions for sin θ, cos θ.**—By the Exponential Theorem (106)

$$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\ldots$$

$$e^{-x}=1-x+\frac{x^2}{2}-\frac{x^3}{2.3}+\frac{x^4}{2.3.4}-\ldots$$

Adding and subtracting, and dividing each result by 2, we have

$$\frac{e^x+e^{-x}}{2}=1+\frac{x^2}{2}+\frac{x^4}{2.3.4}+\ldots[1],\quad \frac{e^x-e^{-x}}{2}=x+\frac{x^3}{2.3}+\frac{x^5}{2.3.4.5}+\ldots[2].$$

As in these identities x may be anything, put $i\theta$ for x, where, as at (61), i stands for the imaginary $\sqrt{-1}$; we then have, after dividing [2] by i,

$$\frac{e^{i\theta}+e^{-i\theta}}{2}=1-\frac{\theta^2}{2}+\frac{\theta^4}{2.3.4}-\ldots[3],\quad \frac{e^{i\theta}-e^{-i\theta}}{2i}=\theta-\frac{\theta^3}{2.3}+\frac{\theta^5}{2.3.4.5}-\ldots[4].$$

Now the developments [3], [4], are those of $\cos\theta$, $\sin\theta$, respectively (252),

$$\therefore \sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i},\quad \cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}\ldots[\mathrm{B}],$$

which are Euler's imaginary expressions for $\sin\theta$, and $\cos\theta$.

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255. **De Moivre's Theorem.**—By Euler's expressions, multiplying the first by i, and then adding and subtracting,

$$e^{\pm i\theta}=\cos\theta\pm i\sin\theta\ldots[1],\text{ in which, since }\theta\text{ is arbitrary, put }n\theta\text{ for }\theta,$$

$$\therefore e^{\pm in\theta}=\cos n\theta\pm i\sin n\theta\ldots[2].\quad\text{But } e^{\pm in\theta}=\left(e^{\pm i\theta}\right)^n=(\cos\theta\pm i\sin\theta)^n \text{ by [1].}$$

Hence by [2], $(\cos\theta\pm i\sin\theta)^n=\cos n\theta\pm i\sin n\theta\ldots[\mathrm{C}]$,

n being any value whatever: which is De Moivre's Theorem. If we put u^n for the first member of [C], we shall have, for n positive,

$\cos n\theta \pm i \sin n\theta = u^n$, and for n negative,

$\cos n\theta \mp i \sin n\theta = \frac{1}{u^n}$; $\therefore$ by adding and subtracting,

$$2\cos n\theta = u^n + \frac{1}{u^n},\quad 2i \sin n\theta = u^n - \frac{1}{u^n} \ldots [5];$$

which, in fact, are no other than Euler's expressions [B], u^n being put for $e^{in\theta}$ (See [2] above).

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256. We would urge upon the student to pay special attention to the remarkable formulæ [B], [C], and to regard them as something more than matters of mere analytical curiosity. The expressions [B] embody, in a form purely algebraical, the entire theory of analytical trigonometry, and every result arrived at in the preceding PART may be reached in a direct manner by their aid: as an illustration, let it be required to prove the formula at page 211, namely, that $\tan\theta + \cot\theta = 2\operatorname{cosec} 2\theta$. Taking Euler's expressions for $\sin\theta$, $\cos\theta$, we have $\tan\theta + \cot\theta =$

$$\frac{e^{i\theta}-e^{-i\theta}}{i(e^{i\theta}+e^{-i\theta})} + \frac{i(e^{i\theta}+e^{-i\theta})}{e^{i\theta}-e^{-i\theta}} = \frac{(e^{i\theta}-e^{-i\theta})^2 - (e^{i\theta}+e^{-i\theta})^2}{i(e^{2i\theta}-e^{-2i\theta})} = \frac{-4}{i(e^{2i\theta}-e^{-2i\theta})}$$

which, since $-\frac{1}{i} = i$, is $= 2\frac{2i}{e^{2i\theta}-e^{-2i\theta}} = 2\frac{1}{\sin 2\theta} = 2\operatorname{cosec} 2\theta$.

257. The student cannot do better, at this stage of his progress, than to pause a little, and returning to the formulæ in the preceding PART, to establish a few of the results anew, by Euler's theorem, after the example here given. Such an exercise, while it will familiarize him with the practical use of the theorem in question, will give him additional confidence in a formula which, in consequence of the peculiar manner in which the imaginary symbol $\sqrt{-1}$ enters, has, at first sight, an appearance of mystery about it. The substitution of the letter i for this symbol—which letter can be very well spared for the purpose, will relieve his work from the apparent complication—for the complication is only apparent—which it would otherwise assume.

Whenever a single angle only is concerned, or whenever multiples only of *one* angle enter, as in the above instance, the still more abridged forms [5] may be employed (see ex. 3, p. 224).

But the formulæ [B] are important, not only on account of the direct and ready manner in which they conduct us to the pre-established truths of analytical trigonometry, but because they enable us to give certain valuable extensions to the subject, which, without their aid, it would be difficult to accomplish: we shall now enter upon some of these applications of Euler's forms.

258. [We shall stop, however, for a moment, to notice an elementary proposition of some importance in itself, and which will enable us to deduce immediately from [C] a result of equal consequence The proposition is this, namely:—Any expression of the form $a \pm b\sqrt{-1}$ is always equivalent to another of the form $r(\cos\theta \pm \sin\theta \cdot \sqrt{-1})$; that is, r may always be determined so that $\frac{a}{r}$ shall express the cosine of some angle θ, and $\frac{b}{r}$ its sine: for we have only to make r satisfy the condition $\left(\frac{a}{r}\right)^2 +$

$\left(\frac{b}{r}\right)^2=1$, that is, to make $r=\sqrt{(a^2+b^2)}$, in order that $\frac{a}{r}$ may be $=\cos\theta$, and $\frac{b}{r}=\sin\theta$. This being done, we have from [*C*], $(a\pm b\sqrt{-1})^n=r^n(\cos\theta\pm\sin\theta\,.\,\sqrt{-1})^n=r^n(\cos n\theta\pm\sin n\theta\,.\,\sqrt{-1})$; which shows that any power, positive, negative, or fractional, of $a\pm b\sqrt{-1}$, is always of *the same form*, namely, $A\pm B\sqrt{-1}$. This general truth could not have been so satisfactorily derived from the binomial development of the power, simply because, when n is negative or fractional, that development is an interminable series, with a supplemental "&c.," about which we know nothing.

The quantity $r=\sqrt{(a^2+b^2)}$ is called, by Cauchy, the *modulus* of $a\pm b\sqrt{-1}$. This distinguished analyst has made considerable use of these moduli: the following are the two fundamental theorems respecting them:—we shall establish them here in a manner much shorter than Cauchy has done ("Cours d'Analyse"). The theorems are 1. The product of two imaginary expressions has for modulus the product of their moduli. 2. The quotient of two imaginary expressions has for modulus the quotient of the two moduli.

Since $a+bi=r(\cos\theta+i\sin\theta)=re^{i\theta}$, and $a'+b'i=r'e^{i\theta'}$, and also $re^{i\theta}\times r'e^{i\theta'}=rr'e^{i(\theta+\theta')}$, $re^{i\theta}\div r'e^{i\theta'}=\frac{r}{r'}e^{i(\theta-\theta')}$, where the multipliers of the exponentials are the moduli, the theorems follow at once].

259. **Development of $\cos^n\theta$.**—In order to develope any positive integral power of a cosine in terms of simple cosines only, we have merely to apply the binomial theorem to Euler's expression for $2\cos\theta$: thus, connecting the terms equidistant from the first and last terms in pairs,

$$(e^{i\theta}+e^{-i\theta})^n=(e^{ni\theta}+e^{-ni\theta})+n(e^{(n-2)i\theta}+e^{-(n-2)i\theta})+\frac{n(n-1)}{2}(e^{(n-4)i\theta}+e^{-(n-4)i\theta})+\ldots$$

$$\text{that is, } 2^n\cos^n\theta=2\cos n\theta+2n\cos(n-2)\theta+2\frac{n(n-1)}{2}\cos(n-4)\theta+\ldots[1]$$

$$\therefore 2^{n-1}\cos^n\theta=\cos n\theta+n\cos(n-2)\theta+\frac{n(n-1)}{2}\cos(n-4)\theta+\ldots[2].$$

It will be observed that, as there are $n+1$ terms in the binomial development, there are but $\frac{1}{2}(n+1)$ pairs of terms, when n is odd, and $\frac{1}{2}n$ pairs, when n is even, with an additional single term:—the middle term, or that involving the product of *equal* powers of $e^{i\theta}$, $e^{-i\theta}$; which product being 1, $\cos\theta$ does not enter the term. As [2] is got from halving the terms in [1], this middle term must also be halved, that is, we are to *take half the coef. connected with cos 0 in the series* [2].

Hence, $2\cos^2\theta=\cos 2\theta+1$, $2^2\cos^3\theta=\cos 3\theta+3\cos\theta$, $2^3\cos^4\theta=\cos 4\theta+4\cos 2\theta+3$, $2^4\cos^5\theta=\cos 5\theta+5\cos 3\theta+10\cos\theta$, &c.

260. **Development of $\sin^n\theta$.**—Developing Euler's expression for $(2i\sin\theta)^n$, and uniting the corresponding terms as before, we have, when n is even,

$$(e^{i\theta}-e^{-i\theta})^n=(e^{ni\theta}+e^{-ni\theta})-n(e^{(n-2)i\theta}+e^{-(n-2)i\theta})+\frac{n(n-1)}{2}(e^{(n-4)i\theta}+e^{-(n-4)i\theta})-\ldots$$

that is, *when* n *is even*, remembering that $i=(-1)^{\frac{1}{2}}$,

$$(-1)^{\frac{n}{2}}2^{n-1}\sin^n\theta=\cos n\theta-n\cos(n-2)\theta+\frac{n(n-1)}{2}\cos(n-4)\theta-\ldots$$

If, however, n be odd, then the development is

$$(e^{i\theta}-e^{-i\theta})^n=(e^{ni\theta}-e^{-ni\theta})-n(e^{(n-2)i\theta}-e^{-(n-2)i\theta})+\frac{n(n-1)}{2}(e^{(n-4)i\theta}-e^{-(n-4)i\theta})-\ldots *$$

that is, *when* n *is odd*,

$$(-1)^{\frac{n-1}{2}}\,2^{n-1}\sin^n\theta=\sin n\theta-n\sin(n-2)\theta+\frac{n(n-1)}{2}\sin(n-4)\theta-\ldots$$

The power of -1, which determines the algebraic sign of $\sin^n\theta$, will be $+$ or $-$ according as the exponent of it is even or odd: hence

$$2\sin^2\theta=-\cos 2\theta+1,\quad 2^2\sin^3\theta=-\sin 3\theta+3\sin\theta$$
$$2^3\sin^4\theta=\cos 4\theta-4\cos 3\theta+3,\quad 2^4\sin^5\theta=\sin 5\theta-5\sin 3\theta+10\sin\theta,\ \&c.$$

NOTE.—As noticed at (255), Euler's expressions, and consequently the preceding general developments, would be abbreviated by writing u for $e^{i\theta}$, for then

$$2i\sin\theta=u-\frac{1}{u},\ 2\cos\theta=u+\frac{1}{u},\ \text{and}\ 2i\sin n\theta=u^n-\frac{1}{u^n},\ 2\cos n\theta=u^n+\frac{1}{u^n},$$

as in [5] at page 214.

261. **Developments of sin $n\theta$, cos $n\theta$.** — By De Moivre's Theorem,

$$\cos n\theta+i\sin n\theta=(\cos\theta+i\sin\theta)^n=\cos^n\theta+ni\cos^{n-1}\theta\sin\theta-\frac{n(n-1)}{2}\cos^{n-2}\theta\sin^2\theta+\ldots$$

where the imaginary i enters with the *odd* powers of $\sin\theta$ only. Hence, equating the real part of the first member with the real part of the second, and the imaginary part with the imaginary part (60), we have

$$\cos n\theta=\cos^n\theta-\frac{n(n-1)}{2}\cos^{n-2}\theta\sin^2\theta+\frac{n(n-1)(n-2)(n-3)}{2.3.4}\cos^{n-4}\theta\sin^4\theta-\ldots$$

$$\sin n\theta=n\cos^{n-1}\theta\sin\theta-\frac{n(n-1)(n-2)}{2.3}\cos^{n-3}\theta\sin^3\theta+$$
$$\frac{n(n-1)(n-2)(n-3)(n-4)}{2.3.4.5}\cos^{n-5}\theta\sin^5\theta-\ldots$$

$\therefore$ $\sin 2\theta=2\cos\theta\sin\theta$
$\sin 3\theta=3\cos^2\theta\sin\theta-\sin^3\theta$
$\sin 4\theta=4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta$
$\sin 5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$
&c. &c.

$\cos 2\theta=\cos^2\theta-\sin^2\theta$
$\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta$
$\cos 4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta$
$\cos 5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$
&c. &c.

262. **Development of θ in powers of tan θ.**—By Euler's forms, we have

$$e^{i\theta}=\cos\theta+i\sin\theta,\ e^{-i\theta}=\cos\theta-i\sin\theta.\quad \text{Dividing the first by the second,}$$
$$e^{2i\theta}=\frac{\cos\theta+i\sin\theta}{\cos\theta-i\sin\theta}=\frac{1+i\tan\theta}{1-i\tan\theta}.\quad \text{Taking the nap. log of each side,}$$
$$i\theta=\frac{1}{2}\log\frac{1+i\tan\theta}{1-i\tan\theta}=i\tan\theta+\frac{1}{3}(i\tan\theta)^3+\frac{1}{5}(i\tan\theta)^5+\frac{1}{7}(i\tan\theta)^7+\ldots$$

by (112). Dividing by i, and remembering that i^2, i^4, i^6, &c., are alternately $-$ and $+$,

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* In these expressions, as also in the analogous expression in the preceding article, the exponentials might have been replaced by the forms [5]. Thus, this last might have been written $(u-u^{-1})^n=(u^n-u^{-n})-n(u^{n-2}-u^{-(n-2)})+\ldots$

$$\left.\begin{aligned}\theta&=\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\frac{1}{7}\tan^7\theta+\ldots\\ \text{or, }\tan^{-1}t&=t-\frac{1}{3}t^3+\frac{1}{5}t^5-\frac{1}{7}t^7+\ldots\end{aligned}\right\}[1].$$

This is called Gregory's series for the arcual measure of an angle in a series of ascending powers of its tangent, it having been first discovered by James Gregory (though certainly not in the above manner), in 1771. [Gregory held the chair of Mathematics in St. Andrew's College, Aberdeen.]

The series is convergent only so long as tan θ does not exceed unit: beyond this limit it is useless for the purposes of calculation. When tan $\theta=1$, that is when $\theta=\frac{\pi}{4}$, we have

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots=\tan^{-1}1\ldots[2],$$

a series which, though convergent, is too slowly so for the numerical determination of π. The following are more suitable.

263. **Euler's Series for $\frac{1}{4}\pi$.** From the general series [3] page 207, putting 1 for t_1, and limiting the number of terms to two, we have $\tan^{-1}1=\tan^{-1}\frac{1-t_2}{1+t_2}+\tan^{-1}t_2$, where t_2 may be any value we please.

Put $t_2=\frac{1}{3}$, then we have

$$\frac{\pi}{4}=\tan^{-1}1=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\text{ by [1], }\left(\frac{1}{2}-\frac{1}{3.2^3}+\frac{1}{5.2^5}-\ldots\right)+\left(\frac{1}{3}-\frac{1}{3.3^3}+\frac{1}{5.3^5}-\ldots\right)\ldots[3].$$

which are Euler's series for $\frac{1}{4}\pi$.

264. **Machin's Series for $\frac{1}{4}\pi$.**—Referring to the same general series at page 207, and taking five of its terms, we have

$$\tan^{-1}1=\tan^{-1}\frac{1-t_2}{1+t_2}+\tan^{-1}\frac{t_2-t_3}{1+t_2t_3}+\tan^{-1}\frac{t_3-t_4}{1+t_3t_4}+\tan^{-1}\frac{t_4-t_5}{1+t_4t_5}+\tan^{-1}t_5,$$

where any values we please may be put for the fractions. Let each of them be $=\frac{1}{5}$; then we easily get

$$t_2=\frac{2}{3},\ t_3=\frac{7}{17},\ t_4=\frac{9}{46}\therefore t_5=-\frac{1}{239}.\quad\text{Consequently }\tan^{-1}1=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239},$$

$$\therefore\frac{\pi}{4}=\tan^{-1}1=\left.\begin{aligned}&4\left(\frac{1}{5}-\frac{1}{3.5^3}+\frac{1}{5.5^5}-\frac{1}{7.5^7}+\ldots\right)\\ &\left(\frac{1}{239}-\frac{1}{3.239^3}+\frac{1}{5.239^5}-\frac{1}{7.239^7}+\ldots\right)\end{aligned}\right\}\ldots[4].$$

These series are so rapidly convergent that, by summing up eight terms of the first, and only three of the second, we find for π the value $\pi=3{\cdot}14159265358979\ldots$.

It is unnecessary to prolong these investigations: from what has now been done, the student must perceive that by aid of the general series at

page 207, and the series [1], above, developments of $\theta=\tan^{-1}t_1$ for any value of θ not exceeding $\frac{1}{4}\pi$, may be multiplied to any extent. In the first series all the tangents t_2, t_3, &c., except one (which may be *any* one) may be chosen at pleasure, or all the quantities, or angles except one, connected with the symbol $\tan^{-1}$, may be chosen at pleasure.

265. **Development of $\frac{1}{2}\theta$ in Sines of its Multiples.**—In the logarithmic theorem [3], at page 81,

$$\text{Nap. log } m=(m-m^{-1})-\frac{1}{2}(m^2-m^{-2})+\frac{1}{3}(m^3-m^{-3})-\frac{1}{4}(m^4-m^{-4})+\ldots$$

put $e^{i\theta}$ for m; then (254), $i\theta=2i\sin\theta-\frac{1}{2}(2i\sin 2\theta)+\frac{1}{3}(2i\sin 3\theta)-\frac{1}{4}(2i\sin 4\theta)+\ldots$

therefore, dividing by $2i$, $\frac{1}{2}\theta=\sin\theta-\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta-\frac{1}{4}\sin 4\theta+\ldots$

which, for $\theta=\frac{1}{2}\pi$, gives the same series as that before found (p. 217).

266. **Development of sin θ, cos θ, in a series of Factors.**—By Algebra (p. 109), if a_1, a_2, &c., be the roots of $A_nx^n+\ldots=0$,

$$1+A_1x+A_2x^2+A_3x^3+\ldots=\left(1-\frac{x}{a_1}\right)\left(1-\frac{x}{a_2}\right)\left(1-\frac{x}{a_3}\right)\ldots$$

Now by [4], page 213, if $\sin\theta=0$, that is, if $\theta=0$, or π, or $-\pi$, or 2π, or -2π, &c., we have, after dividing by θ, $1-\frac{\theta^2}{2.3}+\frac{\theta^4}{2.3.4.5}-\frac{\theta^6}{2.3.4.5.6.7}+\ldots=0$

the roots of which equation, from what has been just said, are

$$a_1=\pi,\ a_2=-\pi,\ a_3=2\pi,\ a_4=-2\pi,\ \&c.,$$

$$\therefore \sin\theta=\theta\left(1-\frac{\theta}{\pi}\right)\left(1+\frac{\theta}{\pi}\right)\left(1-\frac{\theta}{2\pi}\right)\left(1+\frac{\theta}{2\pi}\right)\ldots$$

$$=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{4\pi^2}\right)\left(1-\frac{\theta^2}{9\pi^2}\right)\ldots$$

Similarly, if $\cos\theta=0$, that is, if $\theta=\frac{1}{2}\pi, -\frac{1}{2}\pi, \frac{3}{2}\pi, -\frac{3}{2}\pi$, &c., we have

$$1-\frac{\theta^2}{2}+\frac{\theta^4}{2.3.4}-\frac{\theta^6}{2.3.4.5.6}+\ldots=0,$$

the roots of which equation are $a_1=\frac{1}{2}\pi$, $a_2=-\frac{1}{2}\pi$, $a_3=\frac{3}{2}\pi$, $a_4=-\frac{3}{2}\pi$, &c.,

$$\therefore \cos\theta=\left(1-\frac{2\theta}{\pi}\right)\left(1+\frac{2\theta}{\pi}\right)\left(1-\frac{2\theta}{3\pi}\right)\left(1+\frac{2\theta}{3\pi}\right)\ldots$$

$$=\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{9\pi^2}\right)\left(1-\frac{4\theta^2}{25\pi^2}\right)\ldots$$

267. **Wallis's Expression for $\frac{1}{2}\pi$.**—If in the above expression for $\sin\theta$, we put $\theta=\frac{1}{2}\pi$, it becomes

$$\sin\frac{1}{2}\pi=\frac{1}{2}\pi\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\ldots=\frac{1}{2}\pi\cdot\frac{2^2-1}{2^2}\cdot\frac{4^2-1}{4^2}\cdot\frac{6^2-1}{6^2}\ldots$$

$$\text{that is, } 1=\frac{1}{2}\pi\cdot\frac{1.3}{2.2}\cdot\frac{3.5}{4.4}\cdot\frac{5.7}{6.6}\cdots,\ \therefore\frac{1}{2}\pi=\frac{2.2}{1.3}\cdot\frac{4.4}{3.5}\cdot\frac{6.6}{5.7}\cdots$$

which remarkable expression is due to Wallis.*

Sin θ may also be expressed by factors, as many as we please, by the repeated application of the formula $\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}$, and thence another form of continued product may be got for $\frac{1}{2}\pi$, thus:—

$$\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=2^2\sin\frac{\theta}{2^2}\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}=2^3\sin\frac{\theta}{2^3}\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}=\text{\&c.}$$

$$\therefore\sin\theta=2^n\sin\frac{\theta}{2^n}\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cos\frac{\theta}{2^4}\cdots\cos\frac{\theta}{2^n},\ n\text{ being any positive integer.}$$

$$\text{Let } n=\infty\text{, then (250), since } \sin\frac{\theta}{2^n}\div\frac{\theta}{2^n}=1\ \therefore\sin\frac{\theta}{2^n}\div\frac{1}{2^n}=2^n\sin\frac{\theta}{2^n}=\theta,$$

$$\therefore\sin\theta=\theta\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cos\frac{\theta}{2^4}\cdots 1.$$

Dividing by the second member, omitting the leading factor θ, we have

$$\sin\theta\sec\frac{\theta}{2}\sec\frac{\theta}{2^2}\sec\frac{\theta}{2^3}\cdots 1=\theta.\quad\text{Let } \theta=\frac{\pi}{2}\text{, then } \sin\theta=1\text{, and}$$

$$\therefore\frac{\pi}{2}=\sec 45^\circ\sec\frac{45^\circ}{2}\sec\frac{45^\circ}{2^2}\sec\frac{45^\circ}{2^3}\cdots 1.$$

$$\text{Similarly, } \frac{\pi}{3}=\sec 15^\circ\sec\frac{15^\circ}{2}\sec\frac{15^\circ}{2^2}\sec\frac{15^\circ}{2^3}\cdots 1,$$

which are rather curious results. They are due to Euler.

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268. **Imaginary Logarithms.**—If in the development of e^x, namely,

$$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\frac{x^5}{2.3.4.5}+\frac{x^6}{2.3.4.5.6}+\cdots$$

we substitute $2n\pi i$ for x we shall have

$$e^{2n\pi i}=1+2n\pi i-\frac{(2n\pi)^2}{2}-\frac{(2n\pi)^3 i}{2.3}+\frac{(2n\pi)^4}{2.3.4}+\frac{(2n\pi)^5 i}{2.3.4.5}-\frac{(2n\pi)^6}{2.3.4.5.6}-\cdots$$

$$=\left\{1-\frac{(2n\pi)^2}{2}+\frac{(2n\pi)^4}{2.3.4}-\frac{(2n\pi)^6}{2.3.4.5.6}+\cdots\right\}+\left\{2n\pi-\frac{(2n\pi)^3}{2.3}+\frac{(2n\pi)^5}{2.3.4.5}-\cdots\right\}i.$$

The first of these two series is the development of $\cos 2n\pi$, and the second the development of $\sin 2n\pi$ (page 213); and since $\cos 2n\pi=1$ (n being 0 or any positive or negative whole number), and $\sin 2n\pi=0$ $\therefore e^{2n\pi i}=1\ldots[1]$. Hence taking the nap. logs, $2n\pi i=\log 1$. If $n=0$, this is $0=\log 1$, which being the only real value of $\log 1$, is that employed in arithmetical computation. But when logs are applied to expressions where imaginary quantities are recognized, the imaginary values of $\log 1$, which are comprehended in $2n\pi i$, must not be neglected if our conclusions are expected to be general. Nor, in such a case, may the logs even of *negative numbers* be neglected; for that such logs exist under an imaginary

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* For an interesting deduction from Wallis's approximation to the value of π, which finds its application in the Theory of Probabilities, see De Morgan's Trigonometry and Double Algebra, 1849, p. 61.

form, will appear from putting $2n+1$ for $2n$, in the above development, and remembering that $\cos(2n+1)\pi = -1$, and $\sin(2n+1)\pi = 0$, so that

$$e^{(2n+1)\pi i} = -1 \therefore (2n+1)\pi i = \text{nap. log} -1, \therefore \text{ No real log exists. Multiplying by } N,$$
$$Ne^{(2n+1)\pi i} = -N \therefore \text{nap. log } N + (2n+1)\pi i = \text{nap. log} -N.$$

The logs here are Napierian, but by introducing the modulus M, corresponding to any base a, and considering $2n$ and $2n+1$ to be either positive or negative integers, as we are at liberty to do, we may generalize the foregoing results thus:—

$$\log N = \text{real log } N \pm 2Mn\pi\sqrt{-1}, \log -N = \text{real log } N \pm (2n+1)\pi\sqrt{-1} \ldots [2].$$

269. The conclusions arrived at in the preceding article enable us to give to the development of θ, marked [1] at page 217, a generality which at present it does not possess. It is plain that θ or $\tan^{-1} t$, has any one of the infinite number of values comprehended in $\pm n\pi + \theta$, for the same value of t, whereas the development [1] gives but *one* value, and that the true value only when θ does not exceed $\frac{1}{2}\pi$, even admitting divergent series.

The discrepancy arises from suppressing imaginary logs, when applying logs to an imaginary quantity:—it may be rectified thus:—

$$i\theta = \frac{1}{2}\log\frac{1+i\tan\theta}{1-i\tan\theta} = \pm n\pi i + i\tan\theta + \frac{1}{3}(i\tan\theta)^3 + \frac{1}{5}(i\tan\theta)^5 + \text{ \&c.}$$

$$\therefore \theta = \pm n\pi + \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^3\theta - \text{ \&c.}$$

Rejecting angles whose tangents exceed 1, as leading to diverging series, and putting dots for the "&c.," the second member of this equation represents the first in all its generality.

The development in (265) is, in like manner, defective:—it should be

$$\frac{1}{2}\theta = \pm n\pi + \sin\theta - \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta - \ldots$$

And here we would invite the student's attention to an apparent analytical paradox for which he is perhaps but little prepared: he might think it a fair inference from the above that $\theta = \pm 2n\pi + 2(\sin\theta - \frac{1}{2}\sin 2\theta + \ldots)$ has all the generality of the equation from which it is deduced, but such is not the case: for writing 2θ for θ, in that equation, we find that $\theta = \pm n\pi + \sin 2\theta - \frac{1}{2}\sin 4\theta + \ldots$, which two expressions for θ cannot be equal, inasmuch as $n\pi$ has twice as many values as $2n\pi$:—*odd* multiples of π being excluded from the latter.

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270. **Numbers of Bernoulli.**—There are certain numbers thus called, from the name of the analyst who first invited attention to them, which are of frequent use in certain trigonometrical and exponential developments: we may arrive at them by investigating the development of $\frac{1}{e^x - 1}$. Since (110),

$$e^x-1=x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\frac{x^5}{2.3.4.5}+\ldots[1],$$

unity divided by it must give a series of the form

$$\frac{1}{e^x-1}=\frac{A}{x}+A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\ldots[2]$$

$$\therefore\frac{1}{e^{-x}-1}=\frac{-e^x}{e^x-1}=-\frac{A}{x}+A_0-A_1x+A_2x^2-A_3x+A_4x^4-\ldots\ \therefore\text{ Adding,}$$

$$\frac{1-e^x}{e^x-1}=-1=2A_0+2A_2x^2+2A_4x^4+\ldots\ \therefore(94),\ A_0=-\frac{1}{2},\ A_2=0,\ A_4=0,\ \&c.$$

Now multiplying [1] and [2] together we have

$$1=\left(x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\ldots\right)\left(\frac{A}{x}+A_0+A_1x+A_2x^2+A_3x^3+\ldots\right)\ldots[3],$$

which being an identical equation, the product furnished by the second member of it must be such that the coefficients of the powers of x must each be zero, A being $=1$. We may arrive at the coef. of x by reversing the first two terms $\frac{A}{x}+A_0$ of the multiplier, writing them thus $A_0+\frac{A}{x}$, and then multiplying each by the term above it. In like manner for the coef. of x^2, reverse the first three terms of the multiplier, and then multiply each by the term above it: and so on. We thus get

$$A_0+\frac{1}{2}=0,\ A_1+\frac{A_0}{2}+\frac{1}{2.3}=0,\ A_2+\frac{A_1}{2}+\frac{A_0}{2.3}+\frac{1}{2.3.4}=0,$$

$$A_3+\frac{A_2}{2}+\frac{A_1}{2.3}+\frac{A_0}{2.3.4}+\frac{1}{2.3.4.5}=0,\ \&c.,$$

from which conditions we have to determine only A_0, A_1, A_3, &c., for as before seen, A_2, A_4, &c., are each 0.

$$\therefore A_0=-\frac{1}{2},\ A_1=\frac{1}{6}\cdot\frac{1}{2},\ A_3=-\frac{1}{30}\cdot\frac{1}{2.3.4},\ A_5=\frac{1}{42}\cdot\frac{1}{2.3.4.5.6},\ \&c.$$

Let the factors $\frac{1}{6}, -\frac{1}{30}, \frac{1}{42}$, &c., be represented by B_1, B_3, B_5, &c., we may then write [3] thus, putting for uniformity B_0 for A_0, and remembering that B_2, B_4, &c., are each 0,

$$1=\left(x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\ldots\right)\left(\frac{1}{x}+2B_0\frac{1}{2}+3B_1\frac{x}{2.3}+4B_2\frac{x^2}{2.3.4}+5B_3\frac{x^3}{2.3.4.5}+\ldots\right)$$

from which, by multiplying as before, we get the following relations, namely,

$$2B_0+1=0,\ 3B_1+3B_0+1=0,\ 4B_2+6B_1+4B_0+1=0,$$

$$5B_3+10B_2+10B_1+5B_0+1=0,\ 6B_4+15B_3+20B_2+15B_1+6B_0+1=0,$$

and so on, the numerical coefficients being those of the developed binomial, the leading coef. being absent.

In these equations of condition we already know that B_2, B_4, B_6, &c., being all connected with the even powers of x, are each 0, and determining the others, in order, we readily find that

$$B_1=\frac{1}{6},\ B_3=-\frac{1}{30},\ B_5=\frac{1}{42},\ B_7=-\frac{1}{30},\ B_9=\frac{5}{66},\ \&c.$$

And these are the numbers of Bernoulli. The development of $\frac{1}{e^x-1}$ is therefore

$$\frac{1}{e^x-1}=\frac{1}{x}-\frac{1}{2}+B_1\frac{x}{2}+B_3\frac{x^3}{2.3.4}+B_5\frac{x^5}{2.3.4.5.6}+\ \&c.$$

271. From the results of the foregoing investigation, it appears that for any Bernoullian number we have the general expression

$$(n+2)B_n=-1+\frac{1}{2}(n+2)-\frac{(n+2)(n+1)}{2}B_1-\frac{(n+2)(n+1)(n)(n-1)}{2.3.4}B_3-\ldots$$
$$-\frac{(n+2)(n+1)n}{2.3}B_{n-2}.*$$

[NOTE.—In some works, Bernoulli's numbers, when all taken with *positive signs*, are denoted by B_1, B_3, &c. It may be remarked here that, since $B_3=B_7$, the latter need never be computed.]

Required the development of $\tan\theta$ in terms of θ by Bernoulli's numbers.

$$\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{1}{i}\cdot\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}=\frac{1}{i}\cdot\frac{e^{2i\theta}-1}{e^{2i\theta}+1}=\frac{1}{i}\cdot\left(1-\frac{2}{e^{2i\theta}+1}\right).$$

But $$\frac{2}{e^{2x}-1}=\frac{1}{e^x-1}-\frac{1}{e^x+1}\therefore\frac{1}{e^x+1}=\frac{1}{e^x-1}-\frac{2}{e^{2x}-1}=$$

$$\left(\frac{1}{x}-\frac{1}{2}+B_1\frac{x}{2}+B_3\frac{x^3}{2.3.4}+\ldots\right)-2\left(\frac{1}{2x}-\frac{1}{2}+B_1\frac{2x}{2}+B_3\frac{2^3x^3}{2.3.4}+\ldots\right),\text{ that is,}$$

$$\frac{1}{e^x+1}=\frac{1}{2}-\frac{2^2-1}{2}B_1x-\frac{2^4-1}{2.3.4}B_3x^3-\frac{2^6-1}{2.3.4.5.6}B_5x^5-\text{ \&c.}$$

Substituting $2i\theta$ for x, and multiplying by 2, this becomes

$$\frac{2}{e^{2i\theta}+1}=1-\frac{2^2(2^2-1)}{2}B_1i\theta+\frac{2^4(2^4-1)}{2.3.4}B_3i\theta^3-\ldots\quad\text{Subtracting from 1 and dividing by } i,$$

$$\therefore\tan\theta=\frac{2^2(2^2-1)}{2}B_1\theta-\frac{2^4(2^4-1)}{2.3.4}B_3\theta^3+\frac{2^6(2^6-1)}{2.3.4.5.6}B_5\theta^5-\text{ \&c.}$$

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272. **Summation of Trigonometrical Series.**—1. Let it be required to find S, the sum to n terms, of the sines of a series of angles in arith. prog., namely,

$S=\sin\theta+\sin(\theta+\alpha)+\sin(\theta+2\alpha)+\ldots+\sin\{\theta+(n-1)\alpha\}$. By [IV.], page 200,

$$2\sin\tfrac{1}{2}\alpha\sin\theta\quad=\cos\left(\theta-\tfrac{1}{2}\alpha\right)-\cos\left(\theta+\tfrac{1}{2}\alpha\right)$$

$$2\sin\tfrac{1}{2}\alpha\sin(\theta+\alpha)\ =\cos\left(\theta+\tfrac{1}{2}\alpha\right)-\cos\left(\theta+\tfrac{3}{2}\alpha\right)$$

$$2\sin\tfrac{1}{2}\alpha\sin(\theta+2\alpha)=\cos\left(\theta+\tfrac{3}{2}\alpha\right)-\cos\left(\theta+\tfrac{5}{2}\alpha\right)$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$2\sin\tfrac{1}{2}\alpha\sin\{\theta+(n-1)\alpha\}=\cos\left\{\theta+\tfrac{1}{2}(2n-3)\alpha\right\}-\cos\left\{\theta+\tfrac{1}{2}(2n-1)\alpha\right\}$$

By adding up these, we have $2S\sin\frac{1}{2}\alpha=\cos\left(\theta-\frac{1}{2}\alpha\right)-\cos\left\{\theta+\frac{1}{2}(2n-1)\alpha\right\}$

$$\therefore S=\frac{\cos\left(\theta-\frac{1}{2}\alpha\right)-\cos\left\{\theta+\frac{1}{2}(2n-1)\alpha\right\}}{2\sin\frac{1}{2}\alpha}=\frac{\sin\left\{\theta+\frac{1}{2}(n-1)\alpha\right\}\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}\ldots[1].$$

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* The author is not aware that this formula has ever been given before.

2. Let it be required to find the sum S of n terms of the cosines of a series of angles in arith. prog., namely,

$S=\cos\theta+\cos(\theta+\alpha)+\cos(\theta+2\alpha)+\ldots+\cos\{\theta+(n-1)\alpha\}$. By [IV.], page 200,

$$2\sin\tfrac{1}{2}\alpha\cos\theta = \sin\left(\theta+\tfrac{1}{2}\alpha\right)-\sin\left(\theta-\tfrac{1}{2}\alpha\right)$$

$$2\sin\tfrac{1}{2}\alpha\cos(\theta+\alpha) = \sin\left(\theta+\tfrac{3}{2}\alpha\right)-\sin\left(\theta+\tfrac{1}{2}\alpha\right)$$

$$2\sin\tfrac{1}{2}\alpha\cos(\theta+2\alpha) = \sin\left(\theta+\tfrac{5}{2}\alpha\right)-\sin\left(\theta+\tfrac{3}{2}\alpha\right)$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$2\sin\tfrac{1}{2}\alpha\cos\{\theta+(n-1)\alpha\} = \sin\left\{\theta+\tfrac{1}{2}(2n-1)\alpha\right\}-\sin\left\{\theta+\tfrac{1}{2}(2n-3)\alpha\right\}$$

Adding, as before, $2S\sin\tfrac{1}{2}\alpha=\sin\left\{\theta+\tfrac{1}{2}(2n-1)\alpha\right\}-\sin\left(\theta-\tfrac{1}{2}\alpha\right)$

$$\therefore S=\frac{\sin\left\{\theta+\frac{1}{2}(2n-1)\alpha\right\}-\sin\left(\theta-\frac{1}{2}\alpha\right)}{2\sin\frac{1}{2}\alpha}=\frac{\cos\left\{\theta+\frac{1}{2}(n-1)\alpha\right\}\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}\ldots[2].$$

This expression [2], might have been deduced from [1], by putting in it $\theta+\frac{1}{2}\pi$ for θ.

If in [1] and [2] we put $\alpha=\theta$, 2θ, &c., in succession, we have

$$\sin\theta+\sin2\theta+\ldots+\sin n\theta=\frac{\sin\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta},$$

$$\sin\theta+\sin3\theta+\ldots+\sin(2n-1)\theta=\frac{\sin^2 n\theta}{\sin\theta},\ \text{&c.}$$

$$\cos\theta+\cos2\theta+\ldots+\cos n\theta=\frac{\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta},$$

$$\cos\theta+\cos3\theta+\ldots+\cos(2n-1)\theta=\frac{\sin2n\theta}{2\sin\theta},\ \text{&c.}$$

The sums of these latter series may also be readily obtained from the general forms [5] at page 214, that is, from $2i\sin n\theta=u^n-\frac{1}{u^n}$, $2\cos n\theta=u^n+\frac{1}{u^n}$. For, putting in succession 1, 2, ..., n, for n, and taking the sum of all the results, we have from the first form

$$2iS=(u+u^2+u^3+\ldots+u^n)-\left(\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\ldots+\frac{1}{u^n}\right)=u\frac{u^n-1}{u-1}-\frac{1}{u}\cdot\frac{\frac{1}{u^n}-1}{\frac{1}{u}-1}=$$

$$\frac{u^{n+1}-u+\frac{1}{u^n}-1}{u-1}=\frac{\left(u^{n+1}+\frac{1}{u^n}\right)-(u+1)}{u-1}=\frac{\left(u^{\frac{1}{2}}+\frac{1}{u^{n+\frac{1}{2}}}\right)-\left(u^{\frac{1}{2}}+\frac{1}{u^{\frac{1}{2}}}\right)}{u^{\frac{1}{2}}-\frac{1}{u^{\frac{1}{2}}}}=$$

$$\frac{2\cos\frac{1}{2}(2n+1)\theta-2\cos\frac{1}{2}\theta}{2i\sin\frac{1}{2}\theta}$$

$$\therefore \text{[IV.], p. 200, } S=\frac{\cos\frac{1}{2}\theta-\cos\frac{1}{2}(2n+1)\theta}{2\sin\frac{1}{2}\theta}=\frac{\sin\frac{1}{2}n\theta\sin\frac{1}{2}(n+1)\theta}{\sin\frac{1}{2}\theta}.$$

And in a similar manner may the sums of the other series be found.

3. Find the sum S of n terms of the series

$$x\sin\theta+x^2\sin 2\theta+x^3\sin 3\theta+\ldots$$

By the general forms just referred to,

$$2iS=(ux+u^2x^2+u^3x^3+\ldots+u^nx^n)-\left(\frac{x}{u}+\frac{x^2}{u^2}+\frac{x^3}{u^3}+\ldots+\frac{x^n}{u^n}\right)=$$

$$x\frac{(ux)^n-1}{ux-1}-u^{-1}x\frac{(u^{-1}x)^n-1}{u^{-1}x-1}.$$

Bringing these fractions to a com. dem., $2iS=\frac{ux(u^{n-1}x^{n+1}-u^nx^n-u^{-1}x)+1}{x^2-(u+u^{-1})x+1}-$

$$\frac{u^{-1}x(u^{-n+1}x^{n+1}-u^{-n}x^n-ux+1)}{x^2-(u+u^{-1})x+1}=\frac{(u^n-u^{-n})x^{n+2}-(u^{n+1}-u^{-(n+1)})x^{n+1}+(u-u^{-1})x}{x^2-(u+u^{-1})x+1}$$

$$\therefore S=\frac{x^{n+2}\sin n\theta-x^{n+1}\sin(n+1)\theta+x\sin\theta}{x^2-2x\cos\theta+1}.$$

Examples for Exercise.

(1) Required the cube root of $\cos 3\theta+\sin 3\theta\sqrt{-1}$.

(2) Required the fifth power of $\cos 2\theta-\sin 2\theta\sqrt{-1}$.

(3) Prove that $\sin n\theta=\cos^n\theta\left(n\tan\theta-\frac{n(n-1)(n-2)}{2.3}\tan^3\theta+\ldots\right)$.

$$\cos n\theta=\cos^n\theta\left(1-\frac{n(n-1)}{2}\tan^2\theta+\frac{n(n-1)(n-2)(n-3)}{2.3.4.}\tan^4\theta-\ldots\right)$$

(4) Required the sum of n terms of the series $\sin\theta-\sin(\theta+\alpha)+\sin(\theta+2\alpha)-\ldots$

(5) Find the value of $\frac{\sin\theta+\sin 3\theta+\sin 5\theta+\ldots+\sin(2n-1)\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\ldots+\cos(2n-1)\theta}$.

(6) Find the value of $S=\operatorname{cosec}\theta+\operatorname{cosec}2\theta+\operatorname{cosec}4\theta+\operatorname{cosec}8\theta+\ldots+\operatorname{cosec}2^{n-1}\theta$.

(7) Find $S=\tan\theta+2\tan\theta+2^2\tan 2^2\theta+2^3\tan^3\theta+\ldots+2^{n-1}\tan 2^{n-1}\theta$.

(8) Find $S=x\cos\theta+x^2\cos 2\theta+x^3\cos 3\theta+\ldots+x^n\cos n\theta$.

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* The student will notice that the second of these fractions is got from the first by simply changing u into u^{-1}, and in like manner is the second of the two fractions following got from the first of them.

273. **Imaginary Roots of ± 1.**—Let x stand for either of the nth roots of positive unity indifferently, that is let $x^n=1$. By De Moivre's Theorem,

$$x=(\cos 2m\pi \pm i \sin 2m\pi)^{\frac{1}{n}}=\cos\frac{2m}{n}\pi \pm i \sin\frac{2m}{n}\pi=1^{\frac{1}{n}} \ldots [1],$$

which general expression for $1^{\frac{1}{n}}$ is real only for such values of n as will render $\sin\frac{2m}{n}\pi$ zero, that is, for such values only as will render $\frac{2m}{n}\pi$ either 0, or a multiple of π. Now if n be *odd*, we shall keep clear of these real values by putting, for the arbitrary integer m, the successive values 1, 2, 3, &c., stopping when $2m=n-1$, that is when $m=\frac{1}{2}(n-1)$; we shall thus get $\frac{1}{2}(n-1)$ pairs of imaginary values for $1^{\frac{1}{n}}$, that is $n-1$ imaginary roots, and no more; for if after getting the imaginary value due to $2m=n-1$, we were to continue the substitutions, the former values would recur, as is obvious. Hence, when n is odd, $1^{\frac{1}{n}}$ has $n-1$ imaginary roots, all included in the form [1], and a single real root, namely, $\cos 0=1$.

But if n be *even*, we shall avoid the real values in [1] by putting for m, the successive values 1, 2, 3, &c., stopping when $2m=n-2$, that is when $m=\frac{1}{2}(n-2)$; we shall thus get $\frac{1}{2}(n-2)$ pairs of imaginary values for $1^{\frac{1}{n}}$, that is, $n-2$ imaginary roots, and no more; for if after getting the real value due to $2m=n$, we were to continue the substitutions, the former imaginary values would recur. Hence, when n is even, $1^{\frac{1}{n}}$ has $n-2$ imaginary roots, and two real roots, namely, $\cos 0=1$ and $\cos\pi=-1$.

If we combine the factors of x^n-1, as given by [1], in conjugate pairs (139), recollecting that $\cos^2+\sin^2=1$, we shall have the following decomposition of x^n-1 in quadratic factors, united, when n is odd, to the simple factor $(x-1)$, namely:—

$$\text{For } n \text{ odd, } x^n-1=(x-1)\left\{\left(x^2-2\cos\frac{2\pi}{n}x+1\right)\left(x^2-2\cos\frac{4\pi}{n}x+1\right)\ldots\right.$$
$$\left.\left(x^2-2\cos\frac{(n-1)\pi}{n}x+1\right)\right\}.$$

$$\text{For } n \text{ even, } x^n-1=(x^2-1)\left\{\left(x^2-2\cos\frac{2\pi}{n}x+1\right)\left(x^2-2\cos\frac{4\pi}{n}x+1\right)\ldots\right.$$
$$\left.\left(x^2-2\cos\frac{(n-2)\pi}{n}x+1\right)\right\},$$

the quadratic factors within the braces being, in the first case, $\frac{1}{2}(n-1)$ in number, and in the second case, $\frac{1}{2}n$ in number. From each of these, equated to 0, a pair of imaginary roots will be obtained.

For the n values of $(-1)^n$, that is, for the n roots of x^n+1, we have by De Moivre's Theorem,

$$x=\{\cos(2m+1)\pi\pm i\sin(2m+1)\pi\}^{\frac{1}{n}}=\cos\frac{2m+1}{n}\pi\pm i\sin\frac{2m+1}{n}\pi=(-1)^{\frac{1}{n}}\ldots[2],$$

which general expression for $(-1)^{\frac{1}{n}}$ is real only for such values of n as will make the sine zero. If n be *odd*, then for $m=\frac{1}{2}(n-1)$, we have $x=-1$, the only real value, the $\frac{1}{2}(n-1)$ pairs of imaginary values being got by putting successively 0, 1, 2, &c., up to $\frac{1}{2}(n-1)-1$, for m.

But if n be *even*, there can be no real value included in [2]; for then $\frac{2m+1}{n}\pi$ cannot become either 0, or a multiple of π: all the n values are therefore imaginary, and the $\frac{1}{2}n$ pairs are found by putting in succession 0, 1, 2, &c., up to $\frac{1}{2}(n-2)$ for m. Hence, as above,

$$\text{For } n \text{ odd, } x^n+1=(x+1)\left\{\left(x^2-2\cos\frac{\pi}{n}x+1\right)\left(x^2-2\cos\frac{3\pi}{n}x+1\right)\ldots\left(x^2-2\cos\frac{(n-2)\pi}{n}x+1\right)\right\}.$$

$$\text{For } n \text{ even, } x^n+1=\left\{\left(x^2-2\cos\frac{\pi}{n}x+1\right)\left(x^2-2\cos\frac{3\pi}{n}x+1\right)\ldots\left(x^2-2\cos\frac{(n-1)\pi}{n}x+1\right)\right\}.$$

Ex. Required the cube roots of 1, and -1. From $x=\cos\frac{2m\pi}{n}\pm i\sin\frac{2m\pi}{n}$, by putting for m, 0 and 1 successively, we have $x=1$, and $x=\cos\frac{2\pi}{3}\pm i\sin\frac{2\pi}{3}=\cos 120°\pm i\sin 120°=-\cos 60°\pm i\sin 60°=-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}=\frac{-1\pm\sqrt{-3}}{2}=1^{\frac{1}{3}}$.

Again: From $x=\cos\frac{(2m+1)\pi}{n}\pm i\sin\frac{(2m+1)\pi}{n}$, by putting 0 for m, we have $x=\frac{\pi}{3}\pm i\sin\frac{\pi}{3}=\cos 60°\pm i\sin 60°=\frac{1}{2}\pm\frac{i\sqrt{3}}{2}=\frac{1\pm\sqrt{-3}}{2}$, and by putting 1 for m, $x=-1\ \therefore 1^{\frac{1}{3}}=1$ or $\frac{-1\pm\sqrt{-3}}{2}$, and $(-1)^{\frac{1}{3}}=-1$, or $\frac{1\pm\sqrt{-3}}{2}$.

Examples for Exercise.

(1) Find $N^{\frac{1}{3}}$, or $1^{\frac{1}{3}}\sqrt[3]{N}$.
(2) Find $(-N)^{\frac{1}{3}}$, or $(-1)^{\frac{1}{3}}\sqrt[3]{N}$.
(3) Find the values of $1^{\frac{1}{3}}$.
(4) Resolve $x^{2n}-2\cos\theta\,.\,x^n+1$ into its n quadratic factors, and thence show that all the roots of the equation $x^{2n}-px+1=0$ can be found whenever p is not greater than 2.

---

274. **Construction of Log Sines and Cosines.**—At page 212, we promised to advert to the method of computing log sines, and

log cosines, without first finding the natural sines and cosines. The method is this :—

From $\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots$, at page 218, we have

by putting $\frac{m}{n}\cdot\frac{\pi}{2}$ for θ, and taking the logs of both sides,

$$\log\sin\frac{m}{n}\cdot\frac{\pi}{2}=\log\frac{m}{n}+\log\frac{\pi}{2}+\log\left(1-\frac{m^2}{2^2n^2}\right)+\log\left(1-\frac{m^2}{4^2n^2}\right)+\log\left(1-\frac{m^2}{6^2n^2}\right)+\ldots$$

Now the first three terms of this series are $\log\frac{m}{n}+\log\frac{\pi}{2}+\log\frac{(2n+m)(2n-m)}{2^2n^2}=$

$$\log\pi+\log m+\log(2n+m)+\log(2n-m)-3(\log n+\log 2).$$

The developments of the terms which follow these three, when conveniently arranged, are (p. 81)

$$-M\left(\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\ldots\right)\left(\frac{m}{n}\right)^2-\frac{M}{2}\left(\frac{1}{4^2}+\frac{1}{6^4}+\frac{1}{8^4}+\ldots\right)\left(\frac{m}{n}\right)^4$$
$$-\frac{M}{3}\left(\frac{1}{4^6}+\frac{1}{6^6}+\frac{1}{8^6}+\ldots\right)\left(\frac{m}{n}\right)^6-\ldots$$

And by giving suitable values to m and n, the log sine of any angle may thus be approximated to, to any degree of exactness. Log $\cos\frac{m}{n}\cdot\frac{\pi}{2}$ may be found, in a similar manner, from the factors of the cosine at page 218, and thence the log tangents, and log secants. [On this subject the reader may consult the Appendix to "Woodhouse's Trigonometry."]

END OF THE PLANE TRIGONOMETRY.

III. SPHERICAL TRIGONOMETRY.

275. **Preliminary Theorems.** — Before entering upon the theory of the Spherical Triangle, it will be necessary to establish a few elementary propositions concerning the circles of the sphere, the arcs of which form the sides of every such triangle. The following principles (1, 2, 3,) sufficiently prepare the way for these.

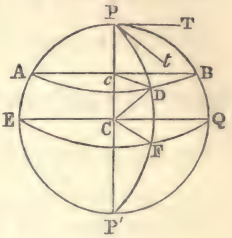
1. Three points anyhow situated in space all lie in the same plane; for having joined any two of the points by a straight line, we may conceive a plane to pass through the line, which plane by being turned round the line, as upon an axis, must at length arrive at the third point: so that in that position all the three points must lie in the plane. Any three points, therefore, not in a straight line, taken in a plane, completely determine the position of that plane. [We thus see, what has hitherto been tacitly assumed, that the figure inclosed by three straight lines is always a *plane* triangle.]

2. The common intersection of two planes is a straight line; for if among the points in the intersection there be three which are not in the same straight line, since these three are in *both* planes, the planes must coincide and form but *one* plane.

3. *Definition.*—A straight line is said to be perpendicular to a plane, when it is perpendicular to every straight line in that plane, drawn through its *foot*, that is, through the point where it meets the plane.

276. THEOREM I. If a sphere be anyhow cut by a plane, the section is a circle.

Let C be the centre of the sphere, and ADB any plane section. Draw Cc perpendicular to the plane, and from c, the foot of this perpendicular, draw any line cD in the section, terminating in some point D in the surface of the sphere: then the angle CcD must be a right angle (by 3, above). Draw CD; then wherever on the surface the point D may be, CD is always of the same constant length, namely, the radius R of the sphere, $\therefore cD = \sqrt{(R^2 - Cc^2)}$: hence the length of cD, that is, of any straight line drawn from c to the boundary ADB, &c., is constant, $\therefore$ c is the centre, and ADB, &c., the circumference of a *circle*, cD being the radius r.



The less Cc is, the greater does $r = \sqrt{(R^2 - Cc^2)}$ become: it is, therefore, the greatest possible when $Cc = 0$, that is, when the section passes through the centre C of the sphere. On this account a circle, the plane of which passes through the centre of the sphere, is called a *great circle* of the sphere, while every circle of which the plane does not pass through the centre C is called a *small* circle of the sphere.

Through any two points on the surface a *great* circle may be drawn, because these two, with the centre, make *three* points, which (by 1 above) are all in one plane. A circle of the sphere of one kind or other may always be drawn through *three* points on the surface, because a plane may be made to pass through them; and every plane, as just shown, produces a circular section of the sphere.

If the line Cc, perpendicular to the circular section, be prolonged to pierce the surface in the points P, P', these points are called the *poles* of the circle; if the centre c of this circle do not coincide with C, the centre of the sphere, the poles of it are unequally distant from its plane; but if c coincides with C, that is, if the section be a great circle, its poles are equidistant from the plane of that circle.

Since PP' is a diameter of the sphere, and since planes may pass in all directions through PP', cutting the surface in great circles, it follows that an indefinite number of *great* circles may be drawn through the poles of any other circle. The distance of any circle from either one of its poles—measured upon any one of the great circles through both poles—is constantly the same, that is, the arcs PB, PD, &c., are equal, because Pc is the common versed sine of all those arcs: hence the other arcs $P'B$, $P'D$, &c., must be equal. Every point, therefore, in the circumference of a great circle is 90° distant from each pole of that circle: so that if a circular radius equal to the quadrant of a great circle be applied to the surface of the sphere from any fixed point, and the other extremity be made to describe a line, that line will be the great circle of which the fixed extremity of the circular radius is one of the poles.

277. II. Two great circles always intersect in two points at the distance of a semicircle, that is, the circumferences bisect each other.

For since the plane of each passes through the centre of the sphere, the intersection of these planes must be a diameter of the sphere common to both circles, and it is at the extremities of this diameter that the circumferences cross each other.

Hence, if from any point on a sphere two quadrantal arcs can be drawn

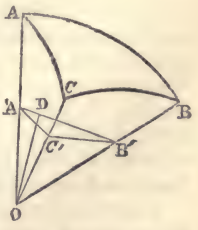
to two points in any great circle—which points are less than a semicircle, or 180° apart—then the first point must be a pole of this great circle. For it is necessarily a pole of *some* great circle passing through the proposed points, and as only *one* great circle can pass through two points less than 180° apart, the pole must belong to the great circle mentioned.

278. **Measure of a Spherical Angle.**—The sides of the spherical triangles considered in Trigonometry are all arcs of *great circles* only, and the angle included between two such arcs, that is, a spherical angle, is measured in a manner similar to that in which a plane angle is measured. For the measure of a plane angle we take the intercepted arc of the circle whose centre is at the vertex of the angle: the number of degrees in this arc is the gradeal measure of the angle: the number of units in it (radius being 1 unit) is the arcual measure. Similarly, for the measure of a spherical angle we take the intercepted arc of that great circle whose pole is at the vertex of the angle, the linear radius of this great circle being that of the sphere: thus, the spherical angle BPD is measured by the arc QF, of which P is the pole, and CQ, the radius of the sphere, the linear radius: expressed in degrees, this arc is the gradeal measure: expressed in units of the radius, it is the arcual measure. But the plane angle QCF is also measured by this same arc: hence the plane angle at C, formed by perpendiculars to the line of intersection PP', of the two planes through C, in which the sides of the spherical angle are situated—a perpendicular being in each plane: this plane angle C has the same measure as the spherical angle P.

It is obvious that the plane angle at C, here referred to, is the angle which would be presented by a section of the two planes in which the arcs PB, PD, are situated, by a third plane through C, and perpendicular to the common edge PP' of the two former. Every plane parallel to this third plane, that is, perpendicular to the edge PP', would give for section an equal plane angle, however the two planes meeting in PP' be prolonged. But the plane of the angle TPt is equally perpendicular to PP': hence the angle TPt is also equal to the spherical angle P, to the sides of which the sides of the plane angle are tangents: so that the measure of the plane angle formed by two tangents to the sides of a spherical angle, at its vertex, is equally the measure of the spherical angle itself.

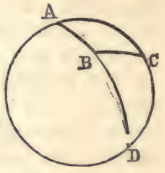
279. III. Any one side of a spherical triangle is less than the sum of the other two sides.

Let O be the centre of the sphere, and ABC a spherical triangle on its surface. Draw the radii OA, OB, OC; then the three angles at O will be in three distinct planes: we may consider the angle AOB to be in the plane of the paper, and the other two to be in inclined planes above the paper. These angles will be measured by the arcs AB, AC, BC, respectively, the radius, OA, or OB, OC, being *unit*. Let AB be the greatest of these three arcs: then it will be only necessary to prove that $AB < (AC + CB)$, or that the angle $AOB < (AOC + BOC)$. In the plane of AOB draw any line $A'B'$, and make an angle BOD in that plane equal to BOC in the inclined plane: make also $OC' = OD$, and draw $A'C'$, $B'C'$. Then by construction, the two sides OB', OD, and the included angle, are respectively equal to the two sides OB', OC', and the



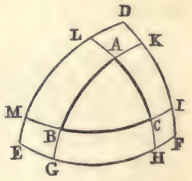
included angle, $\therefore B'D = B'C'$. But in the plane triangle $A'C'B'$, $A'B' < (A'C' + B'C')$: take away the equals $B'D$, $B'C'$, $\therefore A'D < A'C'$: hence, though the two sides OA', OD, of the triangle $A'OD$, are equal respectively to the two OA', OC', of the triangle $A'OC'$, yet as the third side $A'D$ of the former is less than the third side $A'C'$ of the latter, the angle $A'OD < A'OC'$. Now $B'OD$ was made equal to $B'OC'$, $\therefore (A'OD + B'OD) = A'OB < (A'OC' + B'OC')$, $\therefore AB < (AC + CB)$.

280. IV. The three sides of a spherical triangle taken together make less than a whole circumference of a great circle.



Let ABC be any sph. triangle. Prolong the sides AB, AC, till they meet again in D: then the arcs ABD, ACD, will be semicircumferences, because two great circles always bisect each other (II). Now in the triangle BCD, $BC < (BD + CD)$, $\therefore$ adding $AB + AC$, $(AB + AC + BC) < (ABD + ACD)$, $\therefore$ the sum of the three sides is less than the circumference of a great circle. Also no one side can equal 180°, since any one side is less than the sum of the other two. [It may be easily proved from this proposition that the sum of the sides of any spherical polygon, which has no re-entering angles, is less than the whole circumference of a great circle, the arcs themselves all being arcs of great circles.]

281. V. If from the vertices of a sph. triangle, as poles, arcs be described, forming a new triangle, then the vertices of this new triangle will be poles of the sides opposite to them of the original triangle.



Let ABC be any sph. triangle. With pole A, and circular radius AG = a quadrant, describe the arc EF. With pole B, and same radius, describe the arc FD. And lastly, with pole C, and same radius, describe the arc ED, thus completing the sph. triangle DEF. Then, since the arcs whose poles are A, C, intersect at E, the points A, C, are each 90° from E. And since the arc AC is less than 180°, E must be a pole of AC (277). In like manner, F is a pole of AB, and D a pole of BC.

The triangle DEF is called the *polar triangle* in reference to the *primitive triangle* ABC.

282. VI. Each angle of the primitive triangle is the supplement of the side opposite to it of the polar triangle, and each angle of the polar triangle, is the supplement of the side opposite to it of the primitive triangle.

For EH being the quadrantal radius of HL is 90°: also $FG = 90°$, $\therefore$ the sum of these radii, that is, $EH + HF + GH = EF + GH = 180°$, $\therefore GH$, the measure of the angle A, is the supplement of the side EF opposite to it. Similarly, B is the supplement of DF, and C of DE. Again: BI being the circular radius of ID, and CM that of MD, the sum of these, $MI + BC = 180°$, $\therefore BC$ is the supplement of MI, the measure of the angle D opposite to it. The triangles ABC, DEF, are hence usually called *supplemental triangles*.

Legendre has properly remarked that, besides the triangle DEF, three others might be formed by prolonging the sides till they again intersect; but that the proposition above applies exclusively to the central one of the four triangles thus formed, and which is distinguished from the others by the circumstance that the two angles A, D, are on the same side of

BC; the two B, E, on the same side of AC; and the two C, F, on the same side of AB.

283. VII. The three angles of every sph. triangle are together greater than two right angles, and less than six.

For the sides of the supplemental triangle are together less than four right angles (280), which sides are supplements of A, B, C, and therefore, when added to the angles, make six right angles. The angles alone must, therefore, make together more than two right angles; but they cannot amount to six, since to make this number the sides of the supplemental triangle must be added to them. When thus speaking of adding sides to angles, we of course imply the addition of their measures, either arcual or gradeal.

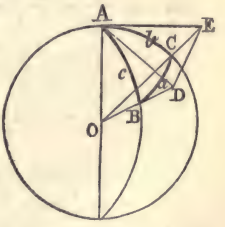
It follows from the above, that if each side of a sph. triangle be less than 90°, the sum of the angles must be less than four right angles.

So far as the *sides* are concerned, there is an analogy between plane triangles and spherical triangles: in both the sum of any two sides is greater than the third side; but as regards the *angles*, there is a wide distinction: the angles of a sph. triangle may be all right angles, or they may be all obtuse. If the sides of a sph. triangle be straightened, they will form the sides of a *plane* triangle; but this will have very different angles. The amount by which the sum of the sph. angles exceeds the sum of these plane angles, called technically the *spherical excess*, is a very important element in trigonometrical surveying (Geodesy), as we shall see further on. It will be hereafter shown, also, that, with the exception here noticed, the two kinds of triangles have fundamental properties in common.

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284. **Fundamental Formulae.**—The fundamental formulæ of Spherical Trigonometry, embodying the analytical conditions which connect together the sides and angles, may be investigated as follows:—

Let ABC be a triangle on the surface of the sphere, the sides being arcs of great circles, and the centre of the sphere O, the radius being the linear unit. From the vertex of the angle A, draw the two tangents AD, AE, meeting the radii OB, OC, produced in D and E. Then, as in plane triangles, representing the sides opposite to A, B, C, by a, b, c, we have



$$AD=\tan c,\ AE=\tan b,\ OD=\sec c,\ OE=\sec b \ldots [1].$$

Join D, E: then, since DE is the base of each of the plane triangles, whose vertices are at O and A, and since the angle A of the plane triangle is equal to the angle A of the spherical triangle (278), and the arc a the measurement of the angle O, we have (228)

$$DE^2=OE^2+OD^2-2\,OE\,.\,OD\cos a=AE^2+AD^2-2\,AE\,.\,AD\cos A;$$

that is, [1], $DE^2=\sec^2 b+\sec^2 c-2\sec b\sec c\cos a=\tan^2 b+\tan^2 c-2\tan b\tan c\cos A$;

therefore, since $\sec^2-\tan^2=1$, $1+1+2\tan b\tan c\cos A-2\sec b\sec c\cos a=0$,

or, dividing by $2\sec b\sec c$, recollecting that $\frac{1}{\sec}=\cos$, and $\frac{\tan}{\sec}=\sin$,

$$\cos b\cos c+\sin b\sin c\cos A-\cos a=0\ \therefore\ \cos a=\cos b\cos c+\sin b\sin c\cos A \ldots [2].$$

We thus have a general expression for the cosine of any side of a spherical triangle in terms of the other two sides and their included angle; so that, taking the sides a, b, c, in succession, we have the following equations, containing among them all the six parts of the triangle, and from which, any three parts being given, the remaining three may be found:

$$\left.\begin{aligned}\cos a &= \cos b \cos c + \sin b \sin c \cos A\\ \cos b &= \cos a \cos c + \sin a \sin c \cos B\\ \cos c &= \cos a \cos b + \sin a \sin b \cos C\end{aligned}\right\} \ldots\ldots[A].$$

These, like the corresponding formulæ of plane trigonometry (p. 195), would supply solutions to every possible case; but as the expressions thus deduced would be ill adapted to logarithmic computation, it is necessary, with a view to practice, to investigate suitable forms in a special manner. The foregoing equations, however, suggest the following geometrical truths:—

1. From the first two, we see that if $a=b$, then $\cos A=\cos B \therefore A=B$, that is, if two sides are equal the angles opposite to them are also equal.

2. If two triangles have two sides and the included angle in one, equal respectively to two sides and the included angle in the other, the bases or third sides are equal.

3. If two triangles have the three sides in one respectively equal to the three sides in the other, the angles of the one are respectively equal to those of the other: the equal angles being opposite to the equal sides.

NOTE.—We should not be justified in making these inferences if in [A] the first members were sines instead of cosines, or if sines instead of cosines of the angles entered the second members, because to the sine of an arc or angle there correspond *two* arcs or angles, supplemental to each other: but to a cosine, there corresponds but a single arc or angle, for an arc or angle of a sph. triangle never reaches 180°.

285. Since [2],

$$\cos A=\frac{\cos a-\cos b\cos c}{\sin b\sin c} \therefore \sin^2 A=1-\cos^2 A=\frac{\sin^2 b\sin^2 c-(\cos a-\cos b\cos c)^2}{\sin^2 b\sin^2 c}=$$

$$\frac{(1-\cos^2 b)(1-\cos^2 c)-\cos^2 a+2\cos a\cos b\cos c-\cos^2 b\cos^2 c}{\sin^2 b\sin^2 c}=$$

$$\frac{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c}{\sin^2 b\sin^2 c}$$

$$\therefore \frac{\sin A}{\sin a}=\frac{\sqrt{(1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c)}}{\sin a\sin b\sin c}.$$

This expression is symmetrical: it remains the same, however a, b, c, be interchanged; it follows, therefore, that

$$\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}\ldots[3].$$

Hence—In any sph. triangle the sines of the sides are as the sines of the angles opposite to them.

286. **Formulae for Angles in terms of Sides.**—Add 1 to each side of the above equation for $\cos A$, then (XII., p. 201) $1+\cos A=2\cos^2\frac{1}{2}A=\frac{\cos a-(\cos b\cos c-\sin b\sin c)}{\sin b\sin c}$. The numerator of this is the

same as $\cos a - \cos(b+c)$, and $b+c$ and a being respectively the sum and difference of $\frac{1}{2}(a+b+c)$ and $\frac{1}{2}(b+c-a)$, or of s and $s-a$, we have, by [VI.], p. 200,

$$\cos\tfrac{1}{2}A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}} \ldots\ldots [4].$$

If $\cos A$ be subtracted from 1, instead of being added to it, then (p. 200) we shall have

$$\sin\tfrac{1}{2}A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}} \ldots\ldots [5];$$

and consequently

$$\tan\tfrac{1}{2}A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}} \ldots\ldots [6].$$

And either of these formulæ enables us to compute the angles from the sides by logs, for any one of the three angles may be called A. We thus get

$$\frac{\sin\frac{1}{2}A}{\sin\frac{1}{2}B} = \sqrt{\frac{\sin a \sin(s-b)}{\sin b \sin(s-a)}}, \quad \frac{\tan\frac{1}{2}A}{\tan\frac{1}{2}B} = \frac{\sin(s-b)}{\sin(s-a)}, \quad \tan\tfrac{1}{2}A \tan\tfrac{1}{2}B = \frac{\sin(s-c)}{\sin s} \ldots [7].$$

Multiplying [4] and [5] together, and recollecting that $\sin\frac{1}{2}A \cos\frac{1}{2}A = \frac{1}{2}\sin A$, we have

$$\sin A = \frac{2}{\sin b \sin c}\sqrt{\{\sin s \sin(s-a)\sin(s-b)\sin(s-c)\}} \ldots [8].$$

287. **Formulae for Sides in terms of Angles.**—Let the formulæ in last article refer to the triangle supplemental to that before us: then marking the letters for the supplemental triangle with accents, we have (282)

$$A' = 180°-a,\ a' = 180°-A,\ b' = 180°-B,\ c' = 180°-C,\ s' = 270°-s$$
$$\therefore s'-a' = (270°-s)-(180°-A) = 90°-(s-A).$$

Also, $\cos\frac{1}{2}A' = \cos\left(90°-\frac{1}{2}a\right) = \sin\frac{1}{2}a \therefore \sin\frac{1}{2}A' = \cos\frac{1}{2}a$

$$\sin b' = \sin(180°-B) = \sin B,\ \sin c = \sin(180°-C) = \sin C$$
$$\sin s' = \sin(270°-s) = \sin[180°-(s-90°)] = \sin(s-90°) = -\cos s$$
$$\sin(s'-a') = \sin[90°-(s-A)] = \cos(s-A).$$

Hence, by substitution in [4], [5], [6], writing S for s', we have

$$\sin\tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos(S-A)}{\sin B \sin C}}, \quad \cos\tfrac{1}{2}a = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B \sin C}},$$

$$\tan\tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos(S-A)}{\cos(S-B)\cos(S-C)}} \ldots [9].$$

NOTE.—The minus sign properly appears before $\cos S$ in these formulæ, for this cosine is itself negative, inasmuch as S always lies between 90° and 270° (283). Not only is $\cos S$ thus necessarily negative, but the other cosines are necessarily positive; for from the supplemental triangle, and from (279), $(180°-A)+(180°-B) > 180°-C$, $\therefore 180° > A+B-C$ $\therefore 90° > S-C$, and so of the others. Neither is there any ambiguity about the radical signs: they are always positive, because $\frac{1}{2}a$ is always $< 90°$, (280).

The product of the first and second of [9] gives for the sine of a side

$$\sin a = \frac{2}{\sin B \sin C}\sqrt{\{-\cos S \cos(S-A)\cos(S-B)\cos(S-C)\}} \ldots [10].$$

288. The preceding formulæ for sides in terms of angles, have been derived from the like formulæ for angles in terms of sides, by a reference to the supplemental triangle: and by availing ourselves of this latter, we may always pass from any expression involving whole angles and sides only to one similarly involving sides and angles. All we have to do is to write for *sin*, *cos*, in the one formula, *sin*, *—cos*, of the opposite quantity, whether side or angle, in the other; thus the fundamental relations [A] become in this way changed into the following, which are equally true:—

$$\left.\begin{aligned} \cos A &= -\cos B \cos C + \sin B \sin C \cos a \\ \cos B &= -\cos A \cos C + \sin A \sin C \cos b \\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c \end{aligned}\right\} \ldots [B],$$

from which we infer, like as at p. 232, the following geometrical truths, namely,

1. If two angles are equal, the sides opposite to them are also equal.

2. If two triangles have two angles and an interjacent side of one, equal respectively to those of the other, the remaining angle in the one will be equal to the remaining angle in the other.

3. If the three angles in one be equal, respectively, to the three in the other, the sides of the one will be respectively equal to those of the other: the equal sides being opposite to the equal angles.

4. From the first of these inferences, combined with its converse at p. 232, it may be easily proved that in every triangle the greater side is opposite to the greater angle, and conversely. Let the angle A, in the triangle DAB' at page 239, be greater than the angle B'; and conceive AB to be drawn, making the angle $B'AB = B'$; then $BA = BB'$. Add DB to each, then $DB + BA = DB'$, but (279), $DB + BA > DA$, $\therefore DB' > DA$. Conversely: let $DB' > DA$, in the triangle DAB'; then $A > B'$. For if these angles were equal, then (by 1) the opp. sides would be equal, which they are not; and if B' were greater than A, then, by what is proved above, DA would be greater than DB', which it is not, hence $A > B'$.

289. As in what follows we shall have frequent occasion to pass from a triangle to its supplemental one, it will save trouble to the student if we here place before him, for the purpose of ready reference, the changes necessary to be made in thus passing from one triangle to the other, in the cases that will be required. We have noticed above—what is sufficiently obvious—that when only *whole* sides and angles enter, unconnected with their multiples, or submultiples, or with their sums and differences, the only change necessary is to write —cos for cos, and *consequently*, —sec for sec, —tan for tan, —cot for cot. But as seen at (287), these are not always the changes when other functions of the angles enter: we here give the directions necessary for what follows:—

1. For sin a write sin A, and for cos a write —cos A, and *vice versâ*.

2. For sin $\frac{1}{2}a$ write cos $\frac{1}{2}A$, and for cos $\frac{1}{2}a$, sin $\frac{1}{2}A$, and *vice versâ;*

and *consequently*, for tan $\frac{1}{2}a$ write cot $\frac{1}{2}A$, and *vice versâ*.

3. For $\sin(a \pm b)$ write $-\sin(A \pm B)$, and for $\cos(a \pm b)$ write $\cos(A \pm B)$, and *vice versâ*.

4. For sin or cos of $\frac{1}{2}(a+b)$ write sin or $-\cos$ of $\frac{1}{2}(A+B)$, and for sin or cos of $\frac{1}{2}(a-b)$ write $-\sin$ or cos of $\frac{1}{2}(A-B)$,

and $\therefore$ $-\tan\frac{1}{2}(A \pm B)$ for $\tan\frac{1}{2}(a \pm b)$.

290. **Napier's Analogies: Two parts and the part between them given.**—In the expression [A] for $\cos a$, substitute for $\cos c$ its value as given by the third equation: we shall then have

$$\cos a = \cos a \cos^2 b + \sin a \sin b \cos b \cos C + \sin b \sin c \cos A$$
$$\therefore \cos a (1 - \cos^2 b) = \cos a \sin^2 b = \sin a \sin b \cos b \cos C + \sin b \sin c \cos A$$
$$\therefore \cos a \sin b = \sin a \cos b \cos C + \sin c \cos A$$
$$\therefore \sin c \cos A = \cos a \sin b - \sin a \cos b \cos C. \quad \left.\begin{array}{l}\\ \\ \\ \end{array}\right\} \ldots[11].$$
$$\text{Similarly, } \sin c \cos B = \cos b \sin a - \sin b \cos a \cos C. \quad \left.\begin{array}{l}\\ \\ \\ \end{array}\right\}$$
$$\text{Adding, } \sin c (\cos A + \cos B) = \sin(a+b)(1 - \cos C).$$
$$\text{Now [3], } \sin A \sin c = \sin a \sin C, \ \sin B \sin c = \sin b \sin C$$
$$\therefore (\sin A \pm \sin B) \sin c = (\sin a \pm \sin b) \sin C \ldots\ldots[12].$$

Dividing [12] by the preceding equation, we have

$$\frac{\sin A \pm \sin B}{\cos A + \cos B} = \frac{\sin a \pm \sin b}{\sin(a+b)} \cdot \frac{\sin C}{1 - \cos C}.$$

But (p. 200), $\dfrac{\sin A \pm \sin B}{\cos A + \cos B} = \tan\frac{1}{2}(A \pm B)$; and [VI.], $\dfrac{\sin a + \sin b}{\sin(a+b)} = \dfrac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}$,

$$\frac{\sin a - \sin b}{\sin(a+b)} = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}. \quad \text{Also } \frac{\sin C}{1 - \cos C} = \frac{2\sin\frac{1}{2}C\cos\frac{1}{2}C}{2\sin^2\frac{1}{2}C} = \cot\frac{1}{2}C.$$

$$\text{Hence, } \tan\tfrac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C, \ \tan\tfrac{1}{2}(A-B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C.$$

And from the supplemental triangle (see art. 289) we have the corresponding forms

$$\tan\tfrac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c, \ \tan\tfrac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c.$$

These four equations furnish Napier's four analogies, namely,

$$\left.\begin{array}{l}\cos\frac{1}{2}(a+b) : \cos\frac{1}{2}(a-b) :: \cot\frac{1}{2}C : \tan\frac{1}{2}(A+B) \\ \sin\frac{1}{2}(a+b) : \sin\frac{1}{2}(a-b) :: \cot\frac{1}{2}C : \tan\frac{1}{2}(A-B)\end{array}\right\} \ldots[13],$$

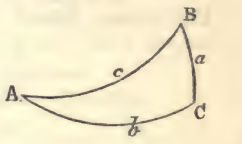
$$\left.\begin{array}{l}\cos\frac{1}{2}(A+B) : \cos\frac{1}{2}(A-B) :: \tan\frac{1}{2}c : \tan\frac{1}{2}(a+b) \\ \sin\frac{1}{2}(A+B) : \sin\frac{1}{2}(A-B) :: \tan\frac{1}{2}c : \tan\frac{1}{2}(a-b)\end{array}\right\} \ldots[14],$$

which supply solutions to the cases in which the three given parts are two sides and the included angle, or two angles and the interjacent side.

From the first of these analogies it is plain that $\cos\frac{1}{2}(a+b)$, $\tan\frac{1}{2}(A+B)$, must have the same sign: hence $a+b$, $A+B$, are either both greater or both less than 180°, an inference which will be useful hereafter.

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291. **Solution of Right-angled Triangles.**—Let ABC be a spherical triangle, right-angled at C: then, since $\sin C=1$, and $\cos C=0$, we have from the formulæ [3], [A], and [B], the five following equations, namely,



$$\sin a=\sin c\sin A,\quad \sin b=\sin c\sin B,\quad \cos c=\cos a\cos b,$$
$$\cos A=\cos a\sin B,\quad \cos B=\cos b\sin A.$$

Let these be written in order one under another, and supply a third member to each, thus: Substitute in the second member of each equation the values for its two factors as given by the other equations: for instance, for $\sin c$, $\sin A$, in the first equation $\sin a=\sin c\sin A$, substitute $\frac{\sin b}{\sin B}$, $\frac{\cos B}{\cos b}$, as given by the second and fifth equations, and so on: we shall thus have

1. $\sin a=\sin c\sin A=\tan b\cot B.$
2. $\sin b=\sin c\sin B=\tan a\cot A.$
3. $\cos c=\cos a\cos b=\cot A\cot B.$
4. $\cos A=\cos a\sin B=\tan b\cot c.$
5. $\cos B=\cos b\sin A=\tan a\cot c.$

It is plain that these five formulæ supply solutions to all the varieties that can occur as to the three given parts (*two* besides the right angle) in a right-angled sph. triangle: but the whole of them may be comprehended in two short and easily-remembered precepts which we shall give in the next article: we shall here deduce one or two useful inferences from them.

1. The shortest great-circle arc that can be drawn from any point on a sphere to a given great circle, is the perpendicular from that point. It is evident that this perpendicular (a) cannot exceed 90°: suppose it less than 90°, and c any other arc from the point making the angle A with the great circle; then, since from (1) above $\sin a<\sin c$, because $\sin A<1$, $\therefore a<c$. Suppose $a=90°$, then $1=\sin c\sin A$, and as neither of these factors can exceed 1, each must be *equal* to 1: hence, when the point is a pole of the great circle, all the great-circle arcs drawn from it are equal and perpendicular arcs, so that then no one is shorter than another.

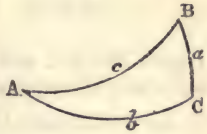
2. From (5) it appears that in a right-angled triangle, $\cos B$, $\cos b$, always have the same sign, $\therefore B$ and b are either both acute or both obtuse, or as it is usually expressed, both are of the *same affection*.

3. From (3) we see that the hypothenuse is acute if the sides, or if the angles opposite to them, are of the same affection, otherwise it is obtuse.

292. **Napier's Rules for the Circular Parts.**—These are the two rules or precepts adverted to above: they are derived as follows:

In every right-angled triangle we are to recognize but five parts,

namely, the three sides and the two angles A, B. If either of these be regarded as a *middle* part, the two which lie next to it—one on each side—are called *adjacent* parts: thus, taking A for the middle part, b, c, will be the adjacent parts: taking c for the middle part, A, B will be the adjacent parts: but if we take a for the middle part, then as C is not recognized, B, b are the adjacent parts. In like manner if b be the middle part, the adjacent parts are A, a.



The two parts which immediately follow the adjacent parts, one on each side, still disregarding the right angle, are called the *opposite* parts: thus, if A be the middle part, then a, next to the adjacent part b, is one opposite part, and B, next to the adjacent part c, is the other. These designations being agreed upon, Napier's two rules are as follows, in which it is to be specially observed that the *complements* of the two angles, and of the intervening hypothenuse are *always* to be used instead of those parts themselves.

Rule I. Sin middle part = Product of tangents of adjacent parts.

Rule II. Sin middle part = Product of cosines of opposite parts.

The vowels in *tangent*, and *adjacent*, and those in *cosine* and *opposite*, as also the *i* in *sine* and *middle*, help to fix these most useful rules in the memory. That they suffice for all the cases of right-angled triangles will be evident upon testing them by a reference to the five formulæ in the preceding article; for, taking for middle part a, b, comp. c, comp. A, comp. B, in succession, Napier's rules give the relations following, which are identical with those referred to:

$$\sin a = \tan b \tan (90°-B) = \cos (90°-c) \cos (90°-A)$$
$$\sin b = \tan a \tan (90°-A) = \cos (90°-c) \cos (90°-B)$$
$$\sin (90°-c) = \tan (90°-A) \tan (90°-B) = \cos a \cos b$$
$$\sin (90°-A) = \tan b \tan (90°-c) = \cos a \cos (90°-B)$$
$$\sin (90°-B) = \tan a \tan (90°-c) = \cos b \cos (90°-A).$$

As in the solution of a right-angled triangle two of the five parts are given to find a third part, it is necessary to choose for the middle one of these three parts that which causes the other two to become either adjacent parts or opposite parts, as in the following examples.

NOTE.—To the log of the *quotient* of two parts, 10 must be added. From the log of the product of two parts, 10 must be subtracted. As the radius (or unit of measure) of the table of log sines, log cosines, &c., is not 1, but log 10^{10} or 10, the tables give log SIN = log $(10^{10} \sin)$ = $10 + \log \sin$, log cos = $10 + \log \cos$, &c. (See p. 176.)

1. In the right-angled triangle ABC, above, are given $c = 64° 40'$, $b = 42° 12'$: required the other parts. Here taking b for the middle part, B and c become opposite parts: hence, by Rule II.,

$$\sin b = \cos \text{comp. } B \cos \text{comp. } c, \text{ that is, } \sin b = \sin B \sin c \therefore \sin B = \frac{\sin b}{\sin c}.$$

Also, taking c for middle part, a and b become opposite parts, therefore

$$\sin \text{comp. } c = \cos a \cos b, \text{ that is, } \cos c = \cos a \cos b \therefore \cos a = \frac{\cos c}{\cos b},$$

and the logarithmic operations for sin B, and cos a, are as follows:—

From log sin 42° 12′ = 9·827189	From log cos 64° 40′ = 9·631326
take log sin 64° 40′ = 9·956089	take log cos 42° 12′ = 9·809704
log sin 48° 0′ = 9·871100	cos 54° 43′ = 9·761622

Hence $B=48°$, and $a=54°\ 43'$. Make now A the middle part, then b and c are adjacent parts, $\therefore$ sin comp. A=tan b tan comp. c, that is

$$\cos A=\tan b \cot c \therefore \log\cos A=9{\cdot}957485+9{\cdot}675237=9{\cdot}632722 \therefore A=64°\ 35'.$$

Note.—The result of the first operation, being a *sine*, applies equally to 48°, and its supplement 132°: but in a right-angled triangle, an angle is acute or obtuse, according as the side opposite to it is acute or obtuse (291): hence there can here be no ambiguity, because b is *acute*.

2. In the right-angled triangle ABC, are given $b=29°\ 12'\ 50''$, and $B=37°\ 26'\ 21''$, to find the side a. Taking a for the middle part, B, and b become adjacent parts, $\therefore$ sin a=tan b tan comp. B, that is, sin a=tan b cot B, and the operation is as follows:—

		D.	(*Parts for secs.*)
tan b, 29° 12′ 50″.....	9·747319	494	24700
cot B, 37° 26′ 21″.....	10·116066	−436	−9156
Cor. for seconds...	155		155,44=Correction for secs.
sin a, 46° 55′ 1″.....	9·863540		

38 D=197)200(1″

2

Here a is ambiguous: it may be either 46° 55′ 1″, or 133° 4′ 59″; in the former case the angle A must be acute, in the latter case, obtuse. Whenever, as here, the two given parts are an angle and the side opposite to it, there are always two right-angled triangles having these given parts the same. They are situated with respect to each other, as the triangles ABC, DBC, in the diagram at page 230; the angle D being $=A$, and the side BC common. There is no ambiguity when the two given parts are other than these, the character of the required parts being fixed from the conditions at page 236.

Examples for Exercise.

(1) Given $a=48°\ 24'\ 16''$, $b=59°\ 38'\ 27''$ to find c.

(2) Given $A=34°\ 27'\ 39''$, $b=46°\ 18'\ 23''$ to find B.

(3) Given $a=116°\ 30'\ 43''$, $b=29°\ 41'\ 32''$ to find A.

(4) Given $a=27°\ 48'$, $c=71°\ 39'\ 37''$ to find b, B.

(5) Given $a=40°\ 30'\ 20''$, $A=47°\ 54'\ 20''$ to find b.

(6) Given $a=48°\ 24'\ 16''$, $c=70°\ 23'\ 42''$: find b, B.

(7) Given $A=23°\ 27'\ 42''$, $b=10°\ 39'\ 40''$: find a, c, B.

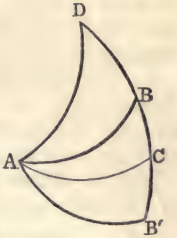
(8) Given the equal sides of an isosceles right-angled triangle each 30°: find the hypothenuse.

(9) Given $a=40°\ 30'\ 20''$, $A=47°\ 54'\ 20''$ to find B.

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293. **Solution of Quadrantal Triangles.**—A quadrantal triangle is one of which one side is a quadrant or 90°. The triangle supple-

mental to it will therefore be right angled, and the solution of this will supply the solution of the former. Thus the sides of the quadrantal triangle being a, b, and $c=90°$, and its angles A, B, C, the sides of the supplemental triangle will be $180°-A$, $180°-B$, and $180°-C$; this latter being the hypothenuse; and the angles opposite to these sides will be $180°-a$, $180°-b$, and $180°-c=90°$, the right angle. But the solution of the quadrantal triangle may be obtained independently of its supplemental triangle thus. Let AD be the quadrantal side, produce DB, if necessary, to make the arc DC also $=90°$, and draw the arc AC, which will be the measure of the angle D, at the pole of this arc (278). The triangle DAC will have its angles at A and C both right angles, so that $DAB+CAB$, or $DAB-CAB$ will be 90°; and the angle ABC will be either the angle ABD itself, or the supplement of it, according as DB exceeds or falls short of 90°: hence all the parts of the quadrantal triangle ABD, are easily determinable from those of the right-angled triangle ABC, and the following inferences are obvious. [NOTE.—B' in the diagram marks the second position of B.]



1. If $DAB<90°$, then $DB<90°$; and if $DAB>90°$, then $DB>90°$, and conversely. Hence the angles adjacent to the quadrantal side are of the same affection as the sides opposite to them.

2. The triangle will be ambiguous whenever the given parts in the right-angled triangle ABC are one of the perpendicular sides, AC, BC, and the angle opposite to it (p. 238).

(1) In the triangle DAB, where $DA=90°$, are given $A=54°\ 43'$, and $D=42°\ 12'$, to find AB and B. Here since $DAB<90°$, it is $<DAC$ $\therefore DB<90°$, BC is therefore the prolongation of DB. In the right-angled triangle ABC, there are given $A=90°-54°\ 43'=35°\ 17'$, and $b=42°\ 12'$, to find c and B. Take A for middle part, then b, c, will be adjacent parts,

$$\therefore \text{ sin comp. } A=\tan b \tan \text{ comp. } c, \text{ that is, } \cos A=\tan b \cot c, \therefore \cot=\frac{\cos A}{\tan b}.$$

Again: take B for middle part, then A, b, will be opposite parts,

$\therefore$ sin comp. $B=\cos b$ cos comp. A, that is, $\cos B=\cos b \sin A$, and the logarithmic work is as follows:

cos A, 35° 17′ .....9·911853	cos b, 42° 12′ .....9·869704
tan b, 42° 12′ .....9·957485	sin A, 35° 17′ .....9·761642
cot c, 48° 0′ 16″...9·954368	cos B, 64° 39′ 55″...9·631346
437	593
$D=423)$ 6900(16″	$D=445)$24700(55″

Hence $AB=48°\ 0'\ 16''$, and $ABD=180°-ABC=180°-64°\ 39'\ 55''=115°\ 20'\ 5''$.

EXAMPLES FOR EXERCISE.

(1) Given, in a quadrantal triangle, the angle opposite to the quadrantal side $=115°\ 20'$, and the vertical angle (D in the diagram above)$=42°\ 12'$, to find the other sides.

(2) Given $AB=67°\ 3'\ 14''$, and the angle A between AB and the quadrantal side $=112°\ 2'\ 9''$, to find the other parts.

(3) Given a side $=90°$, and the two other sides, $=32°\ 57'\ 6''$, $66°\ 32'$, respectively: required the angles opposite to the latter.

(4) In the quadrantal triangle ADB are given $D=69°\ 13'\ 46''$, $A=72°\ 12'\ 4''$: required the other parts.

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294. **Solution of Oblique-Angled Triangles.**—By means of a perpendicular from one of the vertices to the side opposite to it, the solution of every oblique-angled triangle may be reduced to the solution of a pair of right-angled triangles; but it is better to employ distinct formulæ instead. These are six in number, for there are six separate cases to be discussed in all, the given parts always being one or other of the following sets of three, namely:—

1. The three sides. 2. The three angles. 3. Two sides and the included angle. 4. Two angles and the interjacent side. 5. Two sides and an opp. angle. 6. Two angles and an opp. side. These six cases we shall consider in order.

295. I. **The Three Sides Given.**—The formulæ for A, either of the three angles, are (286)

$$\sin\tfrac{1}{2}A=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}},\quad \cos\tfrac{1}{2}A=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}},\quad \tan\tfrac{1}{2}A=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}.$$

Note.—When *all* the three angles are to be computed, it will be best to use the last of these formulæ for one of them, A, and then to deduce the remaining angles B, C, from the conditions [7] at page 233, namely,

$$\tan\tfrac{1}{2}B=\frac{\sin(s-a)}{\sin(s-b)}\tan\tfrac{1}{2}A,\ \tan\tfrac{1}{2}C=\frac{\sin(s-a)}{\sin(s-c)}\tan\tfrac{1}{2}A,$$

as additional references to the tables will thus be dispensed with, the logarithmic formulæ for $\tan\frac{1}{2}B$, $\tan\frac{1}{2}C$, being as follows (See ex. 2):—

$$\log\tan\tfrac{1}{2}B=\log\tan\tfrac{1}{2}A-\{\log\sin(s-b)-\log\sin(s-a)\}=$$

$$\log\tan\tfrac{1}{2}A-\{\log\sin(s-b)+\operatorname{colog}\sin(s-a)\}+10.$$

And similarly, $\log\tan\frac{1}{2}C=\log\tan\frac{1}{2}A-\{\log\sin(s-c)+\operatorname{colog}\sin(s-a)\}+10.$

(1) Given $a=68°\ 46'\ 2''$, $b=43°\ 37'\ 38''$, $c=37°\ 10'$: required A.

					D. (*Parts for secs.*)	
a,	68° 46′ 2″					
$\sin b$,	43 37 38	colog	0·161258		−221	− 8398 ☞ [For a colog, D takes the opposite sign.]
$\sin c$,	37 10 0	colog	0·218866			
	2)149 33 40					
s,	74 46 50					
$\sin(s-b)$,	31 9 12		9·713726		+348	+ 4176
$\sin(s-c)$,	37 36 50		9·785433		+273	+13650
Correction for the seconds			94			94,28
			2)19·879377			
$\sin\frac{1}{2}A$,	60° 29′ 54″		9·939689			
			25			
			D=119)6400(54″		$\therefore A=120°\ 59'\ 48''$	

(2) Given $a=76°\ 35'\ 36''$, $b=50°\ 10'\ 30''$, $c=40°\ 0'\ 10''$: required all the angles. Since all the angles are required, we shall work by the third formulæ for one of these angles A, and shall then derive B and C as recommended in the note above.

					D	(Parts for seconds.)	
a,	76° 35′ 36″						
b,	50 10 30						
c,	40 0 10						
	2)166 46 16						
sin s,	83 23 8	colog	0·002902	−	23	− 184	=−56760
sin $(s-a)$,	6 47 32	colog	0·927694		−1768	−56576	
sin $(s-b)$,	33 12 38		9·738434	+	322	+12236	=+25170
sin $(s-c)$,	43 22 58		9·836745	+	223	+12934	
			20·505775				−315,90
Correction for seconds			−316				
			2)20·505459				
$\tan\frac{1}{2}A$,	60° 48′ 10″		10·252730				
			681				

$D=494)\ 4900(10''$

sin $(s-a)$, colog	0·927694	sin $(s-a)$, colog	0·927694
sin $(s-b)$,	9·738434	sin $(s-c)$,	9·836745
	10·666128		10·764439
Correction for secs.	−443	Correction for secs.	−436
Subtract	10·665685	*Subtract*	10·764003
$\tan\frac{1}{2}A$,	10·252730	$\tan\frac{1}{2}A$,	10·252730
$\tan\frac{1}{2}B$, 21° 7′ 37″	9·587045	$\tan\frac{1}{2}C$, 17° 7′ 31″½	9·488727
	6815		492
$D=626)$	23000(37″	$D=748)$	23500(31″½

Hence the angles are $A=121°\ 36'\ 20''$, $B=42°\ 15'\ 14''$, $C=34°\ 15'\ 3''$.

NOTE.—The correction for seconds is got from the "Parts for seconds" above.

EXAMPLES FOR EXERCISE.

(1) Given $a=47°\ 45'\ 51''$, $b=73°\ 54'\ 13''$, $c=44°\ 30'\ 22''$ to find A.
(2) Given $a=48°\ 27'\ 8''$, $b=53°\ 55'\ 21''$, $c=48°\ 28'\ 57''$ to find A.
(3) Given $a=60°\ 18'\ 48''$, $b=32°\ 37'\ 20''$, $c=47°\ 47'\ 14''$ to find A.
(4) Given $a=63°\ 50'$, $b=80°\ 19'$, $c=120°\ 47'$ to find A, B, and C.
(5) Given $a=40°\ 0'\ 10''$, $b=50°\ 10'\ 30''$, $c=76°\ 35'\ 36''$ to find A, B, and C.

296. II. **The Three Angles given.**—The formulæ for a, any one of the three sides are (295)

$$\sin\frac{1}{2}a=\sqrt{\frac{-\cos S\cos(S-A)}{\sin B\sin C}},\quad \cos\frac{1}{2}a=\sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}},$$

$$\tan\frac{1}{2}a=\sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}},$$

when all the sides are to be determined, apply the third formula to find one, a, and find b, c, by the equations

$$\tan\frac{1}{2}b=\frac{\cos(S-B)}{\cos(S-A)}\tan\frac{1}{2}a,\ \tan\frac{1}{2}c=\frac{\cos(S-C)}{\cos(S-A)}\tan\frac{1}{2}a,$$

like as in the former case, the logarithmic formulæ being as follows (See the example).

$$\log\tan\frac{1}{2}b=\log\tan\frac{1}{2}a-\{\log\cos(S-A)+\operatorname{colog}\cos(S-B)\}+10,$$

$$\log\tan\frac{1}{2}c=\log\tan\frac{1}{2}a-\{\log\cos(S-A)+\operatorname{colog}\cos(S-C)\}+10.$$

(1) Given $A=51°\ 30'$, $B=59°\ 16'$, $C=131°\ 30'$, to find a, b, c.

A,	51° 30′		
B,	59 16		
C,	131 30		
	2)242 16		
$-\cos S$,	121 8		9·713517
$\cos(S-A)$,	69 38		9·541613
$\cos(S-B)$,	61 52	colog	0·326495
$\cos(S-C)$,	−10 22*	colog	0·007148
			2)19·588773
$\tan\frac{1}{2}a$,	31° 55′ 1″		9·794387
			3
			D=469)400(1″

$\cos(S-A)$,		9·541613
$\cos(S-B)$,	colog	0·326495
	Subtract	9·868108
$\tan\frac{1}{2}a$,		9·794387
$\tan\frac{1}{2}b$,	40° 9′ 37″	9·926279
		122
		D=427)15700(37″
$\cos(S-A)$,		9·541613
$\cos(S-C)$,	colog	0·007148
	Subtract	9·548761
		9·794387
$\tan\frac{1}{2}c$,	60° 24′ 7″	10·245626
		591
		D=490) 3500(7″

Hence the sides are. $a=63°\ 50'\ 2''$, $b=80°\ 19'\ 14''$, $c=120°\ 48'\ 14''$.

Examples for Exercise.

(1) Given $A=34°\ 15'\ 4''$, $B=42°\ 15'\ 12''$, $C=121°\ 36'\ 20''$, to find the side b.
(2) Given $A=130°\ 3'\ 11''$, $B=31°\ 34'\ 26''$, $C=30°\ 28'\ 12''$, to find the side a.
(3) Given $A=44°\ 10'\ 40''$, $B=33°\ 22'\ 45''$, $C=119°\ 55'\ 6''$, to find a, b, c.

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* The cosine of a negative angle is the same as the cosine of that angle taken positively.

297. III. **Two Sides and the included Angle given.**—Let a, b, be the given sides, and therefore C the given angle, then to determine A, B, we have by Napier's Analogies

$$\cos\tfrac{1}{2}(a+b) : \cos\tfrac{1}{2}(a\sim b) :: \cot\tfrac{1}{2}C : \tan\tfrac{1}{2}(A+B),$$

$$\text{and } \sin\tfrac{1}{2}(a+b) : \sin\tfrac{1}{2}(a\sim b) :: \cot\tfrac{1}{2}C : \tan\tfrac{1}{2}(A\sim B).$$

And A, B, being thus found, the side c is found by either of the other two analogies.

$$\cos\tfrac{1}{2}(A\sim B) : \cos\tfrac{1}{2}(A+B) :: \tan\tfrac{1}{2}(a+b) : \tan\tfrac{1}{2}c.$$

$$\text{or } \sin\tfrac{1}{2}(A\sim B) : \sin\tfrac{1}{2}(A+B) :: \tan\tfrac{1}{2}(a\sim b) : \tan\tfrac{1}{2}c.$$

NOTE.—In the first of the above analogies, if the leading term is minus, the final term will of course be also minus, and conversely; that is, $\frac{1}{2}(A+B)$ will be acute or obtuse according as $\frac{1}{2}(a+b)$ is acute or obtuse: there can, therefore, never be any ambiguity in this case.

(1) Given $a=80°\ 19'$, $b=120°\ 47'$, and $C=51°\ 30'$, to find the other parts.

1. *The angles A, B, computed.*

$\cos\frac{1}{2}(a+b)$,	100° 33′	colog 0·737327	.....	$\sin\frac{1}{2}(a+b)$,		colog 0·007404	
$\cos\frac{1}{2}(a\sim b)$,	20 14	9·972338	.....	$\sin\frac{1}{2}(a\sim b)$,		9·538880	
$\cot\frac{1}{2}C$,	25 45	10·316644	.....	$\cot\frac{1}{2}C$,		10·316644	
$\tan\frac{1}{2}(A+B)$,	95° 22′ 37″	11·026309		$\tan\frac{1}{2}(A\sim B)$,	36° 6′ 17″	9·862928	
		5791				854	
		$D=2257)$ 51800(23″				$D=442)$ 7400(17″	

By addition and subtraction of the half-sum and half-difference we have

$$A=131°\ 28'\ 54'' \text{ and } B=59°\ 16'\ 20''.$$

Observe that $\frac{1}{2}(A+B)$ is obtuse because $\frac{1}{2}(a+b)$ is.

The angles A, B, being now known, we may proceed to find c by one or other of the second pair of analogies. But when the third side only is required to be computed, it is not the shortest way to arrive at it by first computing the two angles adjacent to it, as above; it is better to employ for the purpose a subsidiary angle, as explained in the next article.

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298. **Side computed by a Subsidiary Angle.**—In the general formulæ [A], we have

$$\cos c=\cos a\cos b+\sin a\sin b\cos C=\cos a\,(\cos b+\tan a\sin b\cos C).$$

$$\text{Assume } \tan a \cos C = \cot \omega = \frac{\cos \omega}{\sin \omega},$$

$$\therefore \cos c = \cos a \frac{\sin \omega \cos b + \sin b \cos \omega}{\sin \omega} = \frac{\cos a \sin (b+\omega)}{\sin \omega}.$$

Hence, knowing a, b, and c, we have the following formulæ for computing c, namely,

$$1.\ \cot \omega = \tan a \cos C,\quad 2.\ \cos c = \frac{\cos a \sin (b+\omega)}{\sin \omega} \ldots [1].$$

We shall compute c in the last example, first by Napier's Analogy, assuming A, B, to have been found as above, and then, by the formula just deduced, independently of A and B.

2. *The side* c *computed by Napier's Analogy.*

			(*Parts for secs.*)
$\cos \frac{1}{2}(A \sim B)$,	36° 6′ 17″	colog 0·092594	— 153 — 2601
$\cos \frac{1}{2}(A+B)$,	95 22 37	8·972289	−2238 −51474
			−488,73
$\tan \frac{1}{2}(a+b)$,	100 33 0*	10·729923	
		9·794806	
Correction for seconds		−489	
$\tan \frac{1}{2} c$,	31° 54′ 46″	9·794317	
	2	101	
$\therefore c =$	63 49 32	D=469)21600(46″	

☞ The table is, of course, entered with the supplement of $\frac{1}{2}(A+B)$, and the difference $D=-2238$, is multiplied by $60-37=23$. It would have come to the same thing if we had entered the table with $180°-(95°\ 22')$, and had then multiplied the same D, taken with opposite sign, by 37. See (226).

* This tangent need not be taken from the tables: for,

log tan=log sin−log cos+10=(10−log cos)−(10−log sin)+10=colog cos−colog sin+10,

and these cologs are already exhibited above, in the work for A and B. Or the tangent may be taken out at the same time with the sine and cosine, in anticipation of the work which is to follow: whenever seconds occur in the arc, it is better to do so.

We take this opportunity of remarking that, in the solution of triangles, whether plane or spherical, the work will be expedited if a *blank form* or skeleton of the whole operation be prepared before the tables are touched: This done, the blanks left for the several tabular quantities should then be filled up, so that all the use possible may be made of the tables when once in hand. If two or more extracts are to be made from the same page, although these may have to be inserted in remotely-different parts of the work, yet they should all be taken out with the first extract, and each put in its proper place in the blank form, in anticipation of the future demand for it. Room should always be left towards the right of the blank form for the corrections for seconds, in imitation of the plan adopted in this work. Such an orderly and systematic mode of arranging this part of the operation will preclude all confusion, as well as facilitate a revision of the entire process.

It may be mentioned here, that when, as in the case of $\frac{1}{2}(A+B)$ above, we have to take out the difference D for an *obtuse* angle, it will be prudent, directly the angle is written down, to put some mark (as an asterisk) against the seconds, as a reminder that since we enter the table with the *supplement*, the D taken out is to be multiplied by what the seconds want of 60″. We consider this to be the preferable mode of proceeding in all such cases, as there will be then no risk of taking the contiguous D for that which is really the correct one.

The same computed by formulæ [1].

$\tan a$,	80° 49′	10·767935	$\cos a$,	9·225833			
$\cos C$,	51 30	9·794150	$\sin \omega$, colog	0·578143	−768	−33024	
			$\sin (b+\omega)$,	9·840854	219	+ 3723	
$\cot \omega$,	15° 19′ 43″	10·562085					
b=	120 47	437		9·644830		−293,01	
		— Correction for secs		−293			
$(b+\omega)$=	136 6 43	D=826)35200(43″					
			$\cos c$, 63° 49′ 33″	9·644537			
				680			

D=428)14300(33″

The small difference of 1″, in the two results, arises chiefly from ω being 15° 19′ 42″½, instead of 15° 19′ 43″, as we have here taken it. Where fractions of a second are disregarded in the several items of the work, small discrepancies in results obtained by different methods are to be expected.

EXAMPLES FOR EXERCISE.

(1) Given two sides 50° 10′ 27″, and 40° 0′ 14″, also the included angle=121° 36′ 24″, to find the remaining angles.

(2) Given a=38° 30′ b=70°, C=31° 34′ 26″, to find the remaining parts.

(3) Given a=84° 14′ 29″, b=44° 13′ 45″, C=36° 45′ 28″, to find the other parts.

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299. IV. **Two Angles and Interjacent Side Given.**— This case is also solved by Napier's analogies (290), which need not be here repeated. The second pair determines the sides, after which the remaining angle may be found by either of the other two analogies, or independently, by means of a subsidiary angle: thus, A and B being the given angles, and c the given side:—

Since $\cos C = \cos A(\tan A \sin B \cos c - \cos B)$, put $\tan A \cos c = \cot \omega = \frac{\cos \omega}{\sin \omega}$,

$$\therefore \cos C = \cos A \frac{\sin B \cos \omega - \sin \omega \cos B}{\sin \omega} = \frac{\cos A \sin (B-\omega)}{\sin \omega}.$$

Hence knowing A, B, and c, we have the following formulæ for C, namely,

$$1.\ \cot \omega = \tan A \cos c\,;\ 2.\ \cos C = \frac{\cos A \sin (B-\omega)}{\sin \omega} \ldots [2].$$

(1) Given A=39° 23′, B=33° 45′ 3″, c=68° 46′ 2″, to find a, b, C.

1. *The sides* a, b, *computed by Napier's analogies.*

$\cos \frac{1}{2}(A+B)$,	36° 34′ 1″·5	colog 0·095198		$\sin \frac{1}{2}(A+B)$,	colog	0·224926	
$\cos \frac{1}{2}(A\sim B)$,	2 48 58·5	9·999475		$\sin \frac{1}{2}(A\sim B)$,		8·691374	
$\tan \frac{1}{2}c$,	34 23 1	9·835243		... ...		9·835243	
$\tan \frac{1}{2}(a+b)$,	40° 23′ 49″	9·929916		$\tan \frac{1}{2}(a\sim b)$,	3° 13′ 48″	8·751543	
		708				49740	
		D=427)20800(49″				D=3749) 180300(48″	

Adding and subtracting, a=43° 37′ 37″, b=37° 10′ 1″.

The angle C computed by Napier's analogy and by formule [2].

$\sin\frac{1}{2}(a\sim b)$,	3° 13′ 48″	colog	1·249150
$\sin\frac{1}{2}(a+b)$,	40 23 49		9·811628
$\tan\frac{1}{2}(A\sim B)$,	2 48 58·5*		8·691899
$\cot\frac{1}{2}C$,	60° 29′ 53″		9·752677
	2		937
	C=120° 59′ 46″		D=491)26000(53″

tan A, 9·914302 cos A,..... 9·888134
cos c, 9·558898 sin ω, colog 0·018392
cot ω, 9·473200 sin $(B-\omega)$, 9·805268
457
ω= 73° 26′ 33″ D=771)25700(33″ cos C, 9·711794
B= 33 45 3 839
$(B-\omega)$=−39° 41′ 30″ Sup. of C=59° 0′ 18″ D=350) 4500(13″
180°
∴ C=120° 59′ 47″

* A reference to the Tables is not necessary for this tangent: we have only to subtract the log cos from the log sin, both of which are already before us, and to increase the remainder by 10. (See the foot note, p. 244).

Since, in the second operation, cos C is negative, inasmuch as sin $(B-\omega)$ is negative, it follows that C must be the supplement of the angle in the table, against cos C.

Examples for Exercise.

(1) Given A= 34° 15′ 3″, B= 42° 15′ 13″, c=76° 35′ 36″, to find a and b.

(2) Given c = 51° 6′ 12″, A=130° 5′ 22″, B=32° 26′ 6″, to find the other parts.

(3) Given A=121° 36′ 20″, C= 42° 15′ 13″, b=40° 0′ 10″, to find the other parts.

300. V. **Two Sides and an Opposite Angle Given.**—Let the two given sides be a, b, and let A the angle given be opposite to a: then for B we have (285)

$$\sin a : \sin b : : \sin A : \sin B \ldots\ldots [1].$$

And for C and c we have by Napier's analogies

$$\cos\tfrac{1}{2}(a\sim b) : \cos\tfrac{1}{2}(a+b) : : \tan\tfrac{1}{2}(A+B) : \cot\tfrac{1}{2}C,$$

$$\cos\tfrac{1}{2}(A\sim B) : \cos\tfrac{1}{2}(A+B) : : \tan\tfrac{1}{2}(a+b) : \tan\tfrac{1}{2}c.$$

Since two angles answer to the same sine, the result of the proportion [1] is ambiguous. It is necessary, therefore, to ascertain from other considerations what the character of the angle B must be when only one triangle can exist with the proposed data, and under what circumstances two triangles may involve them. The formulæ [A] at page 232 give

$$\cos B = (\cos b - \cos a \cos c) \div \sin a \sin c,$$

from which we may derive the following conclusions, namely,

1. If $\cos b$ be numerically greater than $\cos a$, the second member of the equation must of necessity take the same sign as $\cos b$. In other words, B and b must be of the same affection if $\sin b < \sin a$. Hence: *An angle is of the same affection as its opp. side, if the sine of this side is less than the sine of the other given side.*

2. If $\cos b$ be numerically less than $\cos a$, then the sign of the second member of the equation will depend upon the value of $\cos c$: this may be such as to render the product $\sin a \cos c$ still less than $\cos b$, or such as to make this product greater than $\cos b$: thus $\cos B$ will be + or − according to the value of c, or $\cos B$ may be taken either + or − consistently with some admissible value of c. Hence: *An angle is ambiguous if the sine of its opposite side is greater than the sine of the other given side.**

In addition to these the following principles also will be found useful.

3. If $a+b>180°$ then $A+B>180°$, and if $a+b<180°$ then $A+B<180°$ (p. 236).

4. The greater side is always opposite to the greater angle, and conversely (p. 234).

(1) Given $a=80°\ 5'\ 4''$, $b=70°\ 10'\ 30''$, $A=33°\ 15'\ 7''$, to find the other parts.

1. *Computation of the angle B.*

		D.	(*Parts for secs.*)	
sin a, 80° 5′ 4″	colog 0·006538	−37	−148	*** From either of the
sin b, 70 10 30	9·973444	76	2280	principles (1), (4),
sin A, 33 15 7	9·739013	321	2247	above, it follows that
	44			the angle B, here de-
			43,79	termined, is *acute.*
sin B, 31° 34′ 38″	9·719639			
	8909			
	D=343) 13000(38″			

* The author conceives that these tests will be found more simple and convenient than those, from Legendre, given by most modern writers on Spherical Trigonometry.

2. *Computation of the angle C.*

			D.	(*Parts for secs.*)
$\cos\frac{1}{2}(a\sim b)$,	4° 57′ 17″	colog 0·001623	18	306
$\cos\frac{1}{2}(a+b)$,	75 7 47	9·409682	−793	−37271
$\tan\frac{1}{2}(A+B)$,	32 24 52	9·802513	465	24180
		−28		
				−27,85
$\cot\frac{1}{2}C$,	80 42 38½	9·213690		
		4198		
$\therefore C=$	161° 25′ 17″	D=1321) 50800(38″½		

3. *Computation of the side c.*

			D.	(*Parts for secs.*)
$\cos\frac{1}{2}(A\sim B)$,	0° 50′ 15″	colog 0·000046	03	045
$\cos\frac{1}{2}(A+B)$,	32 24 52	9·926511	−134	−6968
$\tan\frac{1}{2}(a+b)$,	75 7 47	10·575497	848	39856
		329		
				329,33
$\tan\frac{1}{2}c$,	72 32 30½	10·502383		
		159		
$\therefore c=$	145° 5′ 1″	D=736)22400(30″½		

NOTE. Cos $\frac{1}{2}(A+B)$, and tan $\frac{1}{2}(a+b)$, employed in this last operation, are both to be taken from the table when the extracts for tan $\frac{1}{2}(A+B)$, and cos $\frac{1}{2}(a+b)$, used in the previous operation are made. (See the footnote, p. 244.)

EXAMPLES FOR EXERCISE.

(1) Given a=63° 50′, b=80° 19′, A=51° 30′, to find the other parts.

(2) Given a=40° 36′ 37″, b=91° 3′ 25″, A=35° 57′ 15″: show that B is ambiguous, admitting of two values, and find the angle C for the case of B obtuse.

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301. **Subsidiary Angle for computing** C, c.—If either of these alone be required, we may effect the solution, without first finding B, by aid of a subsidiary angle, thus:

By [11] page 235, $\sin c\cos A=\cos a\sin b-\sin a\cos b\cos C$.

Divide each side by sin a, and then in the first put $\frac{\sin C}{\sin A}$ for $\frac{\sin c}{\sin a}$, and we shall have

$\sin c \cot A = \cot a \sin b - \cos b \cos C$. Multiply by $\tan C$ and transpose,

$\therefore \sin C + \tan A \cos b \cos C = \cot a \sin b \tan A$. Assume $\tan \omega = \tan A \cos b$, then

$$\sin C + \tan \omega \cos C = \cot a \sin b \tan A \text{ ; that is, since } \tan A = \frac{\sin \omega}{\cos \omega \cos b},$$

$$\sin C + \frac{\sin \omega}{\cos \omega} \cos C = \cot a \frac{\sin b \sin \omega}{\cos b \cos \omega}, \text{ or } \sin C \cos \omega + \sin \omega \cos C = \sin(C+\omega) = \cot a \tan b \sin \omega.$$

Hence, for computing $(C+\omega)$, we have

1. $\tan \omega = \tan A \cos b$; 2. $\sin(C+\omega) = \cot a \tan b \sin \omega$;

and then subtracting ω, the angle C is found.

Again: By [A], page 232, $\sin b \sin c \cos A + \cos b \cos c = \cos a$. Divide by $\cos b$, then

$$\tan b \sin c \cos A + \cos c = \frac{\cos a}{\cos b}. \quad \text{Assume } \tan \omega = \tan b \cos A, \text{ and we have}$$

$$\sin c \tan \omega + \cos c = \sin c \frac{\sin \omega}{\cos \omega} + \cos c = \frac{\cos a}{\cos b}$$

$\therefore \sin c \sin \omega + \cos c \cos \omega = \cos(C-\omega) = \frac{\cos a}{\cos b} \cos \omega$. Hence, for computing $(c-\omega)$ we have

1. $\tan \omega = \tan b \cos A$; 2. $\cos(c-\omega) = \frac{\cos a}{\cos b} \cos \omega$;

and adding ω, the side c is found.

---

302. VI. **Two Angles and an Opposite Side Given.**— The side opposite to the other given angle is found by the proportion [1] of last case, and the remaining two parts by Napier's analogies.

Since the proportion [1] gives a sine, there may be a doubt as to whether the side to which it belongs is acute or obtuse, or whether it may be either indifferently. To decide this it is sufficient to observe that in the formula $\cos b = (\cos B + \cos A \cos C) \div \sin A \sin C$, if $\cos B$ be numerically greater than $\cos A$, B and b must be of the same affection: but if $\cos B$ is numerically less than $\cos A$, then values may exist for C that will render $\cos b$ either positive or negative. Hence

1. *A side will be of the same affection as its opp. angle, if the sine of this angle be less than the sine of the other given angle.*

2. *A side is ambiguous if the sine of its opposite angle is greater than the sine of the other given angle.*

(1) Given $A = 32° 26' 7''$, $B = 130° 5' 22''$, $a = 44° 13' 45''$ to find the rest.

1. *The side b computed.*

			D.	(*Parts for secs.*)	
$\sin A$,	$32° 26' 7''$	colog 0·270578	−331	−2317	*** Here b is ambiguous,
$\sin B$,	130 5 22	9·883617	177	6726	because $\sin B > \sin A$,
$\sin a$,	44 13 45	9·843466	216	9720	its values are either
		141		141,29	$b = 84° 14' 25''$, or
$\sin b$,	84 14 25	9·997802			$b = 95° 45' 35''$.
		797			The former value is
					taken in next page.

$D = 20) 500 (25''$

2. *The side c computed.*

			D.	(*Parts for secs.*)
$\cos\frac{1}{2}(A\sim B)$,	48° 49′ 37″½	colog 0·181464	241	9038
$\cos\frac{1}{2}(A+B)$,	81 15 44½	9·182196	−1369	−60921
$\tan\frac{1}{2}(a+b)$,	64 14 5	10·316321	538	2690
		−492		
				−491,93
$\tan\frac{1}{2}c$,	25° 33′ 5″⅛	9·679499		
		71		
$\therefore c=$	51° 6′ 10″⅛	D=541)2800(5″⅛		

3. *The angle C computed.*

			D.	(*Parts for secs.*)
$\cos\frac{1}{2}(a\sim b)$,	20° 0′ 20″	colog 0·027014	77	1540
$\cos\frac{1}{2}(a+b)$,	64 14 5	9·638197	−436	−2180
$\tan\frac{1}{2}(A+B)$,	81 15 44½	10·812720	1402	62389
		617		
				617,49
$\cot\frac{1}{2}C$,	18° 22′ 42″¾	10·478548		
		849		
$\therefore C=$	36° 45′ 25″½	D=704)30100(42″¾		

NOTE. Cos $\frac{1}{2}(a+b)$, and tan $\frac{1}{2}(A+B)$ are to be taken from the table at the same time that tan $\frac{1}{2}(a+b)$ and cos $\frac{1}{2}(A+B)$ are taken in the preceding operation. (See foot-note, page 244.)

EXAMPLES FOR EXERCISE.

(1) Given A=70° 39′, B=48° 36′, a=89° 16′ 53″½, to find the rest.
(2) Given A=103° 59′ 57″·5, B=46° 18′ 7″·25, a=42° 8′ 48″, to find C.
(3) Given B=59° 16′ 20″, C=131° 28′ 54″, c=120° 47′, to find a.

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303. **Subsidiary Angle for computing** C, c.—As in the last case, these two parts may each be computed, independently of the part b, thus:

In article (301) it was seen that $\sin C+\tan A\cos b\cos C=\cot a\sin b\tan A$. Apply this to the supplemental triangle, remembering (289) that cosines, tangents, and cotangents change sign, and there results

$\sin c-\tan a\cos B\cos c=\cot A\sin B\tan a$. Assume $\tan\omega=\tan a\cos B$, then

$$\sin c-\frac{\sin\omega}{\cos\omega}\cos c=\frac{\cot A\sin B\sin\omega}{\cos B\cos\omega}\quad\therefore\ \sin c\cos\omega-\sin\omega\cos c=\cot A\tan B\sin\omega.$$

Hence, for computing $(c-\omega)$, we have

$$1.\ \tan\omega=\tan a\cos B;\quad 2.\ \sin(c-\omega)=\cot A\tan B\sin\omega.$$

Again: $\tan b\sin c\cos A+\cos c=\dfrac{\cos a}{\cos b}$, applied to the supplemental triangle, gives

$$\tan B\sin C\cos a-\cos C=\frac{\cos A}{\cos B}.$$ Assume $\cot\omega=\tan B\cos a$, then

$$\frac{\cos\omega}{\sin\omega}\sin C-\cos C=\frac{\cos A}{\cos B}\therefore\cos\omega\sin C-\sin\omega\cos C=\sin(C-\omega)=\frac{\cos A}{\cos B}\sin\omega.$$

Hence, for computing $(C-\omega)$, we have

$$1.\ \cot\omega=\tan B\cos a;\quad 2.\ \sin(C-\omega)=\frac{\cos A}{\cos B}\sin\omega.$$

NOTE.—In applying the several formulæ involving subsidiary angles, care must be taken that the influence of algebraic signs be not overlooked: suppose, for instance, that of the three given parts A, B, a, the only one that is obtuse is B; then cot ω in the formulæ just deduced would be negative, and if we were therefore to take ω obtuse, we should evidently have a negative value for sin $(C-\omega)$, to avoid which (though the avoidance is not absolutely necessary), we must regard ω as negative, so that $C-\omega$ will in this case be $C+\omega$.

As an illustration, let C in the example at p. 250 be computed by these formulæ:

		D.	(Parts.)
tan B,	10·074648	427	16226
cos a,	9·855342	−205	−9225
	70		70,01
cot ω,	9·930060		
	220		

D=427)16000(37″·47

$\therefore\ \omega=-49°\ 35'\ 37''\frac{1}{2}$.

Here ω is negative because tan B is negative, and because sin ω, in next operation, in order to give a positive value for sin $(C-\omega)$, must be negative, seeing that cos B is negative.

		D.	(Parts.)
cos B, colog	0·191031	250	9500
cos A,	9·926351	−134	−938
sin ω,	9·881584	179	6707
	152·69		152,69
sin $(C-\omega)$,	9·999118·69		
	8		

D=13) ·69(5″

$\therefore\ C-\omega=$ 86° 21′ 5″

$\omega=-49°\ 35'\ 37''\frac{1}{2}$

$\therefore\ C=$ 36° 45′ 27″½, as before, [nearly.

NOTE.—The angle $C-\omega$ is ambiguous, being given by a sine, and sin $B>$ sin A:—the acute value of it is taken. The difference of 2″, between the former determination and that here arrived at, is easily accounted for: the value of D, namely 13, is so small, that an error even in the figures usually rejected from the "Parts," is sufficient to affect the accuracy of the correction for seconds. It would have been over-refinement to have included these figures—as they are nearly always inaccurate—every D being only an approximation, and consequently, when multiplied by the seconds, giving a product the last two figures of which may be widely erroneous:—it would have been over-refinement to have preserved these, but for the purpose of actually giving practical proof to the student of the desirableness of avoiding sines of very large acute angles, and cosines of very small ones, in fact, all such trigonometrical ratios, of which the differences (D) are very small (see art. 229).

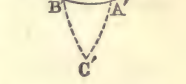
We may here notice, in conclusion, that the student will find, upon

examination, that all the formulæ involving the subsidiary angle ω, may be obtained by dividing the triangle into two right-angled triangles, and then applying Napier's rules.

---

304. **Area of Triangle: Spherical Excess.**—Two great circles of the sphere always intersect each other in two equal angles, at the distance of a semicircumference (277), and the portion of the spheric surface inclosed by the two semicircumferences is called a *Lune*. It is plain that whatever part the angle of the lune is of four right angles, the same part is the lune itself of the whole spheric surface; that is, if $A°$ be the angle, $360° : A° :: \text{surf. of sphere} : \text{surf. of lune} = \frac{A}{360} \times \text{surf. of sphere}$.

It will be proved hereafter that if R be the radius of the sphere, the expression for its surface is $Surface = 4\pi R^2$, so that the surf. of a lune, of angle $A°$, is $Lune = \frac{A}{180}\, 2\pi R^2$.....[1]. Let now ABC be any sph. triangle, and let the sides AC, BC be produced till they again meet. In the diagram, the circle $ABA'B'$ is the base of a hemisphere *above* the plane of the paper: the prolonged sides will therefore meet in a point C' in the remaining hemisphere, *beneath* the plane of the paper: the concealed portions of the meeting semicircles are represented by the dotted lines. Now the triangle $A'B'C'$, in the lower hemisphere, will evidently be equal to the triangle ABC in the upper; for AA', BB', CC', are all equal, each being a semicircle; therefore taking away the part CA', we have $AC = A'C'$. In like manner, $BC = B'C'$, and the angles at C, C', are equal: the two triangles are therefore equal. The triangle ABC is a portion of the lune BB', it is equally a portion of the lune AA'; and the equal triangle $A'B'C'$ is a portion of the third lune CC'. Hence the whole surface of the hemisphere is equal to the sum of these three lunes diminished by the two angles ABC, $A'B'C'$, that is, by twice the triangle ABC. Putting therefore S for the surface of the triangle, we have [1],



$$2\pi R^2 = \frac{A+B+C}{180°} 2\pi R^2 - 2S \therefore S = \frac{A+B+C-180°}{180°} \pi R^2 .....[2],$$

which is the numerical expression for the surface of any sph. triangle, in terms of its three angles and the rad. of the sphere. It follows from it that two triangles are equal in surface, provided only that they are on equal spheres, and that the *sum* of the angles of one is equal to the sum of those of the other. The numerator of the above expression is called the Spherical Excess: it is the amount by which the three angles of a triangle exceed 180°, and is an element of considerable importance in trigonometrical surveying on an extensive scale.

This spherical excess is, we see, an angular magnitude; and we know that magnitudes of this kind admit of two distinct modes of measurement—the gradeal and the arcual: it begets confusion to represent the thing measured by the same symbol when different units of measure are employed, and when it is at the same time of importance that we know

which unit is exclusively taken in the case before us: the expression for the SPHERICAL EXCESS,

$$\left.\begin{array}{l} \textit{in gradeal measure, is } E=\dfrac{S}{R^2}\cdot\dfrac{180^\circ}{\pi} \\ \textit{in arcual measure, it is } \epsilon=\dfrac{S}{R^2} \end{array}\right\} \ldots[3].$$

If, as in all the operations of trigonometry, we regard the radius to be *unit*, then we shall have $\epsilon=S$, that is, the *number* representing the spherical excess, in reference to the angles of a triangle on the surface of a sphere of radius 1, is the *number* representing the surface of the portion of that sphere which the triangle occupies; the latter number being of course so many *square* units. This explanation is necessary, in order that the student may know what writers on this subject mean, when they say that "the area of a triangle on the surface of a sphere, of radius unity, is equal to the excess of its three angles above two right angles."

Given the surface S of a spherical triangle on the surface of the earth, in square feet, to determine the number of seconds in the spherical excess.

Calling here the excess E, we have [3] $E=\dfrac{S}{R^2}\cdot\dfrac{180^\circ}{\pi}$: and taking (as found from measurement) 365154·6 feet for the length of a degree of the earth, we have

$$R=365154{\cdot}6\,\frac{180}{\pi}\text{ feet}\ \therefore\ E=\frac{\pi S}{180(36514{\cdot}6)^2}\text{ degrees}=\frac{3{\cdot}14159\,S\times 60^2}{180(36514{\cdot}6)^2}\text{ seconds},$$

$$\therefore \log E=\log S+\log 62{\cdot}83185\ldots-2\log 365154{\cdot}6=\log S-9{\cdot}3267737.$$

Hence the following RULE. If from the log of the area of the triangle in feet, the constant log, 9·3267737 be subtracted, the remainder will be the log of the spherical excess in seconds.

[This rule, usually called "General Roy's rule," is due to the late Professor Dalby, who communicated it to General Roy while engaged by him on the great Trigonometrical Survey of England and Wales].

305. It will be easy to compute the spherical excess by the above formula when the surface of the spherical triangle in feet is known. In geodesical operations, the triangle on the surface of the earth—or rather the triangle a little above it, whose surface is parallel to the still surface of the ocean—is necessarily so small a portion of the whole surface of the sphere, that its area, computed as that of a plane triangle, from the actual measurements, cannot be affected with any practical error of consequence. Now it is obvious that *horizontal* angles, taken at three different stations, lie in different planes, for the horizons are different, and moreover that each is the angle of a *spherical* triangle:—the spherical triangle formed by great circle arcs of the earth joining the points of observation (like the angle A, for instance, in the diagram at p. 231). The three observed angles therefore—if taken with perfect accuracy—ought, together, to exceed two right angles by the spherical excess E, computed as above: and the observer is thus furnished with a ready method of testing the correctness of his observations. The following is an example, taken from the "Trigonometrical Survey" before referred to.

Names of Stations.	*Observed angles.*
A. Buxter Hill	76° 12′ 22″
B. Dean Hill	48 4 32·25
C. Dunnose	55 43 7
∴ By observation, E=1·25″	180 0 1·25

Distance of Dunnose from	
Buxter Hill	140580·4 feet=AC
Dean Hill	183496·2 feet=BC

Now considering ABC as a *plane* triangle, we have given the two sides AC, BC, and the included angle C, to find the area S.

By Mensuration (see art. 308), $S=\frac{1}{2}AC\,.\,BC\sin C$, which is computed by logs as in the margin, and from the result we infer that the total amount of error in the three observed angles is 5″−1″·25=3″·75. Now, if all the angles have been observed under equally favourable circumstances, so that there appears no reason why one should be more erroneous than another, the correction thus found is to be equally distributed among them, that is, one-third of the total correction is to be applied to each; but if there be a suspicion that one angle is less to be depended upon than the others, then to this is to be applied the greater correction.

$\frac{1}{2}AC$, 70290·2	4·8468947
BC, 183496·2	5·2636271
sin 55° 43′ 7″,	9·9171279
∴ log S=	10·0276497
Subtract	9·3267737
log E=	0·7008760

∴ E=5″·022, or 5″

The corrections of the angles being thus made, the required sides of the triangle may then be computed by Spherical Trigonometry, or we may accomplish the object by Plane Trigonometry, by aid of a theorem for the purpose, first given by Legendre.* If each of the true spherical angles be diminished by one-third of the spherical excess, their sum, thus diminished, will amount to 180°, and will therefore belong to a *plane* triangle; and Legendre has shown that we may employ this plane triangle, instead of the spherical triangle: the sides of the former being regarded as equal to those of the latter, since the difference of length will be practically inappreciable, in reference to triangles connecting stations on the surface of the earth.

And this leads us to remark that in geodesical operations we need never be concerned about ambiguous cases, in reference to our *spherical* triangles: since, in regard to these matters, we may treat them as plane triangles.

Another suggestion also of some interest offers itself. The spherical excess, in a geodesical triangle, is very small, simply because the area of the triangle itself is very small—comparatively with that of the whole spheric surface. In *plane* triangles the area has nothing to do with the magnitude of the angles: *these* always amount to 180°. The sum of the angles of a spherical triangle may have any range between 180° and 540° (283), which range, however, becomes gradually more and more contracted as the area of the triangle diminishes, and by continuing this diminution, may be made as small beyond 180° as we please.

[In what is done above, the spherical excess has been found from the three angles of the triangle: it may also be found (and of course the area of the triangle) from two sides and the included angle, or from the three sides; but it would be quite out of place to enter at any length here into

* See the Appendix to Brewster's translation of Legendre's Geometry.

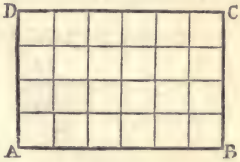
these matters: the subject belongs to *Geodesy*, and the student is referred for ample information on this important branch of practical science, to Captain Yolland's able treatise in the third volume of the Woolwich Course, to the Géodésie of Francœur, and to the valuable and profound articles, TRIGONOMETRY, and FIGURE OF THE EARTH, by the Astronomer Royal, in the ENCYCLOPÆDIA METROPOLITANA.]

END OF THE SPHERICAL TRIGONOMETRY.

IV. MENSURATION: PART I. SURFACES.

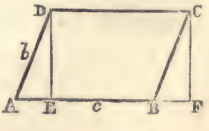
306. **Mensuration** is the practical science which enables us to compute the surfaces and solid contents of bodies when certain of their linear dimensions are known and expressed in figures: it has therefore two principal divisions:—the mensuration of surfaces, and the mensuration of solids. Each of these, too, may be further subdivided under two heads, according as the surfaces and solids are bounded by straight lines and planes, or by curve lines and curve surfaces; but these, and other distinct portions of the subject, will be sufficiently indicated by the headings of the following articles.

307. **Plane Rectilinear Surfaces.**—The amount of surface inclosed by the boundaries of any figure, is called the *area* of that figure, and is estimated in *square* measure, that is the area is so many square inches, square feet, square yards, &c. The simplest of all rectilinear figures is the rectangle: its area is found by multiplying the number of units of length in one side, by the number of units of length in the other side, the product being the number of square units in the surface. Thus, if the side AB, of the annexed rectangle, measure 6 inches, or feet, &c., and the side AD, 4 of these linear units, it is plain that the whole surface might be cut up, by the cross lines in the figure, into just $6 \times 4 = 24$ *squares*, the side of each square being the unit of length: hence the area is 24 square inches, or feet, &c. To prevent confusion, linear feet will be denoted by f, and square feet by F; so that if the sides of the rectangle in the margin were *a feet*, and *b feet*, respectively, we should have Area$=ab$ *Feet*. In algebraic investigations, however, of the various formulæ of mensuration, the denomination *Feet*, or *Inches*, &c., though always understood, is usually suppressed; so that we should say that the area of the rectangle whose sides are a, b, is ab.



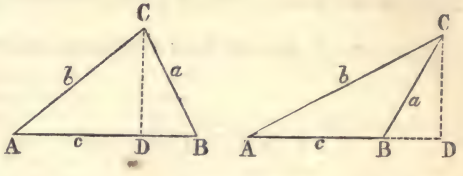
From the expression for the area of a rectangle, we may easily deduce that for any parallelogram, and thence that for a plane triangle, thus:—

308. **Areas of Parallelogram and Triangle.**—The determinations of these merely require that we know the base and altitude of each. For 1, let AC be any parallelogram, AB being its base, and the perp. DE or CF its altitude, then (Euc. 35, I.) the parallelogram $AC=$ the rectangle DF. The area of this latter is $EF \times ED = DC \times ED = AB \times ED$, that is, the area of the proposed parallelogram is the product of its



base and altitude. 2. Let either DB, or AC be joined: then each of the triangles, having the common base AB, will be half the parallelogram (Euc. 41, I.): hence the area of a triangle ABC will be $\frac{1}{2}\ AB \times CD$.

Though the altitude be not directly given, yet it is as good as given if the base, an adjacent side, and the angle between them be given. For in each of these figures, the perpendicular, or altitude p, is $p = b \sin A$: hence



1. Area of parallelogram $AC = bc \sin A$. 2. Area of triangle $ABC = \frac{1}{2}\ bc \sin A$...[1] which expressions become simply bc, and $\frac{1}{2}\ bc$, when A is a right angle.

NOTE.—When the measures of the lines are given in more denominations than one, the multiplication of the two measures may be performed either by decimals or duodecimals (see Weale's Arithmetic, p. 172). But when the given sides include an oblique angle, decimals are to be preferred, since sines are given in decimals.

Ex. (1) Given the base and altitude of a parallelogram 23 f. 9 in., and 11 f. 3 in.: required the area.

1. *By Decimals.*

23 f. 9 in. = 23·75 f.
11 f. 3 in. = 11·25

Product 267·1875 F. = area
12
2·2500
12
27·00

∴ Area = 267 F. 27 In.

2. *By Duodecimals.*

23 f.	9 in.
11	3
261	3
5	11 3
267	2 3

∴ Area = 267 F. 2 P. 3 In. = 267 F. 27 In.

(2) Given two sides of a parallelogram 23 f. 9 in. and 11 f. 3 in.; and the angle between them 58°: required the area.

log 23·75 = 1·375664
log 11·25 = 1·051152
log sin 58° = 9·928420
log 227·634 = 2·355236
172

D=191) 6400(34

∴ Area = 227·634 F.

(3) Two sides of a triangle are 36 f., and 25½ f., and the angle between them 58°: required the area.

nat. sin 58° = ·84805

51 × 9 = 459 = ½ prod. of sides

Area = 389·255 F.
12
3·060
12
36·72

∴ Area = 389 F. 36·7 In.

In using natural sines, contracted multiplication may be employed.

Examples for Exercise.

(1) How many square feet of flooring are there in a room 30 ft. by 25 ft. 4 in. ?

(2) How many square yards of painting are required for a parallelogram whose length is 37 ft. and the perpendicular breadth 5 ft. 3 in. ?

(3) What is the area of a triangle whose base is 18 ft. 4 in. and altitude 11 ft. 10 in. ?

(4) How many square yards are there in a triangle, two sides of which are 25 ft. and 21¼ ft. respectively, and the angle between them 45° ?

(5) How much length of carpeting, ¾ of a yard wide, will suffice for the room in ex. 1 ?

[Note.—In purchasing a carpet for a room the outlay will in general be greater than what calculation would give from the dimensions of the room: for it must be remembered that a roll of carpeting is never cut in the direction of its *length*. If so many breadths and a fraction of a breadth would suffice, an additional breadth must be paid for, and where the length of the carpeting necessary is considerable, the expense may thus be a good deal increased. In general one dimension of the floor is more favourable for the demand of an integer number of breadths than the other, and where economy is studied, and the way in which the pattern runs is not considered of consequence, this should be attended to: suppose, for example, a room is 9 yards by 5. If the pattern is to run along the room, then the breadths are to make up 5 yards; the number of them will therefore be $5 \div \frac{3}{4} = 6\frac{2}{3}$, so that 7 breadths will be required, $\therefore 7 \times 9 = 63$ will be the number of yards of carpeting. But if the pattern run across the room, the number of breadths will be $9 \div \frac{3}{4} = 12$, and $12 \times 5 = 60$ will be the number of yards necessary, so that 3 yards will be saved. Of course in either case some little waste may be expected in adjusting the pattern.]

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309. **Area of Triangle: Three Sides Given.**—Let the sides be a, b, c, and A the angle between b, c, then (p. 256), area $= \frac{1}{2} bc \sin A$; therefore, substituting for sin A its value in terms of the sides as given by [5] at p. 186, we have

$$\text{Area} = \sqrt{\{s(s-a)(s-b)(s-c)\}} \ldots\ldots [2],$$

where $s = \frac{1}{2}(a+b+c)$, which formula, except when the sides are small, should be computed by logs, to which it is well adapted.

Expressed in words, the formula is this, namely:—Rule. From the half sum of the three sides subtract each side separately. Multiply the half sum and the three remainders together, and the square root of the product will be the area.

Ex. Required the area of a triangle whose sides a, b, c, are 47·8, 31·2, 39·6:

1. *By Logs.*

a,	47·8	log s, 59·3	1·773055
b,	31·2	,, $s-a$, 11·5	1·060698
c,	39·6	,, $s-b$, 28·1	1·448706
	2)118·6	,, $s-c$, 19·7	1·294466
s,	59·3		2)5·576925
$s-a$,	11·5	Area 614·416	2·7884625
$s-b$,	28·1		51
$s-c$,	19·7		D=71) 1150(16

2. *Without Logs.*

$59{\cdot}3\times11{\cdot}5\times28{\cdot}1\times19{\cdot}7$
$=377507{\cdot}0615$, and
$\sqrt{37507{\cdot}0615}=614{\cdot}416=$Area.

EXAMPLES FOR EXERCISE.

(1) How many square yards are there in a triangular pavement, the sides of the triangle being 30, 40, and 50 feet respectively?

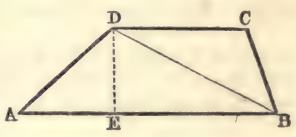
(2) Find the area of a triangle whose sides are $4\frac{1}{2}$, $5\frac{1}{4}$, and $6\frac{1}{3}$ feet.

(3) How many acres are there in a triangular field whose sides are 2569, 4900, and 5025 links?

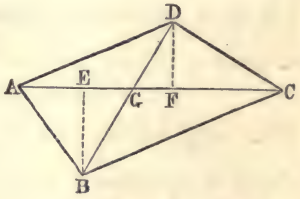
(4) How many acres are there in a triangle whose sides are 20, 30, and 40 chains?

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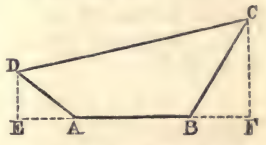
310. **Areas of Quadrilaterals.**—The simplest four-sided figure, after the parallelogram, is the *trapezoid:* this has two opposite sides parallel, but not the other two. Let the diagonal DB be drawn, dividing the figure into two triangles: then if AB, DC, be the two parallel sides, these two triangles will have a common altitude DE —the perp. distance between those sides: hence the area is $\frac{1}{2}(AB+DC)\,DE$...[1].



But if no two sides be parallel, the figure is a *trapezium,* in which case it will be necessary to draw two perpendiculars to the diagonal—one from each of the vertices opposite to it, and the area will obviously be $\frac{1}{2}(BE+DF)\,AC$...[2].



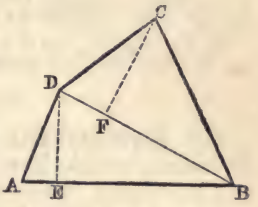
We may, however, dispense with the measurement of the diagonal, which it is sometimes desirable to do, by either adding to, or taking from, the figure, two right-angled triangles, such that the result is a trapezoid. Thus, in the trapezium AC, if perpendiculars be drawn to AB, or AB prolonged, from C, D, twice the area of the trapezoid DF will be equal to $(FC+ED)\,EF$, and twice the area of the two triangles, will be equal to $BF\,.\,FC+AE\,.\,ED$; so that twice the area of the trapezium will be $(EF\mp BF)\,FC+(EF\mp AE)\,ED$; the upper sign having place when the triangles both lie without the figure, as in the diagram, and the lower when they lie within it. But a single rule, comprehending both



cases, as also that in which one triangle is within and the other without the figure, may be easily deduced; it is this:

RULE.—Having measured the two perpendiculars upon the side—or side produced—chosen for base, proceed thus: Measure along the base from one extremity up to the *farther* perpendicular, and multiply that measure by the other perp. Measure, in like manner, from the other extremity up to the farther perpendicular, and multiply that measure by the other perp. Half the sum of these products will be the area. Of course to get these measures the base need be measured but once, and then the bases of the two triangles.

311. There is still another way of computing the area of a trapezium, which is a variation of the first method. A diagonal is measured as before, but only one perp. dropped upon it: the other being drawn in the remaining triangle as in the annexed diagram. *DB*, *AB*, and the two perpendiculars, being measured, the areas of the two triangles are found and added together. In land surveying, complicated figures are always divided into triangles and quadrilaterals, which are computed separately, and the sum of the whole taken for the area of the figure. In what has been explained above, lengths only are measured, and not angles; but it is plain that if the diagonal *AC* (see fig. at p. 258), as also the two sides *AB*, *AD*, are measured and the two angles at *A*, the pair of triangles may be computed by [1] at p. 256. Or, instead of this, we may measure both diagonals *AC*, *BD*, and the angle *G*, where they intersect; any one of the angles at *G* will do as well as any other, because the sine of one is the same as the sine of any other one. Twice the areas of the four triangles, having *G* for their common vertex, is



$$DG \cdot GC \sin G + DG \cdot GA \sin G + BG \cdot GC \sin G + BG \cdot GA \sin G;$$

or, adding together the first and third, and the second and fourth,

$$DB \cdot GC \sin G + DB \cdot GA \sin G = BD \cdot AC \sin G \ldots [3];$$

so that the rule is to multiply the product of the two diagonals by the sine of the angle at which they intersect.

(1) In a four-sided figure two of the opposite sides are parallel, and their lengths are 32 feet, and 47 feet, respectively: also the perp. dist. between them is 28 feet: required the area. Here by [1], the area is

$$\frac{1}{2}(32+47)28 = 79 \times 14 = 1106 \text{ Feet.}$$

(2) The diagonal of a trapezium is 132 feet, and the perpendiculars upon it from the opposite corners, 17 feet and 26 feet, respectively: required the area. Here by [2], the area is

$$\frac{1}{2}(17+26)132 = 43 \times 66 = 2838 \text{ Feet.}$$

(3) In the quadrilateral field *AC* (last fig. at p. 258), perpendiculars are drawn from *A*, *B*, to the side *DC*: the distance of *C* from the farther perp. is 745 links: the distance of *D* from the other perp. is 1000 links:

moreover the perp. from A is 595, and that from B, 352. What is the area of the field in acres? By the rule p. 259, the area is

$$\frac{1}{2}(745.352+1000.595)=428620 \text{ links}=4 \text{ acres } 1 \text{ rood } 5{\cdot}792 \text{ perches.}$$

Examples for Exercise.

(1) How many square feet are there in a plank 12½ feet long, the breadth at one end being 15 inches and that of the other 11 inches?

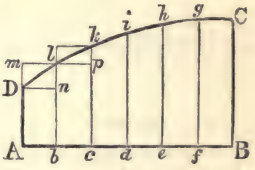
(2) In the trapezium $ABCD$ (p. 258), the measures were as follows: AC=161, DF=30·1, BE=24·5: required the area.

(3) In the quadrangular field $ABCD$ (p. 258), only the following measures could be taken, namely, BC=265 yards, AD=220 yards, AC=378 yards, AE=100 yards, and CF=70 yards: required the area in acres.

(4) Required the area of a quadrangular paddock of which the two diagonals are 30 and 40 yards, and the angle at which they intersect 60°.

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312. **Equidistant Ordinates.**—Every plane figure, with all its boundaries straight lines, may be cut up into triangles and quadrilaterals; and by suitable measurement of the parts of these, all the component areas may be obtained, as explained above. But when the boundary at any part is a curve, other methods must of course be adopted. Should the curve be of a kind not subject to any geometrical law, the area of the portion of surface of which it is a boundary can be found approximately only: the approximation may, however, be brought so close as to involve no error of practical consequence, by the method of *Equidistant Ordinates*. Let DC be any curve boundary, and let it be required to find the area of the portion of surface AC, where DA, CB are perpendiculars upon AB. Let AB be divided into any even number of equal parts by the perpendiculars lb, kc, id, &c.: these perpendiculars are called *ordinates:* they are *here* equidistant ordinates. Draw through D, l, the parallels Dn, mp, to AB, and complete the little rectangles Dl, lk. Then the area of the two rectangles An, bp, will be



$$(AD+bl)Ab \ldots\ldots [1],$$

and the area of the two Al, bk, will be

$$(bl+ck)Ab \ldots\ldots [2].$$

Now it is plain that the curved area $ADkc$ is greater than the first of these and less than the second, that is, it is intermediate between the two; taking, therefore, half the sum of the two, we have $\frac{1}{2}(AD+2bl+ck)Ab$ for the approximate area of the curved portion of surface referred to, and which approximation is the nearer to the truth the closer the ordinates are together. Applying a similar method of proceeding to the portion ch, and then to eC, &c., till the last ordinate is reached, we have

$$\text{Approx. area}=\left(\frac{1}{2}AD+bl+ck+di+\ldots+\frac{1}{2}BC\right)Ab \ldots [3].$$

Again: Referring to the expressions [1], [2], above, we see that the rectangle mc is greater than the first and less than the second; the dif-

ference being in the one case the rectangle mn in excess, and in the other, the rectangle lk in defect. Consequently the rectangle mc, as well as the curve surface $ADkc$, is intermediate between [1] and [2], and is evidently a closer approximation to the curve surface than either of those areas. Hence, an approximation closer than [3] will, in general, be had by taking, instead of one-half the sum of the former two, one-third of the sum of all three. Now, the rectangle mc, like the rectangles before considered, is made up of two rectangles Al, bp; so that, adding these to the former pairs, and taking one-third, we have

$$\frac{1}{3}\{(AD+2bl+ck)+(bl+bl)\}Ab=\frac{1}{3}(AD+4bl+ck)Ab=\text{area } ADkc.$$

$$\text{Similarly, } \frac{1}{3}(ck+4di+eh)Ab=\text{area } ckhe,\ \frac{1}{3}(eh+4fg+BC)Ab=\text{area } ehCB, \text{ \&c.}$$

$$\therefore \frac{1}{3}\{(AD+BC)+4(bl+di+fg+\ldots)+2(ck+eh+\ldots)\}Ab=\text{area } AC,$$

which, expressed in words, gives the following rule, namely,

RULE.—1. Measure an odd number of equidistant ordinates between the extremities of the curve boundary—the more numerous they are the better.

2. To the sum of the extreme ordinates or boundaries add 4 times the sum of the *even* ordinates, that is, of the second, fourth, &c., and *twice* the sum of all the others. Multiply the result by the common distance between the ordinates, and one-third of the product will be the area bounded by the curve, the extreme ordinates, and the line connecting these extremes, and to which the ordinates are perpendicular.

(1) Given nine equidistant ordinates, 14, 15, 16, 17, 18, 20, 22, 23, and 25 feet, and the common distance between them 2 feet: required the area of the surface.

$$14+25=39,\ 4(15+17+20+23)=300,\ 2(16+18+22)=112,$$

$$\text{and } \frac{2}{3}(39+300+112)=300\frac{2}{3} \text{ F.}$$

EXAMPLES FOR EXERCISE.

(1) Given seven equidist. ordinates, 2, 3, 5, 6, 9, 10, and $10\frac{1}{2}$ ft., and the distance between them 6 inches: required the area of the surface.

(2) Given eleven equidist. ordinates, 5, 9, 12, 14, 20, 20, 20, 18, 16, 11, 8 in., and the common distance 6 in.: required the area.

(3) The ordinates of a curve are 0, 2, $3\frac{1}{4}$, $5\frac{1}{4}$, 6, 7, $8\frac{3}{4}$, 9, $7\frac{1}{4}$, $6\frac{3}{4}$, 5, 3, $2\frac{1}{2}$, $1\frac{1}{2}$, 0, feet: required the area, the common interval being 1 ft. 3 in.

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313. **Regular Polygons.**—Let ab, in the diagram at p. 191, be the side of a regular polygon of n sides, and let O be the common centre of the circumscribed and inscribed circles: the radius of the former will be $R=OM$, that of the latter $r=OD$. The angles subtended at O, by all the n sides, will be equal, the measure of each being $\frac{2\pi}{n}$. Call each of the equal sides, a: then $R\sin\frac{\pi}{n}=\frac{1}{2}a$, and $r\tan\frac{\pi}{n}=\frac{1}{2}a$,

$$\therefore R=\frac{1}{2}a\operatorname{cosec}\frac{\pi}{n}, \text{ and } r=\frac{1}{2}a\cot\frac{\pi}{n}\ldots[1].$$

Now the area of the triangle Oab being $\frac{1}{2}ar$, or (p. 256) $\frac{1}{2}R^2 \sin\frac{2\pi}{n}$, if we put S for the areas of all the n triangles, that is, for the area of the whole polygon, we shall have

$$S=\frac{1}{2}nar, \text{ or } S=\frac{1}{4}na^2\cot\frac{\pi}{n}, \text{ or } S=nr^2\tan\frac{\pi}{n}, \text{ or } S=\frac{1}{2}nR^2\sin\frac{2\pi}{n}; \ldots[2]$$

where r is the perp. from the centre of the polygon to one of the sides a, R the line from the centre to one of the vertices, and $\frac{2\pi}{n}$ the whole, and $\frac{\pi}{n}$ the half of the angle subtended at the centre by any side.

The first of the expressions [2], namely, $S=\frac{1}{2}nar$, is the semiperimeter of the polygon multiplied by the radius of the circle it circumscribes.

(1) Required the area of a pentagon, each side of which is 25 feet.

$$\text{Here } S=\frac{1}{4}na^2\cot\frac{\pi}{n}=\frac{1}{4}5\times25^2\cot 36^\circ=\frac{5^3}{4}\times1{\cdot}37638=1075{\cdot}297 \text{ Feet.}$$

(2) Required the area of a regular octagon circumscribed about a circle of 20 feet radius.

$$\text{Here } S=nr^2\tan\frac{\pi}{n}=8\times20^2\times\tan 22^\circ\,30'=3200\times{\cdot}41421=1325{\cdot}472 \text{ Feet.}$$

(3) Required the area of a regular octagon inscribed in a circle whose radius is 20 feet.

$$\text{Here } S=\frac{1}{2}nR^2\sin\frac{2\pi}{n}=4\times20^2\times\sin 45^\circ=1600\times\frac{1}{2}\sqrt{2}=800\sqrt{2}.$$

Examples for Exercise.

(1) Required the area of a regular hexagon, each side of which is $20\frac{1}{2}$ feet.

(2) Required the area of an equilateral triangle, a side of which is a.

(3) Required the area of a regular polygon of 20 sides, the radius of its inscribed circle being 20 feet.

(4) Required the area of a polygon of 50 sides, the radius of the circumscribing circle being 25 feet.

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314. **The Circle and its Sectors.**—When the sides of a regular polygon, circumscribing a circle of radius r, are indefinitely great in number, the perimeter of the polygon becomes confounded with the circumference of the circle, the semiperimeter and the semicircumference being alike represented by π (233), and the area of each being $\frac{1}{2}$ perimeter $\times r$. If $r=1$, the $\frac{1}{2}$ perimeter or semicircumference is $\pi=3{\cdot}1415926\ldots$ and so many *square* units is also the area (233). And since the circumferences of circles are as their diameters, and their areas as the squares of the diameters, by putting d for the diameter ($2r$) of any circle, we have

$$\left.\begin{aligned} 1 : \tfrac{1}{2}d :: \pi : \tfrac{1}{2}d\pi = \text{the semicircumference,} \\ \text{and } 1^2 : \left(\tfrac{1}{2}\right)^2 :: \pi : \tfrac{1}{4}\pi d^2 = \pi r^2 = \text{the area} \end{aligned}\right\} \ldots[1].$$

Hence, to find the circumference of a circle we have only to multiply the diameter by 3·14159..., or rather by 3·1416. And to find the area, to multiply the square of the diameter by $\frac{1}{4}\pi = \cdot 7854$, or the square of the radius by 3·1416. For the length of an arc of a circle subtending $A°$ at the centre, or containing itself $A°$, the radius being $\frac{1}{2}d = r$, the expression is obviously $\frac{r\pi}{180}A = \cdot 0175rA$; and for the area of the sector, bounded by this arc, the expression is $\frac{1}{2}(\cdot 1075rA) \times r = \cdot 0175r^2A \div 2$, or $r \times$ half the length of the arc: hence calling the length l, whether it be that of a whole circumference or not, we have

$$l = \frac{r\pi}{180}A, \text{ area} = \frac{1}{2}\frac{r^2\pi}{180}A = \frac{1}{2}rl \ldots[2],$$

where A is the number of degrees in the circular arc.

(1) Required the circumference and area of a circle whose diameter is 50 feet. Here [1], circumference $= d\pi = 3{\cdot}1416 \times 50 = 157{\cdot}08$ feet, and area $= \cdot 7854d^2 = \cdot 7854 \times 2500 = 1963{\cdot}5$ feet.

(2) Required the area of a circular sector, the chord of which is 24·24, the radius of the circle being 15 feet. Referring to the diagram at page 191 we have

$$\sin aOD = \frac{12{\cdot}12}{15} = \frac{4{\cdot}04}{5} = \cdot 808 \therefore aOD = 53° 54' : \text{ hence [2],}$$

$$\text{Area } aOM = \frac{1}{2}\frac{r^2\pi}{180}A = \frac{15^2 \times 3{\cdot}1416}{360} \times 53\frac{54}{60} = \frac{5 \times 3{\cdot}1416}{8} \times \left(54 - \frac{1}{10}\right) = 105{\cdot}83265 \text{ Feet,}$$

and the double of this, 211·6653 Feet, is the area of the sector.

NOTE.—Since the circumference of a circle is $c = \pi d$, $\therefore d = \frac{c}{\pi}$,

$$\therefore \text{Area} = \frac{\pi}{4}d^2 = \frac{\pi}{4}\left(\frac{c}{\pi}\right)^2 = \frac{c^2}{4\pi} = \cdot 07958c^2 \ldots[3],$$

an expression which is useful in finding the area when the circumference only is given.

EXAMPLES FOR EXERCISE.

(1) How many square yards are there in a circle whose diameter is 3½ feet?

(2) What length of arc is there in 12° 10′ of a circle of 10 feet radius?

(3) Required the area of a circle whose circumference is 12 feet.

(4) Required the area of a sector of a circle of 25 feet radius, the arc being 147° 29′.

(5) If the diameter of the earth at the equator be 7924·9 miles, what is the length of one degree of the equator?

(6) How many square yards are there in the top of a circular table of which the circumference is 178 inches?

NOTE.—The number 3·1416, which so frequently occurs in practical inquiries about the circle, may be replaced by the fraction $\frac{22}{7}=3{\cdot}142857$, when much accuracy is not necessary. This fraction for the ratio of the circumference of a circle to its diameter, was first given by Archimedes. But the fraction $\frac{355}{113}$, proposed by Metius, is a much closer approximation, closer even than the ratio 3·1416 to 1, since $\frac{355}{113}=3{\cdot}1415929$, which agrees with 3·1415926 as far as six decimals.

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315. **Segment of a Circle.**—To find the area of a segment *aMb* (p. 191), we have first to find the area of the *sector*, and then the area of the triangle *Oab*, and to take the sum or difference of the two, according as the segment is greater or less than a semicircle.

In the diagram at page 191, *ab* is the common chord of the *two* segments into which that line divides the circle; the smaller of these, that exhibited in the diagram, is the sector *minus* the triangle; the remaining part of the circle *includes* this triangle.

(1) Required the area of the smaller segment of a circle cut off by a chord 12 feet in length, the radius of the circle being 10 feet.

Here the semichord $\div r=6\div 10=\cdot 6=\sin\frac{1}{2}A^\circ=\sin 36^\circ\ 52'\frac{1}{2}$ $\therefore A^\circ=73^\circ{\cdot}74$,

$$\therefore \text{Area of Sector}=\frac{1}{2}\frac{r^2\pi}{180}A=\frac{5\times 3{\cdot}1416}{18}73{\cdot}74=5\times 1{\cdot}0472\times 12{\cdot}29=64{\cdot}35044 \text{ Feet.}$$

$$\text{Area of Triangle}=\frac{1}{2}r^2\sin A^\circ=50\sin 73^\circ\ 44'\ \tfrac{2}{3}=50\times{\cdot}96000=48 \text{ Feet.}$$

$$\therefore \text{Area of Segment}=64{\cdot}35044-48=16{\cdot}35044 \text{ Feet.}$$

(2) Required the area of a segment of a circle whose height is $46\frac{3}{4}$ feet, and the radius 25 feet. Here the segment is obviously greater than a semicircle, and

$\therefore$ its height minus the radius is the height of the triangle $=21\frac{3}{4}$.

$$\text{Also } 21\tfrac{3}{4}\div 25=\cdot 87=\cos\frac{1}{2}A^\circ \therefore A^\circ=59^\circ\ 5'=59^\circ\tfrac{1}{12}$$

$$\text{and } 360-59\tfrac{1}{12}=300\tfrac{11}{12}=\text{number of degrees in the arc of the segment.}$$

$$\therefore \text{Area of sector}=\frac{1}{2}\frac{r^2\pi}{180}\,300\ \tfrac{11}{12}=\frac{125\times{\cdot}1309}{3}(301-\tfrac{1}{12})=1641{\cdot}2 \text{ Feet.}$$

$$\text{Area of triangle}=\frac{1}{2}r^2\sin A=312{\cdot}5\times{\cdot}85792=268{\cdot}1 \text{ Feet.}$$

$$\therefore \text{Area of segment}=1641{\cdot}2+268{\cdot}1=1909{\cdot}3 \text{ Feet.}$$

316. There is another and more speedy way of finding the area of a segment, when the height is given, by aid of a table of segments to diameter 1, the area being then found by the following rule, namely:—

RULE. Divide the height of the segment by the diameter; the quotient will be the height of the similar segment in the table, against which the area of that similar segment will be found. Multiply this tabular area by the square of the given diameter, and the product will be the area sought.

NOTE 1.—When the proposed segment exceeds a semicircle, the similar segment will not be found in the table, which is here annexed. In this case the quotient mentioned above must be subtracted from 1 (the tabular diameter), and the segment corresponding to the remainder taken out. This segment must then be subtracted from the whole tabular circle, namely, from ·785398, the remainder will be the tabular area corresponding to that sought. *This*, therefore, must be multiplied by the square of the given diameter, as the rule directs.

2. If the quotient, or the remainder here spoken of, be not found exactly in the table, we take out the nearest less segment, and correct it as we do a logarithm. The following application of the rule to the example just worked will sufficiently explain this:—

$$1-\frac{46\cdot75}{50}=1-\cdot935=\cdot065.$$

Against ·06 in the table we find ·01924: the difference between this and the next number is ·00493, which multiplied by the neglected 5, (and divided by 10) gives ·00246 for correction; therefore the correct tabular segment is ·02170:

The theory of the annexed operation is obvious; the heights of similar segments are as the diameters of the circles, and their areas, as the squares of the diameters.

NOTE.—Instead of multiplying by 2500, it is better to divide 7637·0 by 4.

Hence, tab. seg.	= ·02170
To be subtracted from	·78540
Area of sim. segment	·76370
Square of diameter .	2500
	38185
	15274
Area of segment .	1909·25 Feet.

EXAMPLES FOR EXERCISE.

(1) Required the area of a segment of a circle, of which the height is 6 feet, and the diameter of the circle 32 feet.

(2) Required the area of the segment of a circle, the arc being a quadrant, and the radius 24.

(3) What is the area of a segment, the arc of which is 280°, the diameter of the circle being 50 feet?

(4) Required the area of a segment, whose height is 18 feet, and the diameter of the circle 50 feet.

(5) Required the area of a segment, less than a semicircle, whose chord is 16 feet, the diameter being 20 feet.

(6) From a circle, whose diameter is 50 feet, a segment, containing 54·1475 Feet, is to be cut: required the length of the chord.

Areas of Segments.	
Height.	Area, $d=1$.
·01	·00133
·02	·00375
·03	·00687
·04	·01054
·05	·01468
·06	·01924
·07	·02417
·08	·02944
·09	·03502
·10	·04088
·11	·04701
·12	·05339
·13	·06000
·14	·06683
·15	·07387
·16	·08111
·17	·08853
·18	·09613
·19	·10390
·20	·11182
·21	·11990
·22	·12811
·23	·13646
·24	·14494
·25	·15354
·26	·16226
·27	·17109
·28	·18002
·29	·18905
·30	·19817
·31	·20738
·32	·21667
·33	·22603
·34	·23547
·35	·24498
·36	·25455
·37	·26418
·38	·27386
·39	·28359
·40	·29337
·41	·30319
·42	·31304
·43	·32293
·44	·33284
·45	·34278
·46	·35274
·47	·36272
·48	·37270
·49	·38270
·50	·39270

317. **Circular Ring.**—If a smaller circle be placed wholly within a larger, the space between the two is called a circular ring. If D be the

diameter of the outer circle, and d that of the inner, the difference of their areas, that is the area of the ring, will obviously be

$$\text{Area} = \cdot 7854\,(D^2 - d^2) = \cdot 7854\,(D+d)(D-d),$$

whether the two circles be concentric or not, so that we have only to multiply together the sum and difference of the diameters, and the product by ·7854.

(1) The diameters of two circles, one within the other, are 10 and 6: required the area of the space between their circumferences.

Here $\cdot 7854\,(10+6)(10-6) = \cdot 7854 \times 64 = 50 \cdot 2656$, the area required.

Examples for Exercise.

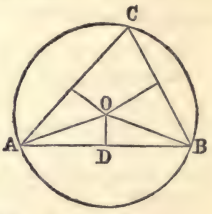
(1) Required the area of the ring between two circumferences, the diameter of which are 10 and 20.

(2) In a circular slab of marble, of which the diameter is 73·25 inches, a circular hole, 3·5 inches in diameter, is cut: how many square inches are there in the upper surface of the remaining ring?

(3) The inner diameter of a circular building is 73 ft. 3 in., and the thickness of the wall is 1 ft. 9 in.: how much ground does the wall stand upon?

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318. **Inscribed and Circumscribed Triangle.**—Let ABC be a triangle inscribed in a circle. If any two of its sides be bisected by perpendiculars, they will meet in O, the centre of the circle (Euc. 5. VI.). Put the radius $OA = R$, then



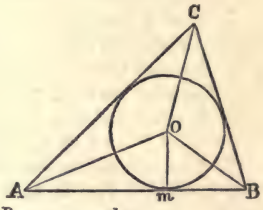
$AD = R \sin AOD = R \sin C$ (Euc. 20. III.): hence

$$\frac{1}{2}c = R \sin C,\ \frac{1}{2}b = R \sin B,\ \frac{1}{2}a = R \sin A;$$

$$\therefore \text{[5] page 186, } R = \frac{abc}{4\sqrt{\{s(s-a)(s-b)(s-c)\}}} = \frac{abc}{4 \text{ area of triangle}} \ldots [1],$$

$$\therefore \text{Area of triangle} = \frac{abc}{4R}.$$

Again, let the triangle circumscribe a circle: then the centre O is the point where the lines AO, BO, bisecting two of the angles, meet (Euc. 4. IV.), and Om, drawn to the point of contact m, is the radius r, and is perpendicular to AB: hence



$$\text{Triangle } OAB = \frac{1}{2}cr, \text{ triangle } OAC = \frac{1}{2}br,$$

$$\text{triangle } OBC = \frac{1}{2}ar.$$

The sum of these is the triangle

$$ABC = \frac{1}{2}bc \sin A = \frac{1}{2}(a+b+c)r = sr,$$

$$\therefore r = \frac{bc}{2s} \sin A, \text{ or, } r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \ldots [2], \quad \therefore \frac{R}{r} = \frac{abc}{4(s-a)(s-b)(s-c)} \ldots [3].$$

It is plain that the expression Area $= sr$, where s is the semiperimeter of the figure, applies to any polygon circumscribing the circle, whether it be

regular or irregular. If the *triangle* be regular, that is, equilateral, $\frac{R}{r}=2$ $\therefore R=2r$. But of the equilateral triangle, the area is

$$\frac{1}{2}a^2 \sin 60°=\frac{1}{4}a^2\sqrt{3}, \text{ and from the expression above, area}=\frac{1}{4}\frac{a^3}{R},$$

$$\therefore \sqrt{3}=\frac{a}{R}, \ \therefore R=\frac{a}{\sqrt{3}}=\frac{1}{3}a\sqrt{3}, \ \therefore r=\frac{1}{2}R=\frac{1}{6}a\sqrt{3}, \text{ also area}=\frac{1}{4}a^2\sqrt{3}.$$

MENSURATION: Part II. Solids.

319. To find the solid content or volume of a parallelopiped, prism, or cylinder, the rule is this: Multiply the area of the base by the height, and the product will be the number of the cubic units in the body.

Note.—This rule applies, whatever be the boundary of the plane base of the solid, provided the sides rising from this base are all upright, or perpendicular to it. The truth of the rule is obvious: Let the base contain m square feet: then an upright solid on this base, 1 foot high, must contain m cubic or solid feet, and therefore a solid h feet high, on that base, must contain mh cubic feet.

(1) The diameter of a rolling stone is 18·7 inches, and its length 4·75 feet: how many cubic feet of stone are there in it?

$18{\cdot}7^2 \times {\cdot}7854$=area of base in sq. *inches*,

$\therefore$ 2 log 18·7+log ·7854+log 4·75−log 144=log volume.

The calculation is annexed, from which it appears that the number of cubic feet is 9·0595.

Note.—The 18·7 being inches, we must divide by 12 to bring them to feet: hence the square of 18·7 must be divided by 144.

2 log 18·7		1·271842
		1·271842
log	·7854	$\bar{1}$·895090 (See p. 279.)
log	4·75	0·676694
colog	144	7·841638
log	9·0595	957106
		080

D=48) 260(5

Examples for Exercise.

(1) How many cubic feet are there in a block of marble of which the length is 3 ft. 2 in., the breadth 2 ft. 8 in., and the thickness 2 ft. 6 in.?

(2) The diameter of a well is 3 ft. 9 in., and its depth 45 ft.; what was the cost of sinking at 7*s.* 3*d.* per cubic foot?

(3) How many cubic feet are there in a triangular prism, whose length is 10 feet, and the three sides of the triangular base 3, 4, and 5 feet?

(4) The length of a hollow iron roller is 4 ft., the diameter of the outer circumference 2 ft., and the thickness of the metal $\frac{3}{4}$ in. Required the solid content.

(5) How many cubic feet are there in a cylindrical column whose height is 20 ft., and circumference $5\frac{1}{2}$ ft.?

(6) Required the volume of an oblique circular cylinder, which, when standing on a horizontal plane, inclines at an angle of 60°, the length being 25 feet, and diameter of the base 30 inches.

---

320. **Surface of Prism and Cylinder.**—The surface of any solid bounded by plane rectilinear faces, is found by computing each face separately by the rules given in the preceding Part, and then taking the

sum of the results. If the solid be an upright prism or parallelopiped, then omitting the two ends, it will be sufficient to multiply the perimeter of the base by the height: if the base be a regular polygon of n sides, then the lateral surface will be found by multiplying n times a side of the base by the height. The curve surface of an upright cylinder is found also by multiplying the circumference of the base by the height, for the base may be regarded as a regular polygon of an infinite number of sides (233). If, therefore, r be the radius of the base, d its diameter, c its circumference, and h the height of the cylinder, the curve surface is $S=ch=\pi dh$. Thus the curve surface of a cylinder whose height is 20 feet, and the diameter of its base 2 feet, is $S=3{\cdot}1416\times 2\times 20=125{\cdot}664$ square feet. If the whole surface be required, we must add to this the areas of the two ends, namely $\cdot 7854\times 8=6{\cdot}2832$; so that the whole surface is 131·9472 sq. feet.

Examples for Exercise.

(1) Required the surface of a cube, the length of each side being 15 feet.

(2) Required the whole surface of a triangular prism, whose length is 20 feet, and each side of the base 18 inches.

(3) The gallery of a church is supported by 16 cylindrical columns of wood, each of which is 4 ft. 8 in. in perimeter, and 12 ft. high: required the cost of painting them at 9*d.* per sq. yard.

---

321. **Pyramid and Cone.**—Since by Euclid (7 and 10, Book XII.) a pyramid is a third part of a prism of the same base and altitude, and a cone is a third part of a cylinder of the same base and altitude, we have this Rule. Multiply the area of the base by the altitude, and take $\frac{1}{3}$ the product.

(1) The diameter of the base of an upright circular cone is $26\frac{1}{2}$ inches, and the altitude $16\frac{1}{2}$ feet: required the solid content, or volume.

$$\text{Area of base}=26{\cdot}5^2\times{\cdot}7854=551{\cdot}547, \text{ and since } \tfrac{1}{3}h=5{\cdot}5,$$

$$\therefore \text{Volume}=\frac{551{\cdot}547\times 5{\cdot}5}{144}=21{\cdot}066 \text{ cubic feet.}$$

Examples for Exercise.

(1) Required the volume of a triangular pyramid, the height being $14\frac{1}{2}$ ft., and the sides of the base 5, 6, and 7 ft.

(2) Required the volume of a cone of which the height is $10\frac{1}{2}$ ft., and the circumference of the circular base 9 ft.

(3) How many cubic feet are there in a pyramid whose base is a pentagon, of which each side is 2 ft., the height of the pyramid being 12 ft.?

(4) The diameter of the base of a circular cone is 42 inches, and its height 94 inches: it is required to cut off two solid feet of material from the top by a plane parallel to the base: what must be the altitude of the cone out off?

---

322. **Surface of Pyramid and Cone.**—The sides of a pyramid are all plane triangles; its lateral surface, the sum of all these, is therefore found by the rules for computing the areas of plane triangles before given. If the base be a regular polygon of n sides, the lateral surface of

an upright pyramid is found by multiplying n times a side by the slant height of the pyramid, and taking half the product. As there is no limit to the number n, it may be indefinitely great, the sides themselves then being indefinitely small, the polygonal base becoming confounded with a circle (233): hence the convex surface of an upright cone also is found by multiplying the circumference of the base by the slant height, and taking half the product. Thus, the convex surface of an upright cone, of which the slant height is 50 feet and the diameter of whose base is 8·5 feet, is $S=3\cdot1416\times8\cdot5\times25=\frac{1}{4}3\cdot1416\times850=667\cdot59$ sq. feet.

Examples for Exercise.

(1) Required the convex surface of a triangular pyramid, each side of its base being 3 feet, and its slant height 20 feet.

(2) Required the convex surface of an hexagonal pyramid, each side of its base being $2\frac{1}{2}$ feet, and its perpendicular height 10 feet.

(3) The diameter of the base of an upright circular cone is $26\frac{1}{2}$ inches, and its altitude $16\frac{1}{2}$ feet: required the number of sq. feet in its entire surface.

---

323. **Frustum of Pyramid or Cone.**—A frustum is the portion of the solid which remains after a solid similar to the whole is cut off by a plane parallel to the base. Let H be the height of the complete pyramid or cone, and h the height of the similar solid cut off: also let A be the area of the base of the larger body, and a the area of the base of the smaller. Then, since the bases of similar pyramids and cones are as the squares of their altitudes, we have

$A:a::H^2:h^2$ $\therefore \sqrt{A}:\sqrt{a}::H:h$ $\therefore$ (79) $\sqrt{A}-\sqrt{a}:\sqrt{A}::H-h:H$, and $\sqrt{A}-\sqrt{a}:\sqrt{a}::H-h:h$. From these two proportions we have

$$H=\frac{\sqrt{A}}{\sqrt{A}-\sqrt{a}}(H-h), \text{ and } h=\frac{\sqrt{a}}{\sqrt{A}-\sqrt{a}}(H-h).$$

Now the difference in solid content between the original body and the part cut off is, (321),

$$\frac{1}{3}(AH-ah)=\text{Volume of the frustum};$$

$$\therefore \text{Volume of frustum}=\frac{1}{3}\frac{A\sqrt{A}-a\sqrt{a}}{\sqrt{A}-\sqrt{a}}(H-h)=\frac{1}{3}(A+a+\sqrt{A}a)(H-h)\ldots[1],$$

where $H-h$ is the height of the frustum. Hence the following rule:—

Rule I. To the areas of the two ends add the square root of the product of those areas: multiply the sum by the height, and one-third of the result will be the volume or solid content.

II. If the solid be a conical frustum, the shortest way of arriving at the final product will be to add together the squares of the diameters of the two ends and the product of those diameters, and to multiply the sum by the height and by ·7854: one-third of the result will be the volume. (See p. 263.)

III. If, however, the diameters are not given but the circumferences, add together the squares of these and their product, multiply the sum by the height and by ·07958, and take one-third of the result. (See p. 263.) Expressed as formulæ, the last two rules are as follow, where D, d, are

the diameters of the ends, and C, c, the circumferences, observing that

$$\cdot 7854 = \frac{\pi}{4}, \text{ and } \cdot 07958 = \frac{1}{4\pi}:$$

$$\text{Volume} = \frac{1}{12}\pi h(D^2 + d^2 + Dd), \text{ or Volume} = \frac{1}{12} \cdot \frac{h}{\pi}(C^2 + c^2 + Cc) \ldots [2].$$

(1) The shaft of Pompey's Pillar is a single stone of granite 90 feet in height: the diameter of its circular base is 9 feet and the diameter of the circle at top is $7\frac{1}{2}$ feet: how many cubic feet of stone does it contain.

Here $\frac{1}{3} \cdot 7854h\,(D^2 + d^2 + Dd) = 30 \times \cdot 7854\,(81 + 56 \cdot 25 + 67 \cdot 5) = 23 \cdot 562 \times 204 \cdot 75 = 4824 \cdot 32$ cubic feet.

Examples for Exercise.

(1) How many solid feet are there in a stick of timber whose ends are squares, each side of the greater end being 15 inches, and each side of the less end 6 inches, the length of the stick being 24 feet?

(2) How many cubic feet are there in a conic frustum 18 feet high, the end diameters being 8 feet and 4 feet?

(3) The ends of the frustum of a pyramid are regular hexagons, a side of the greater of which is 13, and a side of the less 8: the length of the frustum is 24: required its volume.

(4) The largest of the Egyptian Pyramids is only a square frustum; its height is 480 feet; a side of its square base is 745 feet, and a side of its square top is 32 feet: required its weight, allowing 150 lb. to the cubic foot.

---

324. **Surface of a Frustum.**—If the frustum be that of a pyramid, each face will be a trapezoid, the area of which is found by multiplying the sum of the two parallel sides by the perpendicular distance between them, and taking half the product: for the frustum of a regular pyramid of n equal faces, we have only to take one of these trapezoids n times, or, which is the same thing, to multiply n times the sum of the opposite ends of a face by the distance between them, and to take half the product. When the frustum becomes that of an upright cone, by the number n of the sides of the polygonal ends becoming indefinitely great, the sum of the two circumferences multiplied by the slant height of the frustum, and half the product taken gives the curve surface; or, which amounts to the same:—Multiply the circumference of the frustum, taken half-way between the two ends, by the slant height, and the product will be the surface. Let it be required to find the convex surface of a frustum of an upright cone, of which the slant height is $12\frac{1}{2}$ feet, and the circumferences of the two ends 6 ft. and 8·4 ft. Here half the sum of the circumferences, that is, the circumference in the middle, is 7·2 ft. $\therefore S = 7 \cdot 2 \times 12\frac{1}{2} = 90$ F.

Examples for Exercise.

(1) The slant height of a frustum of an upright cone is 45 ft., and the circumferences of the ends 113 ft. 9 in. and 31 ft. 3 in.: required the convex surface.

(2) The slant height of a frustum of an upright cone is 9 ft., and the diameters of the ends 43 in. and 23 in.: required the whole surface.

(3) The perp. height of the frustum of an upright pentagonal pyramid is 11 ft., each side of the base is 34 in., and each side of the top 18 in.: how many sq. feet are there in the convex surface?

---

325. **The Sphere.**—Since a sphere is two-thirds of its circumscribing cylinder [see article 327],

$$\therefore \text{Sphere} = \frac{2}{3}\pi R^2.2R = \frac{4}{3}\pi R^3 = \frac{1}{6}\pi D^3 = \cdot 5236\, D^3 = \frac{1}{6}\frac{C^3}{\pi^2} = \cdot 01689\, C^3;$$

that is: multiply the cube of the diameter by ·5236; or multiply the cube of the circumference by ·01689. [See this value computed at p. 279.]

EXAMPLES FOR EXERCISE.

(1) Required the volume of a sphere whose diameter is 12 feet.

(2) The circumference of a cannon ball is 18½ inches: how many cubic inches does it contain?

(3) The exterior diameter of an iron shell is 10 inches, and the interior diameter 8 inches: how much metal, in cubic feet, is there in the shell?

---

326. **Surface of a Sphere and Segment of a Sphere.**—Let a radius be drawn to any point on the surface, and conceive a small tangent plane at this point: the radius will be perp. to this plane, which plane therefore will form the base of a pyramid of which the altitude is the radius R of the sphere. By thus covering the surface with small tangent planes, we shall have a series of pyramids having these planes for bases, and having R for their common altitude. The volume of the whole assemblage of pyramids will therefore be found by multiplying the sum of their bases by $\frac{1}{3}R$. But by making these bases smaller and smaller—thus increasing their number indefinitely, the polygonal surface will become confounded with the spheric surface: hence the volume of the sphere will be

$$V = \frac{1}{3}RS, \text{ where } S \text{ is the surface of it.}$$

And it is obvious from the above reasoning that the expression $V = \frac{1}{3}RS$ is equally that for a spherical sector (consisting of a segment and an attached cone), the base of it being S, any portion of the spheric surface, and its vertex being at the centre of the sphere, that is, its height being R.

We have seen that for the sphere

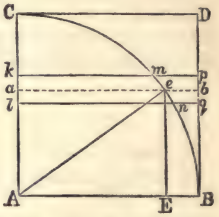
$$V = \frac{4}{3}\pi R^3 \therefore \frac{1}{3}RS = \frac{4}{3}\pi R^3 \therefore S = 4\pi R^2 = \pi D^2 \ldots [1],$$

the surface is therefore equal to four times the area of a great circle of the sphere, or it is equal to the convex surface of a cylinder circumscribing the sphere (320).

These deductions are admissible, however, only on the assumption in (325), that the volume of the sphere is two-thirds that of the circum-

scribing cylinder—the celebrated theorem of Archimedes. Or if the deduction [1] be proved, independently, the theorem itself follows: we proceed therefore to prove [1] without reference to (325).

327. **Theorem of Archimedes.**—Let ABC be a quadrant of a circle, and draw the perpendiculars BD, CD, completing the square CB. If this square and quadrant be made to revolve about the fixed line AC as an axis, the former will evidently describe a cylinder, and the latter a hemisphere inscribed in the cylinder. Now two perpendiculars, km, ln, to the axis may be drawn so close together that the intercepted arc mn may be taken for a straight line, that is, for its chord, in which case mn, in revolving, will generate part of a conic surface, pq generating the corresponding portion of the cylindric surface: the radius of the middle section of the conic surface is represented by the dotted line ae. Now the area of the surface, thus described by mn, is $\pi \times 2ae \times mn$ (324), and the area of the corresponding surface described by pq, is $\pi \times 2ab \times pq$.



Conceive a perpendicular to lq to be drawn from m: this will be equal to pq, so that pq, mn, will be the perpendicular and hypotenuse of a little right-angled triangle similar to the triangle AEe, the angle included between the little hypotenuse (mn) and the perpendicular ($=pq$) being equal to the angle A, since the sides of one angle are perpendiculars to those of the other. Hence by similar triangles,

$$mn : pq :: Ae : AE \text{; but } Ae=ab, \text{ and } AE=ae, \therefore ae \times mn = ab \times pq.$$

Consequently the surface generated by mn is equal to the surface generated by pq. The same reasoning applies to every other arc $m'n'$ of BC, taken so small as to be confounded with its chord: hence, taking the sum of all the indefinitely small arcs into which BC may be conceived to be divided, we conclude that the convex surface of the whole hemisphere is equal to the convex surface of the cylinder on the same base and of the same altitude: and, moreover, that the convex surface of any portion of the sphere, contained between two parallel planes, is equal to the convex surface of that portion of the circumscribing cylinder which is contained between the same planes. Putting therefore S for the whole surface of the sphere, we have $S=4\pi R^2=\pi D^2$; and since it was proved above that

$$V=\frac{1}{3}RS, \therefore V=\frac{4}{3}\pi R^3=\frac{2}{3}\pi R^2.2R,$$

that is, the volume of the sphere is equal to two-thirds that of its circumscribing cylinder. For the portion of surface between two parallel planes at the distance h apart, we have $S=\pi Dh$. The formulæ established above supply the following rules:—

Rule I. *To find the surface of a sphere.*—Multiply the square of the diameter by 3·1416, or the circumference of the sphere by the diameter.

II. *To find the surface of a segment.*—Multiply the circumference of the sphere by the height of the segment.

(1) The surface of a sphere whose diameter is 24 inches is $24^2 \times 3{\cdot}1416 = 1809{\cdot}56$.

(2) How many square inches are there in the convex surface of a segment of a sphere whose circumference is 3 ft. 10 in.: the height of the segment being 6 inches? Here $S=46 \times 6=276$ inches.

EXAMPLES FOR EXERCISE.

(1) How many square miles are there in the surface of the earth, the circumference of which is 25000 miles?

(2) Required the convex surface of a segment of a sphere of 5 feet diameter, the height of the segment being 1 ft. 9 in.

(3) From a sphere 4 ft. 11 in. in diameter two segments, 12 and 14 inches high respectively, are cut: required the convex surface of the portion left.

NOTE.—The portion of surface left is evidently the same whether the planes cutting off the segments are parallel or not; and the same may be said as to the volume left.

---

328. **Volume of Segment and Zone.**—As before, D being the diameter of the sphere, and h the height of the segment, which we shall first regard as less than a hemisphere, we have for its surface $S=\pi Dh$; also for the volume of the spheric sector, we have (326) $V=\frac{1}{6}DS=\frac{1}{6}\pi D^2h=$ $\cdot5236D^2h$. This includes both the segment considered, and the attached cone. Putting r for the radius of the base of this cone, that is, for the radius of the base of the segment, its volume is (321)

$$\pi r^2\times\frac{1}{3}(R-h)\text{; but (Euc. 35. III.), } r^2=(D-h)h,$$

$$\therefore \text{ vol. of cone}=\pi(D-h)h\times\frac{1}{6}(D-2h)=\cdot5236\,(D^2h-3Dh^2+2h^3),$$

which taken from $\cdot5236\,D^2h$, the spheric sector, leaves, for the volume of the segment,

$$V=\cdot5236\,(3D-2h)h^2\ldots[1].$$

This formula may always be employed when the height h of the segment and the radius R of the sphere are given. But if instead of the radius of the sphere, the radius r of the base of the segment is given, then a more suitable formula results from substituting in this for D, its value $D=\frac{r^2+h^2}{h}$, derived from the equation $r^2=(D-h)h$ above. Thus changed, the formula [1] above becomes

$$V=\cdot5236\left\{\frac{3(r^2+h^2)}{h}-2h\right\}h^2=\cdot5236\,(3r^2+h^2)h\ldots[2].$$

These formulæ, in words, give the following rules, which, as shown in the Note, are applicable to all cases.

RULE I. From three times the diameter of the sphere subtract twice the height of the segment. Multiply the remainder by the square of the height, and the product by $\cdot5236$.

RULE II. To three times the square of the radius of the base of the segment, add the square of the height, and multiply the sum by the height and by $\cdot5236$.

NOTE.—A segment V' greater than a hemisphere, may be found by subtracting the remaining segment V from the whole sphere. Calling the height of the greater segment h', we have $h=D-h'$, so that

$$\frac{\pi}{6}\{D^3-[3D-2(D-h')](D-h')^2\}=\frac{\pi}{6}(3D-2h')h'^2=V',$$

which is the same formula as [1] above, h being the height of the segment, whether it be less or greater than a hemisphere: hence the above rules apply without restriction.

If opposite segments be cut off by two parallel planes, the intermediate portion left is called a *zone* of the sphere, the volume of which will be found by subtracting one segment from the larger segment made up of the one to be subtracted and the zone itself. The formula is therefore

$$V=\cdot5236\,(3r'^2h'-3r^2h+h'^3-h^3).\quad\text{But as shown above, } D=\frac{r'^2+h'^2}{h'}=\frac{r^2+h^2}{h},$$

$$\therefore\ r'^2h-r^2h'=h^2h'-h'^2h,\ \text{also } h'^3-h^3=3h'^2h-3h'h^2+(h'-h)^3\text{: hence}$$

$$V=\cdot5236\,\{3(r'^2+r^2)+(h'-h)^2\}(h'-h)\ldots[3],$$

that is:—To three times the sum of the squares of the radii of the two ends or bases, add the square of the height of the zone. Multiply the sum by the height, and the product will be the volume. If the zone be a *middle* zone, the two bases are equal; so that $r'=r$, also the square of the radius of the sphere being

$$R^2=r^2+\left(\frac{h'-h}{2}\right)^2,\ \therefore\ 6R^2=6r^2+\frac{3}{2}(h'-h)^2,$$

$$\therefore\ V=\cdot5236\left\{6R^2-\frac{1}{2}(h'-h)^2\right\}(h'-h)\ldots[4],$$

that is:—From six times the square of the radius of the sphere, take half the square of the height. Multiply the remainder by the height, and by ·5236: the product will be the volume of the middle zone.

Ex. Required the volume of a zone of a sphere, the diameter of the ends being 20 in., and 15 in., and the distance between them 10 in.

$$V=\cdot5236\{3(10^2+7\cdot5^2)+10^2\}10=\cdot5236\times5687\cdot5=2977\cdot975\text{ cubic inches.}$$

Examples for Exercise.

(1) From a sphere whose diameter is 80 inches, two equal segments are cut off, the height of each being 8 inches: required the volume of the portion of the sphere left. (See note, page 273.)

(2) The two end diameters of a spherical zone are 4 ft., and 3 ft., and the height of the zone 4 ft.: required the volume.

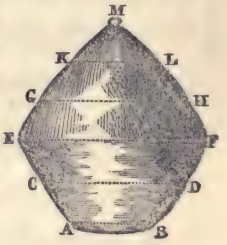
The convex surface of a zone is found in the same way as that of a segment, that is, by multiplying the circumference of the sphere by the height of the zone (see p. 272).

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329. **Equidistant Sections of a Solid.**—The principles explained at (312), for finding the area of an irregular plane surface by the method of equidistant ordinates, may be safely extended to the calculation of the volume of an irregular solid by means of equidistant sections. Regarding the solid to be comprehended between two parallel planes, let equidistant parallel sections—even in number—be severally computed. Call the sum of the sections through the *odd* points of division of the distance between the extreme planes, S_1, and the sum of the sections through the *even* points, S_2: also let the sum of the two end boundaries of the solid, that is, the two surfaces coincident with the extreme planes, be s; then if the thickness of each slice of the body be d, we shall have

for the volume, as at (312), $V=(4S_1+2S_2+s)\frac{d}{3}$. This formula, which, like the corresponding rule at p. 260, is only approximative, may be justified by the following considerations :—

Conceive the sections spoken of to be perpendicular to the plane of the paper. The series of sections may be represented by so many straight lines proportional to them : if these are placed perpendicularly to a straight line, as for instance perpendicularly to AB, in the diagram at page 260, and at the proposed equal distances, their other extremities will be on the line, regular or irregular, which would be marked out could we interpose the continuous series of lines representing all the continuous series of sections of the body : the *plane* surface Ab, thus presented, would then represent the *solid* of which these innumerable ordinates represent the sections : and we have seen that the magnitude of this surface is approximately expressed by the above formula. The approximation will be the closer the nearer the form of the body approaches to symmetry ; and in practice, unsymmetrical forms very seldom occur. The rule is applicable, with but a slight departure from strict accuracy, to the measurement of haystacks, of railway cuttings, of embankments, and of bulging vessels, such as vats, and casks. As an example, take a circular haystack, which is a figure widening upwards from the circular base to a certain height ; it then contracts, the upper portion rising like a conical dome, and terminating in a point, as in the annexed figure. Suppose the following dimensions were taken, namely :—



Circum. or girt at bottom A, 36 ft.
,, at C, 54
,, at E, 66
,, at G, 58
,, at K, 37

The perp. distance between these being 5 feet, and the height of the cone MKL, 4 ft. 6 in. Since the area of a circle is = the square of the circum. $\times\cdot0796$ (Note, p. 263), we may take the multiplier in an inquiry like this to be ·08, and calculate the sections as annexed.

$36^2\times\cdot08=103\cdot68$ area of bottom
$54^2\times\cdot08=233\cdot28$,, section C
$66^2\times\cdot08=348\cdot48$,, ,, E
$58^2\times\cdot08=269\cdot12$,, ,, G
$37^2\times\cdot08=109\cdot52$,, ,, K

$\therefore s=213\cdot2$, $S_1=502\cdot4$, $S_2=348\cdot48$, $d=5$,

$$\therefore (4S_1+2S_2+s)\frac{d}{3}=4866\cdot26 \text{ feet.}$$

Also of the cone MKL, the volume is

$$\frac{1}{3}(109\cdot52\times4\cdot5)=164\cdot28 \text{ feet.}$$

Add 4866·26

27)5030·54(186·31

Hence the volume of the stack is 186·31 cubic yards.

330. **Weight and Dimensions of Shot and Shells.**—The weight and dimensions of any proposed shell or shot are most easily found from comparing it with one of like kind in reference to which these particulars are found by experiment, since the weights are as the quantities of metal, and spheres are as the cubes of their diameters. Experi-

ments show, 1. That an iron shot, 4 inches in diameter, weighs 9 lb. 2. That a leaden shot, 1 inch in diameter, weighs $\frac{3}{14}$ lb.

331. **Iron Shot.**—Let D be the diameter of the shot; then from the above experimental result, $4^3 : D^3 :: 9 : \frac{9}{64} D^3 = \left(\frac{1}{8}+\frac{1}{64}\right)D^3$ = the weight of the shot.

RULE I. Take $\frac{1}{8}$, the cube of the diameter, then $\frac{1}{8}$ of that eighth: add the two results together, and the sum will be the weight of the shot in lbs.

RULE II. Multiply the weight by $7\frac{1}{9}\left(=\frac{64}{9}\right)$, and the cube root of the product will be the diameter.

(1) The diameter of an iron shot is 6·7 inches: what is its weight?

$\frac{1}{8}$ $6{\cdot}7^3 = 37{\cdot}595$, and $\frac{1}{8}$ $37{\cdot}595 = 4{\cdot}699$: the sum of these is 42·294 lb.

(2) What is the diameter of a 42-lb iron shot?

$7\frac{1}{9} \times 42 = 298{\cdot}666\ldots$, and the cube root of this is 6·685 in. the diameter.

EXAMPLES FOR EXERCISE.

(1) Required the weight of an iron shot of diameter 5·54 in.

(2) Required the diameter of a 24 lb. iron shot.

(3) Required the diameters of the several shot weighing 1, 2, 3, 4, 6, 9, 12, 18, 24, 32, 36, and 42 lb., and the calibre of their guns, allowing $\frac{1}{50}$ of the calibre, or $\frac{1}{49}$ of the ball's diameter for windage.

Answer.

Wt. of Shot.	D. of Shot.	Calibre.	Wt. of Shot.	D. of Shot.	Calibre.
1	1·9230	1·9622	12	4·4026	4·4924
2	2·4228	2·4723	18	5·0397	5·1425
3	2·7734	2·8301	24	5·5469	5·6601
4	3·0526	3·1149	32	6·1051	6·2297
6	3·4943	3·5656	36	6·3496	6·4792
9	4·0000	4·0816	42	6·6844	6·8208

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332. **Iron Shell.**—The weight of a shell is the difference of the weights of two shots, one having the external diameter of the shell, and the other the internal diameter for its own diameter. Calling the two diameters D, d, the weight is therefore

$$W = \frac{9}{64}(D^3 - d^3) = \left(\frac{1}{8}+\frac{1}{64}\right)(D^3 - d^3).$$

RULE. Subtract the cube of the internal diameter from the cube of the external diameter; take $\frac{1}{8}$ of the remainder, then $\frac{1}{8}$ of that eighth, and add the two together: the sum will be the weight of the shell in lbs.

Ex. The diameter of the outside of an iron shell is 12·8 inches, and the diameter inside 9·1 inches: required its weight.

$(12{\cdot}8)^3=1343{\cdot}581$, $(9{\cdot}1)^3=753{\cdot}571$, $\frac{1}{8}$ of the diff. is 167·948, and $\frac{1}{8}$ of this is 20·993: the sum of the two is 188·941 lbs. the weight.

EXAMPLES FOR EXERCISE.

(1) What is the weight of an iron shell whose diameters are 10·5 in. and 7·5 in.?
(2) What is the weight of an iron shell, the two diameters of which are 9·8 in. and 7 in.?

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333. **Weight of Powder in Shells, &c.**—The weight of a given bulk of gunpowder depends on the fineness of the grains and also upon the quality of the ingredients in its composition: on the average 30 cubic inches of powder weigh 1 lb. Hence if any receptacle is to be filled with such powder, the weight in lbs. necessary for the purpose will be found by dividing the volume, or solid content of the receptacle, in cubic inches, by 30. The volume of a cylinder of diameter d and height h is $7854d^2h$, which, divided by 30, is $\frac{1}{38{\cdot}2}d^2h$ very nearly: hence these rules for a cylinder.

RULE I. Multiply the square of the diameter by the length of the cylinder and divide by 38·2: the quotient will be the number of lbs. of powder it will contain.

RULE II. Multiply the number of lbs. by 38·2 and divide the product by the square of the diameter: the quotient will be the length of cylinder occupied.

(1) How much powder will fill a cylinder whose diam. is 10 in., and length 20?

$(10^2\times20)\div38{\cdot}2=1000\div19{\cdot}1=52{\cdot}356=52$ lb. 6 oz. nearly.

(2) What length of a 36-pounder gun of $6\frac{2}{3}$ inches calibre (or bore) will be filled by 12 lb. of powder?

$$(12\times38{\cdot}2)\div\left(\frac{20}{3}\right)^2=12\times38{\cdot}2\times\frac{9}{400}=3\times38{\cdot}2\times{\cdot}09=10{\cdot}314 \text{ inches.}$$

EXAMPLES FOR EXERCISE.

(1) How much powder will fill a cylinder whose diameter is 4 in., and length 12 in.?
(2) What length of cylinder of 8-in. diameter may be filled by 20 lb. of powder?

For the weight of powder to fill a shell.—RULE. Divide the cube of the internal diameter by 57·3: the quotient will be the number of lbs. of powder. For the capacity of the shell is $\cdot5236d^3$, which divided by 30, is

$$\frac{\cdot5236}{30}d^3=\frac{1}{57{\cdot}3}d^3.$$

Ex. Supposing the windage of all mortars to be $\frac{1}{60}$ of the calibre, and the internal diameter of the shell to be $\frac{7}{10}$ of the calibre of the mortar; it is required to find the diameter and weight of the shell, and the weight of powder requisite to fill it, for each of the several sorts of mortar, namely, 4·6, 5·8, 8, 10, and 13, inch mortars.*

Calibre of Mortar.	Diameter of Shell.	Weight of Empty Shell.	Weight of Powder.	Weight of Filled Shell.
4·6 in	4·523 in.	8·320 lbs.	0·583 lbs.	8·903 lbs.
5·8	5·703	16·677	1·168	17·845
8	7·867	43·764	3·065	46·829
10	9·833	85·476	5·986	91·462
13	12·783	187·791	13·151	200·942

334. **Scholium.**—It has been sufficiently seen in the course of the foregoing treatise, that whenever the circle or sphere is concerned, the multiplier 3·1416, and its submultiples ·7854, ·5236, are in very frequent request. It has also been observed (p. 264), that the fraction $\frac{22}{7}$, being nearly equal to 3·1416, is employed instead of the latter number when strict accuracy is not of moment, as it is easier to multiply by $\frac{22}{7}$, or rather by $3\frac{1}{7}$, than to multiply by 3·1416.

It suggested itself to the author some years ago that $\frac{22}{7}$ might *always* supply the place of 3·1416, without any sacrifice of accuracy, provided a very simple correction were applied to the product. Thus: the fraction $\frac{22}{7}$ is 3·142857142857..., and if $\frac{4}{10000}$ of this be subtracted from it, the remainder will be 3·1416 *exactly*. The $\frac{4}{10000}$*th* part of the value of $\frac{7}{22}$ is obtained at once by multiplying it by ·0004, that is, by cutting off four places of figures from the end, and then multiplying all the preceding by 4, placing the figures of the product, in order, under the number itself, commencing under the last of those temporarily cut off. Annexed is a verification of the truth of what is here said. Let $\frac{22}{7}$ be taken to any assigned number of decimals—say twelve; then cutting off the last four we multiply the remaining figures by 4 (taking account of the *carrying*

$\frac{22}{7}$=3·142857142857
$\frac{22}{7}\times$·0004=001257142857
Rem. 3·1416

* What concerns shot and shells above is from Dr. Hutton's Course of Mathematics, vol. ii.

from the part rejected), and place the results, in order, under the given figures. We see, upon subtracting, that the remainder is 3·1416 *exactly*.

Suppose the following question were proposed: the diameter of the sun is about 883220 miles: what is his circumference? Multiplying the diameter by $3\frac{1}{7}$, and proceeding as explained above, we get the same result that would be arrived at by multiplying by 3·1416, and with much less numerical work. In like manner, instead of the multiplier ·7854, we may employ $\frac{1}{2}\cdot\frac{11}{7}$, and instead of ·5236, may use $\frac{1}{3}\cdot\frac{11}{7}$ and then introduce the above correction. The worked examples at (pp. 268, 274), for instance, are more briefly solved in this way.*

7)883220	
3	
2649660	
126174·2857	
2775834·2857	
1110·3337	
Circum.	2774723·9520

It may be proper here to apprize the student that in multiplying by 3·1416, or indeed by any number, the last decimal of which is not strictly accurate, our product cannot be regarded as correct, beyond the number of leading figures which amount to one more than the number in the multiplier at furthest. But the student will find ample information respecting the Arithmetic of Decimals in the "Rudimentary Arithmetic," published by Mr. Weale, of Holborn.

The following numbers are of frequent use in Mensuration:—

π = 3·14159 26535 89793...	Log.	·49714 98726 94133...
$\frac{1}{4}\pi$ = ·78539 81633 97448...		$\bar{1}$·89508 98813 66171...
$\frac{1}{6}\pi$ = ·52359 87755 98298...		$\bar{1}$·71899 86223 10490...
$\frac{1}{\pi}$ = ·31830 98861 83790...		$\bar{1}$·50285 01273 05866...
$\frac{1}{4\pi}$ = ·07957 74715 45947...		$\bar{2}$·90079 01359 77903...
π^2 = 9·86960 44010 89358...		·99429 97453 88267...
$\frac{1}{\pi^2}$ = ·10132 11836 42337...		$\bar{1}$·00570 02546 11732...
$\frac{1}{6\pi^2}$ = ·01688 68639 40389...		$\bar{2}$·22754 90042 28088...

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335. **Applications of Maxima and Minima.**—We shall terminate the present treatise on Mensuration with a few miscellaneous problems on finding the greatest or least possible values of certain geometrical quantities: in the solution of them the student will find it

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* See for fuller details the Treatise on Mensuration, by the author of this work, published by Mozley and Co., Paternoster Row.

necessary to keep in mind the following principles established at pp. 147-50:—

1. If an expression $X=u$ is a maximum or a minimum, the equation $X-u=0$ must have equal roots, one of which will be the value of x corresponding to that max. or min.

2. If this equation be a quadratic, the root sought will be found by rendering the coef. of x^2 unit, and then taking half the coef. of x with changed sign; and there will also exist the condition, "square of the middle coef. minus 4 times the prod. of the extremes=0."

3. Whatever be the degree of the equation, the limiting equation will always have one of its roots equal to that sought (162). This equation, therefore, with any factor introduced into it, may always be combined with the primitive, for the purpose of eliminating u, or for any other purpose, and the resulting equation will involve the same root.

4. The value of x, and thence that of u (or conversely), having been determined, attention must be paid to the precepts at p. 148, to ascertain, *in the case of a quadratic*, whether u be a maximum or a minimum. And the rule at p. 150 must be applied, for this purpose, when the expression is of a higher degree than the second, and it may be applied in all cases where the expression is a rational polynomial.

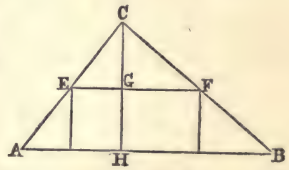
5. An expression, if a maximum or minimum, must continue so whatever constant quantities it be increased or diminished by, and whatever constant factors be introduced or suppressed. But if a constant quantity be diminished by a maximum or minimum, the remainder, on the contrary, is a minimum or a maximum. Any power or root of a maximum or minimum must give a max. or min. result.

6. The reciprocal of a maximum is a minimum, and the reciprocal of a minimum is a maximum.

The last two principles are self-evident.

336. Prob. 1.* To determine the greatest rectangle that can be inscribed in a given triangle.

Let ABC be the given triangle, its height $CH=h$, being known, and let $GH=x$, be the unknown height of the rectangle sought. Then by similar triangles,



$$CH : AB :: CG : EF,$$

$$\text{that is, } h : c :: h-x : EF=\frac{c(h-x)}{h},$$

$$\therefore \text{Area of rectangle}=\frac{c}{h}(hx-x^2)=\text{max.} \quad \therefore hx-x^2=\text{max.}=u \quad \therefore hx-x^2-u=0 \ldots [1],$$

$$\therefore x^2-hx+u=0 \quad \therefore \text{(p. 148)}, \quad x=\frac{h}{2}, \text{ the height required:}$$

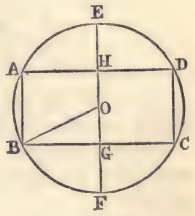
and it is plain that for this height the rectangle is a *maximum*, since for

* These problems are taken from a large collection brought together from various sources, and recently published by W. H. Allen and Co., Leadenhall Street, under the following title: "A Treatise on Problems of Maxima and Minima, solved by Algebra, by Ramchundra, late Teacher of Science, Delhi College." This remarkable work was printed at Calcutta in 1850, and the reprint (1859) here referred to was undertaken by the Court of Directors of the East India Company, by the advice of Professor De Morgan, to whom, therefore, British students are mainly indebted for a very curious and interesting performance. In the present work, the several problems are solved in a manner different from that of the Hindoo algebraist.

$x=0$, [1] is *negative* (see p. 148). The greatest rectangle, therefore, is that whose height is half the height of the given triangle.

PROB. 2. To determine the greatest rectangle that can be inscribed in a given circle.

Let AC be the rectangle, and EF a diameter bisecting BC in G. Put $OG=x$, and the radius $=r$; $OG=OH$, because equal chords are equidistant from the centre; also $BG=\sqrt{(r^2-x^2)}$,



$$\therefore 2BG \,.\, OG=4x\sqrt{(r^2-x^2)}=\text{max.},$$

$$\therefore x^2(r^2-x^2)=\text{max.}=u \;\therefore r^2x^2-x^4-u=0\ldots[1],$$

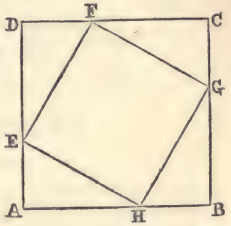
$$\therefore x^4-r^2x^2+u=0 \;\therefore \text{(p. 148)},\; x^2=\frac{r^2}{2},$$

$$\therefore x=\frac{r\sqrt{2}}{2} \;\therefore r\sqrt{2}=\text{the height of the rectangle.}$$

Putting $x=0$ in [1], the result is *negative:* hence the rectangle is a *maximum*.

PROB. 3. To determine the least square that can be inscribed in a given square.

Let EG be a square inscribed in the given square $AC=a^2$: then, because the sides of EG are all equal, and because the opposite sides are parallel, it is plain that the four triangles, cut off by the sides, are all equal.



Let $AH=x$, then $HB=EA=a-x$,

$$\therefore x^2+(a-x)^2=EG=\text{min.}=u \;\therefore 2x^2-2ax+a^2-u=0\ldots[1],$$

$$\therefore \text{(p. 148)},\; x=\frac{a}{2} \;\therefore u=\frac{a^2}{2}.$$

And since for this value of u, and $x=0$, [1] is *positive*, u is the least possible. The least square is $\therefore$ half the given square, and has its vertices at the middle of the sides of it.

PROB. 4. To determine the cone of given volume, whose entire surface shall be the least possible.

Let x be the height of the required cone, and y the radius of its base; then (321) its volume will be $V=\frac{1}{3}\pi y^2x$, which being given, put $y^2x=a^3$.

The surface is, convex surf. + base $=\pi y\sqrt{(x^2+y^2)}+\pi y^2$, but $y^2=\frac{a^3}{x}$,

$$\therefore x^2+y^2=\frac{a^3+x^3}{x} \;\therefore \text{surface}=\pi a^{\frac{3}{2}}\frac{\sqrt{(a^3+x^3)}+a^{\frac{3}{2}}}{x}=\text{min.} \;\therefore \frac{\sqrt{(a^3+x^3)}+a^{\frac{3}{2}}}{x}=\text{min.}=u,$$

$$\therefore a^3+x^3=\left(ux-a^{\frac{3}{2}}\right)^2=u^2x^2-2a^{\frac{3}{2}}ux+a^3 \;\therefore x^2-u^2x+2a^{\frac{3}{2}}u=0\ldots[1],$$

$$\therefore \text{(p. 148)},\; x=\frac{u^2}{2}. \quad \text{Also } \sqrt{U}=\sqrt{\left(u^4-8a^{\frac{3}{2}}u\right)}=0 \;\therefore u=2\sqrt{a},$$

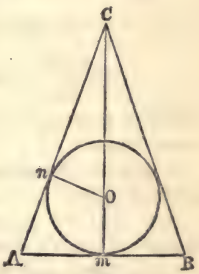
and it is plain that for any value of u less than this, $\sqrt{U}$ will be imaginary: hence $u=2\sqrt{a}$ is the least possible value of u, and the corresponding value of x is $x=\frac{u^2}{2}=2a$: hence the height of the cone containing

the given volume under the smallest amount of surface, is found by dividing that volume by $\frac{1}{3}\pi$, and taking twice the cube root of the result; also the radius of its base is

$$y=\frac{a^{\frac{3}{2}}}{x^{\frac{1}{2}}}=\frac{a^{\frac{3}{2}}}{(2a)^{\frac{1}{2}}}=\frac{1}{2}a\sqrt{2},$$ one-fourth the height multiplied by $\sqrt{2}$.

PROB. 5. Of all cones circumscribing a given sphere, to determine that which has the smallest volume.

Let $On=r$ the radius of the sphere: put $OC=x$, and $Am=y$: then, by similar triangles, CnO, CmA, we have



$Cm : mA :: Cn : nO$, that is, $x+r : y :: \sqrt{(x^2-r^2)} : r$

$$\therefore y=\frac{r(x+r)}{\sqrt{(x^2-r^2)}},$$

$$\therefore \text{Area of base of cone}=\pi y^2=\pi\frac{r^2(x+r)^2}{x^2-r^2}=\pi\frac{r^2(x+r)}{x-r},$$

$$\therefore \text{Volume}=\frac{1}{3}\pi\frac{r^2(x+r)^2}{x-r}=\text{min.}\ \therefore \frac{(x+r)^2}{x-r}=\text{min.}=u,$$

$$\therefore x^2+2rx+r^2=ux-ur\ \therefore x^2-(u-2r)x+r^2+ur=0\ \therefore \text{(p. 148)},\ x=\frac{u-2r}{2}.$$

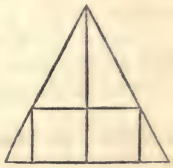
$$\text{Also } \sqrt{U}=\sqrt{\{(u-2r)^2-4(r^2+ur)\}}=\sqrt{\{u(u-8r)\}}=0,$$

$$\therefore u=8r, \text{ the smallest value of } u,\ \therefore x=3r:$$

hence the height of the smallest cone is twice the diameter of the sphere; and the radius (y) of its base is $r\sqrt{2}$.

PROB. 6. To determine the greatest cylinder that can be inscribed in a given cone.

Let h be the height of the cone, and r the radius of its base. Put x for the radius of the cylinder: then it is obvious, by similar triangles, that



$$r : h :: r-x : \frac{h(r-x)}{r}=\text{the height of the cylinder,}$$

the volume of which is $V=\pi x^2\frac{h(r-x)}{r}=\text{max.},$

$$\therefore rx^2-x^3=\text{max.}=u,\ \therefore f(x)=rx^2-x^3-u=0.$$

That this may have equal roots, we must have

$$f_1(x)=2rx-3x^2=0\ \therefore\ 2r-3x=0\ \therefore\ x=\frac{2}{3}r.$$

This value of x, put in $f_2(x)=2r-6x$, renders it negative; the value is therefore such as to render the function V a maximum. Hence, the height of the greatest cylinder is

$$\frac{h\left(r-\frac{2}{3}r\right)}{r}=\frac{1}{3}h=\text{one-third the height of the cone.}$$

PROB. 7. To determine the dimensions of a cylindrical measure that shall hold a given quantity under the smallest extent of surface.

Let a be the given quantity, or the volume of the cylinder, and put x

for the radius of its base, and y for its height. Then the volume of the cylinder is $a=\pi x^2 y$, $\therefore y=\frac{a}{\pi x^2}$; and the entire surface is

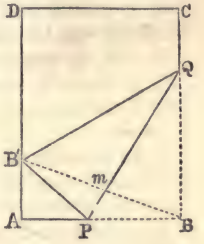
$$\text{convex surf.}+\text{base}=2\pi xy+\pi x^2=\frac{2a}{x}+\pi x^2=\text{min.}=u, \therefore f(x)=\pi x^3-ux+2a=0.$$

That this may have two equal roots, the condition $f_1(x)=3\pi x^2-u=0$ must be satisfied. Subtract the former equation from x times this, then

$$2\pi x^3-2a=0 \therefore x=\sqrt[3]{\frac{a}{\pi}} \therefore y=\frac{a}{\pi x^2}=\frac{a}{\pi}\div\sqrt[3]{\frac{a^2}{\pi^2}}=\sqrt[3]{\frac{a}{\pi}};$$

hence the height of the cylinder must be equal to the radius of the base, and this is the given quantity $\sqrt[3]{\frac{a}{\pi}}$. That this value of x answers to a *minimum*, follows from the fact that $f_2(x)=6\pi x$ is positive for that value (p. 150). The other values of x are imaginary, so that u admits of but one minimum, and no maximum.

PROB. 8. The corner of a leaf is turned back, so as just to reach the other edge; to find when the length of the crease is the smallest possible.



Let the bottom corner of the page now before the reader, represented by AC, be turned down, PQ being the crease; then $PB=PB'$, and $QB=QB'$; also the angles B, B', being right angles, a circle may be described about the figure BB'. And since (Euc. B. VI. Prop. II.) the rectangle of the diagonals is equal to the rectangles of the opposite sides, $PQ\,.\,BB'=2BQ\,.\,PB$...[1].

Let $PB=x$, and $AB=a$; then (Euc. 12. II.),

$$B'B^2=B'P^2+BP^2+2AP\,.\,PB=2x^2+2(a-x)x=2ax,$$

$$\therefore B'B=\sqrt{2ax}. \quad \text{Now } BQ^2=PQ^2-PB^2,$$

$$\therefore (1)\ PQ^2\,.\,B'B^2=4BQ^2\,.\,PB^2=4(PQ^2-PB^2)PB^2=4PQ^2\,.\,PB^2-4PB^4,$$

$$\therefore 4PB^4=(4PB^2-B'B^2)PQ^2, \text{ that is, } 4x^4=(4x^2-2ax)PQ^2,$$

$$\therefore PQ^2=\frac{2x^3}{2x-a}=\text{min.} \therefore \frac{2x-a}{x^3}=\text{max.}=u \therefore -fx=ux^3-2x+a=0,$$

$$\therefore -f_1(x)=3ux^2-2=0. \text{ Multiply this by } \frac{x}{3}, \text{ and subtract, } \therefore \frac{4}{3}x-a=0,$$

$$\therefore x=\frac{3}{4}a \therefore u=\frac{1}{2}a\div\left(\frac{3}{4}\right)^3 a^3=\frac{32}{27a^2}, \text{ and twice the reciprocal of this is } PQ^2=\frac{27a^2}{16},$$

$$\therefore PQ=\frac{3}{4}a\sqrt{3}, \text{ the minimum length of the crease.}$$

That this *is* a minimum, or that u is a maximum, follows from the fact that $f_2(x)$, namely $-6ux$, is *negative* for the determined values of x and u.

NOTE.—Since the sign of $f_2(x)$ is required to be ascertained, in order to enable us to distinguish between a max. and a min., it is necessary to notice whether or not the sign of $f_1(x)$ has been changed when equating it to zero: the expression represented by u above having been transposed, the sign of it has been changed. [In next prob. the sign will be preserved.]

PROB. 9. To determine when the area of the part of the leaf turned down, as in last problem, is the least possible.

It is plain that $B'B$ is bisected at right angles by PQ, therefore $PQ \cdot B'B = 4$ times the area of $B'PQ$. Now by last prob.,

$PQ = \sqrt{\frac{2x^3}{2x-a}}$, and $B'B = \sqrt{2ax}$, $\therefore$ multiplying the squares of these together, we have

$$4a\frac{x^4}{2x-a} = \text{min.}, \quad \therefore \frac{2x-a}{x^4} = \text{max.} = u, \quad \therefore f(x) = 2x - a - ux^4 = 0, \quad \therefore f_1(x) = 2 - 4ux^3 = 0.$$

Multiply this by $\frac{x}{4}$, and subtract, $\therefore \frac{3}{2}x - a = 0$, $\therefore x = \frac{2a}{3}$,

and since $f_2(x) = -12ux^2$ is negative, u, for this value of x, is a maximum; and therefore the proposed expression is a minimum.

EXAMPLES FOR EXERCISE.

(1) Of all right-angled plane triangles having the same hypotenuse, to find that which has the greatest area.

(2) Of all right-angled plane triangles of the same area, to find that the sum of the two perpendicular sides of which is the least possible.

(3) To determine the angle the sum of whose sine and cosine is the greatest possible.

(4) To determine the cone of greatest volume under a given surface.

(5) To inscribe the greatest parallelogram in a given triangle, and having one angle (A) in common with the triangle.

(6) Find the smallest isoceles triangle that can circumscribe a given circle.

(7) To determine the greatest cone that can be inscribed in a given sphere.

(8) To determine the height of the greatest cylinder that can be inscribed in a sphere of radius r.

(9) To determine the greatest cube which, being put with a face downward into a given conical vessel full of water, shall expel the greatest quantity possible.

(10) Required the radius of the sphere, such that, when put into a conical vessel full of water, of depth 6 inches and diameter 5 inches, it shall expel the greatest quantity possible.

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337. **Scholium.**—In applying the theory of maxima and minima to the objects of Mensuration, it almost always happens, as in the foregoing problems, that a slight contemplation of the figure or body under consideration, will be sufficient to inform us whether the solution obtained belongs to a maximum or a minimum, without applying to the result any analytical test. Thus, of a rectangle inscribed in a triangle—of a cylinder in a cone—a cone in a sphere, &c., there is evidently no minimum state; and of a cone circumscribing a sphere there is evidently no maximum state. But as it is not invariably thus evident to which of the two states our result must belong, we have in the preceding solutions always submitted such result to analytical examination. Take, for instance, the third of the "Examples for Exercise:" it is certainly not immediately obvious that the sum of the sine and cosine of an angle cannot be a minimum as well as a maximum; but the analytical examination shows that no minimum can exist. There is another matter, too, to which it is of importance that the attention of the student should be specially directed: a reference to the third ex. at p. 148 will illustrate the point we have in view. Suppose the expression in that example were $x - \sqrt{(a^2 - x^2)}$, instead of $x + \sqrt{(a^2 - x^2)}$; the process there followed would have remained

precisely the same, and without attending to the caution now to be pressed upon his attention, the student would naturally conclude that the maximum value of $x-\sqrt{(a^2-x^2)}$ is $a\sqrt{2}$, the value of x, giving this maximum, being $x=\frac{1}{2}a\sqrt{2}$, which is an absurd conclusion, for $x=\frac{1}{2}a\sqrt{2}$ gives, not $a\sqrt{2}$, but 0. The cause of the absurdity is this, namely, that in reducing $x-u-\sqrt{(a^2-x^2)}=0$ to a rational form, we have virtually replaced this equation by $\{x-u-\sqrt{(a^2-x^2)}\}\{x-u+\sqrt{(a^2-x^2)}\}=0$, which is equally satisfied whether the *first* factor be $=0$, or the second, and consequently have infringed the express restriction that the *first* factor must of necessity be 0; and it turns out that our solution applies exclusively to the *second* factor—the factor extraneously introduced to effect a particular purpose, and not to the factor to which we have presumed it to apply. It is thus seen how important it is, when we have to satisfy the conditions of an equation involving a quadratic radical, and instead of dealing with it in the state in which it is presented we first rationalize it, and then deduce our conclusions:—it is seen how important it is to examine whether those conclusions really apply to the given equation, or only to the new equation unavoidably introduced in changing the irrational form to a rational one. And such examination is equally necessary, whatever be the nature of the inquiry. In the particular example here commented upon, the correct inference from the absurdity which such examination leads to would be, that $x-\sqrt{(a^2-x^2)}$ has neither a maximum value nor a minimum value. (See the author's "Theory of Equations," pp. 26, 40.)

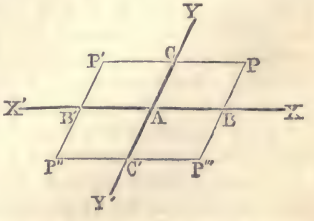
END OF THE MENSURATION.

V. ANALYTICAL GEOMETRY: The Conic Sections.

338. The main object of Analytical Geometry, or, as it is sometimes called, Coordinate Geometry, is, by representing geometrical lines and surfaces by algebraic equations, to deduce their various properties by the processes of algebra, and thus to express them in a purely symbolical form. Of lines situated in a plane, the most important are the straight line, the circle, and the higher curves known as the conic sections: these will be discussed in the order here named.

Part I. The Straight Line and Circle.

339. **Equations of a Point.**—Let P be the position of any point on the paper; the object is to denote this position by the symbols of algebra, so that if the geometrical position P were obliterated, we might recover it from the algebraic representation. In order to do this, two straight AX, AY, intersecting at any angle A, are assumed in the plane of the point: these are called *axes of reference;* then if the parallels to these axes, PB, PC, be of known lengths, and



these lengths be represented by appropriate symbols—such as the figures of arithmetic, and it be also indicated by such representation in which of the four angles about A it is situated, it is plain that the position of P will be sufficiently recorded; for if the given lengths AB, AC, be measured along the axes in the indicated directions and parallels to those axes be drawn from B, C, their intersection P will mark the position of the point.

340. The length AB is called the *abscissa* of the point P, and AC, or BP, its *ordinate;* when spoken of together, AB, BP, or AB, AC, are called the *co-ordinates* of the point P, and the axes of reference are hence more generally called the *axes of co-ordinates:* to distinguish one from the other, AX is referred to as the axis of abscissas, and AY as the axis of ordinates, since it is along these that the abscissas and ordinates are respectively measured in constructing the point. It has already been seen, in the TRIGONOMETRY, how lines drawn from one fixed axis parallel to another may be algebraically indicated both in length and direction. *There* the axes of reference were always two diameters of a circle, at right angles to each other; in reference to these the geometrical sines, cosines, &c., were measured. The conventions found to be necessary and sufficient for the complete representation of lines thus referred to rectangular axes, are equally necessary and sufficient to denote lines referred to oblique axes: thus, let x stand generally for the abscissa of a point, and y for its ordinate, then, if for any particular point P, $x=a$, and $y=b$, and if each of the *four* points P, P', P'', P''', have a, b, for the lengths, or numerical values, of its co-ordinates, any one of the four will be distinguished from the others, and be unambiguously indicated, as in the margin, by two equations. These are called the *equations of the point.* And it is plain that any one of these pairs of equations being given, the point algebraically represented by them—attention being paid to the signs of direction—can always be found, the fixed axes of co-ordinates being given. For instance, suppose the equations $x=5$, $y=-4$, in reference to the axes XX', YY', in the preceding diagram, were given, to find the position of the point indicated by those equations. The first equation informs us that from the *origin* of co-ordinates A we are to measure the distance 5 (inches, feet, or whatever may be the assigned unit of measure), along the axis of abscissas, and as the 5 is *plus*, it must be measured from A towards X: we thus get AB. The second equation informs us that we are to measure 4, from the origin A, along the axis of ordinates; and as the 4 is *minus*, it must be measured from A towards Y': we thus get AC': it then remains only to draw parallels to the axes through B and C': the intersection P''' of these is the point represented by the given equations. And thus it is with propriety said, that a point is given when its equations are given. Saying that its *equations* are given, is no more than saying that its co-ordinates are given; in algebraically representing the position of a point, therefore, the formality of writing down two equations may be dispensed with; it is sufficient to say that the point P is (a, b), the point P', $(-a, b)$, the point P'', $(-a,-b)$, and the point P''' $(a,-b)$, the general symbolical representation of a point being (x, y).

The equations of a point.

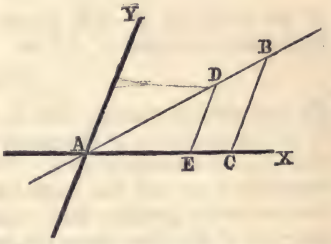
P, $x=a$, $y=b$
P', $x=-a$, $y=b$
P'', $x=-a$, $y=-b$
P''', $x=a$, $y=-b$

A glance at the signs of direction, prefixed to the co-ordinates, is sufficient to enable us to pronounce at once in which of the four compartments about the origin A the point represented is situated; thus the

point $(-2,-7)$, is in the same compartment as P'', the point $(-3, 2)$, in the same compartment as P', and so on. But it may happen that the point is not in *either* compartment; it may be upon one of the axes themselves: it may, in fact, be at the same time upon *both* axes, for it may be identical with the origin A. If it be on the axis of x (as for brevity the axis of abscissas is called), the y of the point is $y=0$, and if the x of it be a, the point is $(a, 0)$; if it be on the axis of y, and the y of it be b, the point is $(0, b)$, and if it be at the origin A, the point is $(0, 0)$.

---

341. **Equation of a Straight Line.**—As before, let AX, AY, be any fixed axes given in position: it is required to express algebraically the position, in reference to these axes, of any proposed straight line. And first, let the proposed line pass through the origin A. From any two points B, D, in the line, let parallels BC, DE, to AY, be drawn, then AC, CB, are the co-ordinates of the point B, and AE, ED, the co-ordinates of D; and since



$AC : CB :: AE : ED \therefore \frac{CB}{AC} = \frac{ED}{AE}$; that is, whatever point be taken on the proposed line, the ordinate and abscissa of the point are always in the same constant ratio, so that calling this invariable ratio a, and, as before, putting x, y, for the co-ordinates of *any* point on the line,

$$\text{we have generally } \frac{y}{x}=a, \therefore y=ax \ldots\ldots [1].$$

And this is the general equation of a straight line passing through the origin of co-ordinates. For any particular one of the innumerable straight lines comprehended in [1], the coefficient a must be given.

342. The beginner is here recommended to execute a few constructions of particular cases of the general equation [1]: let him take, for instance, the individual lines $y=2x$, $y=-3x$, $y=-4x$, &c., and proceed thus, the axes of reference being supposed given, and already drawn on the paper. Measure from the origin on the axis of x, towards the right, any arbitrary length: this length will be the abscissa of *some* point in the first line; and since by the equation the corresponding ordinate is double this, measure its double on the axis of y, above the origin A: draw the parallels through the points thus marked on the axes, and the point, whose co-ordinates are 1, 2, on the line, will be given by their intersection: then a line through this point, and through the origin, will be the line $y=2x$. Or, to show that this line *must* pass through the origin, find *two* points on the line, say the points (1, 2), and (2, 4): draw a straight line through these, and it will be found to pass through A. In like manner, to construct the second line, take x of any length, and make the corresponding y three times that length, measuring it from A downwards, and thus, as before, a point on the line will be found, and the line itself will be that through this point and the origin. Or, find *two* points, as in the last ex., and a line through both will be found to

pass through A. For the third line, the construction is similar to that for the second.

The constant ratio $a=\frac{DE}{AE}$, in the equation of a straight line, expressed in trigonometrical terms, is $\frac{\sin DAE}{\sin ADE}=\frac{\sin DAE}{\sin DAY}$; so that if the fixed angle at which the axes are inclined be called α, and the angle at which the proposed line AB is inclined to the axis of x be called α', we may write the equation of AB thus:

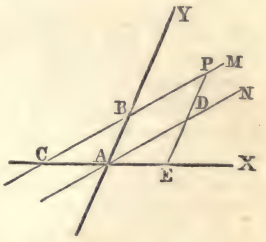
$$y=\frac{\sin \alpha'}{\sin (\alpha-\alpha')}x \ldots\ldots [2].$$

If the axes are at right angles, $\alpha-\alpha'$ will be the complement of α', so that then the equation will be

$$y=\tan \alpha' \,.\, x \ldots\ldots [3].$$

Hence, when a straight line is referred to rectangular axes, the constant a, in the equation of it, always denotes the tangent of the inclination of that line to the axis of x. This constant, whatever be the axes, it is convenient to call *the coefficient of inclination*. If the coefficient of inclination be positive, then, whether the axes be oblique or rectangular, the part of the line above the axis of x will always lie to the right of the axis of y; and if the coef. of inclination be negative, it will always lie to the left. This is only saying that if tan α' be *plus* in [3], α' will be acute, and that if it be minus, α' will be obtuse; and if $\alpha-\alpha'$ in [2], be *plus*, $\alpha>\alpha'$, and if it be *minus*, $\alpha'>\alpha$. If $\alpha=2\alpha'$, then *coef. of inc.* $=1$.

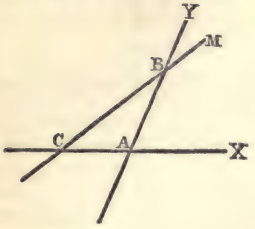
343. The general equation $y=ax$, considered above, belongs only to straight lines passing through the origin; all others are excluded: let the restriction be now removed, and let the line sought to be represented by an equation take any position whatever, as the position CM in the annexed diagram. Let a parallel to CM be drawn from the origin A, and let the ordinate PE, of any point P in the proposed line, cut it in D: then, whatever point in CM, P may be, we must always have $DP=AB$, so that to any abscissa $x=AE$, the corresponding ordinate is $y=ED+AB$; but putting, as usual, a for the coefficient of inclination, $ED=ax$, as shown above: hence, the equation $y=ax+b$, where b is put for the constant length, AB, and a for the constant ratio $\frac{BA}{AC}=\frac{DE}{AE}$, expresses the relation between the x and y of every point in the unlimited straight line CM, and is, therefore, the equation of that line; and since CM is any straight line whatever,



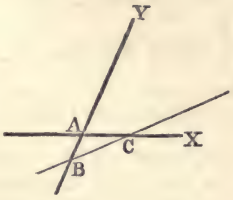
$$y=ax+b \ldots\ldots [1],$$

is the general equation of a straight line: it represents a particular straight line only when particular values are given to the constants a, b. These are called *constants*, because they never change in passing from point to point of *the same* straight line; and since x, y, *do* change their

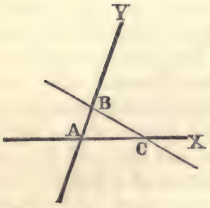
values, in passing from one point to another, these are called *variables*: the general equation of a straight line contains, therefore, two variables, and two constants, the constant b being always the ordinate of the line at the origin, or the value of y corresponding to $x=0$. Writing these constants with their proper signs, as indicated by the different positions of the line CM, in the following diagrams, the several equations of CM, or of CB, prolonged indefinitely, will be as below, each equation being written to the right of the line to which it belongs:—



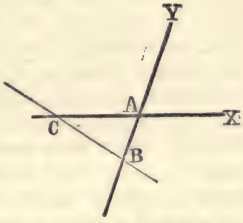
$y=ax+b$,



$y=ax-b$,

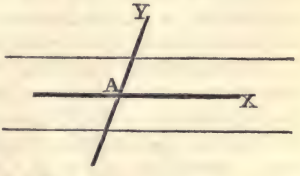


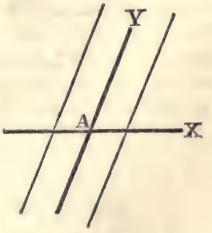
$y=-ax+b$,



$y=-ax-b$,

the coef. of inclination a being $+$ or $-$, according as a parallel from A to BC, drawn above AX, would fall to the right or to the left of AY; and b being $+$ or $-$, according as BC crosses the axis of Y above or below A. The preceding comprehend all the varieties of position of BC, except when it is parallel to one or other of the axes, as in the annexed diagrams: in the first, the equation is $y=0x\pm b$, or $y=\pm b$, according as the line is above or below AX; that is, y is constantly the same, whatever be x: in the second, the equation of the line is $x=\pm c$, that is, x is constantly the same, whatever be y.



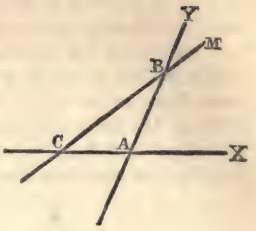


344. As every straight line may thus be represented by a simple equation, involving at most but two variables (x and y), so every simple equation with only two variables is the analytical representation of some straight line. In other words, such an equation being proposed, we can always geometrically interpret it by a straight line drawn in reference to two fixed axes. Thus, every simple equation with two variables may be put in the form $My=Nx+P$, or

$$y=\frac{N}{M}x+\frac{P}{M}, \text{ or more simply, } y=Ax+B\ldots\ldots[1].$$

Now, assuming any axes AX, AY, of which the origin is A, make $AB=B$,

and $AC=\frac{B}{A}$, attending of course to the signs of these quantities, as pointing out the directions from A in which the measures are to be taken: through B, C, draw an unlimited straight line: this will be the geometrical representation of the equation. For the equation of the line thus drawn is



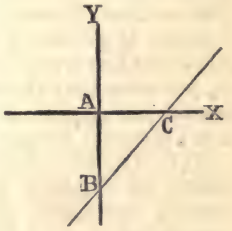
$$y=\frac{AB}{AC}x+AB\ldots\ldots[2],$$

the ratio $\frac{AB}{AC}$ being the coef. of inclination (343); but by construction, $\frac{AB}{AC}=B\div\frac{B}{A}=A$, also $AB=B$, so that the equations [1], [2], are identical; *either* therefore represents the straight line CB.

345. In the foregoing illustrations, the inclination of the axes has been taken at random; but it is best, when the inclination is not overruled by other circumstances, to choose them rectangular, for then the coef. of inclination is simply the tangent of the inclination of the line to the axis of x. In the following examples, which the learner will do well carefully to construct, the axes may always be thus chosen. All that is necessary, in order to construct the line from its equation, is to get two points in that line: those most easily found are the points where the line crosses the axes: putting $x=0$, in the equation, we get the y of the point where it crosses the axis of y, and putting $y=0$, the x of the point where it crosses the axis of x, that is, we get the lengths, AB, AC: the line through B, C, is that required.

Note.—The lengths AB, AC, intercepted between the line and the origin, are called the two *intercepts*; and the line which any equation represents, is called the *locus* of that equation, being the line in which all the points (x, y) are placed.

Ex. Required the locus of the equation $y=x-3$ in reference to the given rectangular axes in the diagram. Put $x=0$, then $y=-3$. Put $y=0$, then $x=3$: we thus have the lengths of the intercepts. Measure, therefore, below A, $AB=3$, and to the right of A, $AC=3$: then the straight line through B, C, will be the locus required. The angle C is 45°, because the coef. of inclination, which, the axes being rectangular, is the *tangent* of the inclination, is 1.



Examples for Exercise.

Construct the locus of each of the following equations:—

(1) $y=2x-1$.	(3) $y+x-1=0$.	(5) $3x-y+2=0$.	(7) $2y+6=4x-y$.
(2) $y=3x+2$.	(4) $2y-x+2=0$.	(6) $4y-6+x=0$.	(8) $6x-4y+3=0$.

Note.—If the general equation of a straight line, $y=ax+b$, to which form all these are of course reducible, be written

$$\frac{x}{A}+\frac{y}{B}=1\ldots\ldots[1],$$

which is merely putting B for b, and A for $-\frac{b}{a}$, the constants A, B, will be the intercepts on the axes of x and y respectively, as we see by putting first $y=0$, and then $x=0$, in the equation.

346. The general equation of a straight line, as we have sufficiently seen, contains two constants—these are *arbitrary constants*—unless an assigned individual straight line is to be represented: that is to say, we may put for these constants any numerical values we please, and some straight line or other will always be represented by the equation: thus, in the foregoing examples, the numerical quantities have all been taken at random. We may therefore impose any two conditions on these constants, so that, by satisfying these conditions, they may become fixed and determinate in value, and thus represent only one straight line, drawn conditionally, to the exclusion of all others. Or, we may introduce but one condition, sufficient to fix the numerical value of only one of the two constants, leaving the other still arbitrary, or open to any numerical interpretation whatever, or else sufficient to fix a numerical relation between the two. There will then be represented, not a single exclusive line, but any one of a group of lines, infinite in number, seeing that one of the constants may take any one of an infinite variety of values: but this group of lines, in virtue of the fixed constant, or the fixed relation between the two constants, will be exclusive of all other groups. We shall here give the different problems which thus suggest themselves.

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347. **The Straight Line subject to Conditions.**—The notation (x, y) means *any* point, x and y being any pair of co-ordinates: if a particular point is exclusively meant, the notation is (x', y'), or (x'', y''), &c.

Problem I. To find the equation of a straight line passing through *one* given point. Let (x', y') be the given point: then $y=ax+b$ representing any straight line indifferently, so long as a and b are arbitrary, if one of these lines is to pass through (x', y'), the equation must be satisfied for these values of x and y; that is, we must have $y'=ax'+b$, $\therefore b=y'-ax'$. A relation is thus fixed between a and b: either may be anything we please, but once chosen, the other is necessarily fixed by the above condition. Substituting this value of b in the general equation, the resulting restricted equation necessarily excludes all lines not passing through the point (x', y'). This equation is

$$y=ax+y'-ax', \text{ or, as it is usually written, } y-y'=a(x-x')\ldots[1],$$

from which we see b is eliminated; but a may take any value whatever, showing that an innumerable variety of straight lines may be drawn through the same given point. If the point were on the axis of x, then $y'=0$, and the equation would be $y=a(x-x')$: if it were on the axis of y, then $x'=0$, and the equation would be $y-y'=ax$, or $y=ax+y'$.

For ex. Suppose it were required to find the equation of a straight line passing through the point $(-2, 3)$, in reference to assigned axes. Here $y-y'=a(x-x')$ is $y-3=a(x+2)$, the required equation; and whatever straight line be drawn at random through the point $(-2, 3)$, *some* value of the arbitrary constant a exists which will render this equation the representative of that single line only; but if a straight line be drawn not passing through the given point, then no value can possibly be

assigned to a, in the above equation, that will render it the equation of that line.

PROB. II. To find the equation of the straight line which passes through two given points. Let the given points be (x', y'), and (x'', y''). All lines passing through the first of these are included in the equation $y-y'=a(x-x')$, and the particular one here required must fulfil the additional condition $y''-y'=a(x''-x')$; that is to say, for this particular line, a must have the particular value $a=\frac{y''-y'}{x''-x'}$, or $a=\frac{y'-y''}{x'-x''}$. Substituting this fixed value of a, therefore, in the equation [1], we have for

Equation of straight line through (x', y'), (x'', y''), $y-y'=\frac{y'-y''}{x'-x''}(x-x')$...[2].

Ex. Suppose the two given points are $(-2, 3)$, $(1, 4)$: then the equation of the line passing through them is $y-3=\frac{3-4}{-2-1}(x+2)$, or $y-3=\frac{1}{3}(x+2)$, or $3y-11=x$. We see by the second form of the equation, that the particular value which must be given to a in the last example, in order that, out of the group of lines passing through $(-2, 3)$, that particular one which passes also through $(1, 4)$, may be exclusively represented, is $a=\frac{1}{3}$.

If one of the given points (x', y') is on the axis of x, then $y'=0$, and the equa. is $y=\frac{y''}{x''-x'}(x-x')$, or if (x'', y'') be the point which is on the axis of x, the equa. is $y-y'=\frac{y'}{x'-x''}(x-x')$. If (x', y') be on the axis of y, then $x'=0$, and the equa. is $y-y'=\frac{y''-y'}{x''}x$; and if (x'', y'') be the point on the axis of y, the equa. is $y-y'=\frac{y'-y''}{x'}(x-x')$. If one of the points be upon both axes, that is, at the origin, then either $x'=0$, and $y'=0$, or $x''=0$, and $y''=0$, the equation being either $y=\frac{y''}{x''}x$, or $y=\frac{y'}{x'}x$, the coef. of inclination being either $\frac{y''}{x''}$, or $\frac{y'}{x'}$, these ratios being *equal* (341).

NOTE.—In the following examples the learner should always write down from memory the equa. [1], or [2], in the general symbols, and then put the particular values for the accented quantities.

EXAMPLES FOR EXERCISE.

Find the equation of a line, or of a group of lines, from the following conditions:—

(1) Through the two points $(1, -2)$, $(-3, -4)$.
(2) ,, ,, ,, $(0, 4)$, $(-5, 2)$.
(3) ,, ,, ,, $(3, 0)$, $(0, -7)$.
(4) Through the point ... $(8, -3)$.
(5) ,, ,, ... $(0, -6)$.
(6) Through the two points $(0, 0)$, $(-3, -4)$.

PROB. III. To find the equation of a straight line passing through a given point, and also parallel to a given straight line. Let (x', y') be the given point, and let a' be the coef. of inclination in the given line; then, since two straight lines must be parallel, provided they have the same coef. of inclination, the equation sought will be $y-y'=a'(x-x')$.

Ex. Find the equa. of a straight line passing through (2, −3) and parallel to the line $5y-3x+4=0$. Putting this equa. in the form

$$y=ax+b, \text{ it is } y=\frac{3}{5}x-\frac{4}{5}, \therefore \frac{3}{5} \text{ is the coef. of inclination.}$$

$$\text{Hence the required equation is } y+3=\frac{3}{5}(x-2).$$

Prob. IV. To find the point where two given straight lines intersect. Let the equations of the two lines be $y=ax+b$, and $y=a'x+b'$, the constants being of course given, and both lines referred to the same assigned axes. The symbols x, y, it must be remembered, stand for the variable co-ordinates of the several points of the line $y=ax+b$ } into whose equation they enter, so that in different lines $y=a'x+b'$ } they are differently related: the x and y of one line must, therefore, never be confounded with the x and y of another, except only at the single point in which they intersect. At *this* point, and at this only, the two equations are simultaneous, and we therefore link them together as in the margin, and find the values of x, y, common to both, as in ordinary algebra: we thus get for the point of intersection

$$x=\frac{b'-b}{a-a'}, \quad y=\frac{ab'-a'b}{a-a'}=\frac{a(b'-b)+b(a-a')}{a-a'}=a\frac{(b'-b)}{a-a'}+b;$$

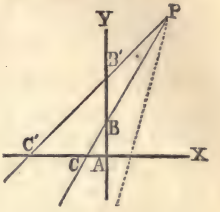
so that, having computed the x of the intersection from the first of these conditions, the y of it is got by multiplying that x by a, and adding b, as indeed we should infer from the first of the original equations. Or if the x of the point be multiplied by a', and then b' added, the y of the point will equally be obtained, as the other equa. shows. If $a=a'$, the given lines will be parallel, and the co-ordinates of their intersection will become infinite, as they ought to do. If $b=b'$, then $x=0$, and $y=b$, showing that the lines intersect on the axis of y, their equations implying that the intercept on this axis is the same for both. If $a=a'$, and $b=b'$, the x of the intersection is $\frac{0}{0}$, any value whatever (191): the lines then being identical, *every* point is a point common to both.

Ex. Required the co-ordinates of the point in which the lines $2y-3x+4=0$, $3y+x-6=0$ intersect. Here, proceeding as in common algebra, multiplying the first equa. by 3, and the second by 2, to eliminate y, and then multiplying the second by 3, to eliminate x, we find $x=2\frac{2}{11}$, $y=1\frac{3}{11}$, the co-ordinates required.

348. Since at the point where two lines intersect, the x, y in the equation of one, is the same as the x, y in the equation of the other, if we add these two equations together, or subtract one from the other, or do so after introducing any numerical factor into either, the resulting equation must be equally satisfied for the co-ordinates of the point; just as in common algebra, though we add or subtract, in like manner, two simultaneous equations, the particular values of the unknowns, x, y, equally satisfy the resulting equation. In fact, every pair of simple simultaneous equations, with two unknowns (x, y), when solved, furnishes single values for the x and y, which are no other than the co-ordinates of the point in which the lines represented by the proposed equations intersect.

Prob. V. To find the expression for the angle of intersection of two given straight lines. In all the foregoing problems the axes may be regarded as unrestricted as to their inclination: in the present problem

they will be assumed to be rectangular, because, unlike the former problems, it is not matter of indifference *here*, as regards simplicity of result, whether the axes be rectangular or oblique. Let the equations of the two given lines CP, $C'P$, referred to the rectangular axes AX, AY, be $y=ax+b$, and $y=a'x+b'$, in which a is the tan of PCX and a' the tan of $PC'X$. Now, since the angle CPC' is the difference of these two, if we call the tangent of it v, we shall have (242),



$$v=\frac{a-a'}{1+aa'}, \text{ or } v=\frac{a'-a}{1+aa'}.....[1],$$

according as this angle lies to the left or right of PC.

If a, a', are so related that $1+aa'=0$, then the tangent v is ∞, showing that the angle of intersection P is a right angle: hence the condition of perpendicularity is $aa'=-1$, or $a'=-\frac{1}{a}$, so that $y=ax+b$, being the equation of any line, $y=-\frac{1}{a}x+b'$ will be the equa. of a line perp. to it.

The perpendiculars are innumerable, since b' may have any value whatever: if one of these be fixed by the condition of passing through a given point (x', y'), then (Prob. I.) the equation of this perp. will be

$$y-y'=-\frac{1}{a}(x-x').....[2].$$

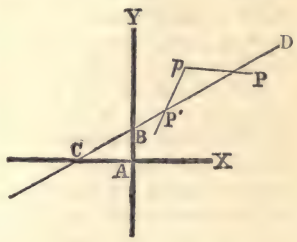
EXAMPLES FOR EXERCISE.

(1) Determine the point where the two lines $3y-x+1=0$, $y+3x-1=0$, intersect, as also the tangent of the angle of intersection.

(2) Required the equation to the straight line which passes through the point $(-5, -3)$, and is parallel to the line $2y=-3x+4$.

(3) Required the equation to the straight line which passes through the point $(-5, -3)$, and is perp. to the line $2y=-3x+4$.

PROB. VI. To find the equation of the straight line which passes through a given point, and which makes a given angle with a given straight line. Let the given line be $y=ax+b$, the tan of the given angle v, and the given point (x', y'), the axes being rectangular; then the equa. of the required line will be of the form $y-y'=a'(x-x')$ in which a' is to be so determined that $v=\pm\frac{a-a'}{1+aa'}$. (Prob. V.),



$$\therefore a'=\frac{a-v}{1+av}, \text{ or } \frac{a+v}{1-av},$$

$$\left.\begin{array}{r}\text{and the required equation either } y-y'=\frac{a-v}{1+av}(x-x'),\\ \text{or else } y-y'=\frac{a+v}{1-av}(x-x')\end{array}\right\}.....[1].$$

There are thus two lines satisfying the required conditions, one forming the assigned angle on the left of the given line, and the other forming it on the right; thus, the first of the above equations applies to the line

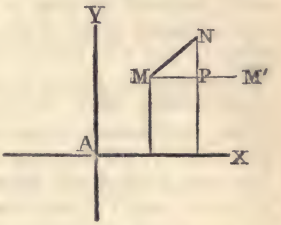
Pp (p being the given point), and the second to pP', the given line being CD. But if the given angle is a right angle, then there is only *one* straight line that will fulfil the conditions: for the two coefficients of inclination,

$\frac{a-v}{1+av}$, and $\frac{a+v}{1-av}$, may be written $\left(\frac{a}{v}-1\right)\div\left(\frac{1}{v}+a\right)$, and $\left(\frac{a}{v}+1\right)\div\left(\frac{1}{v}-a\right)$;

and if $v=\tan 90°=\infty$, these become $\frac{-1}{a}$, and $\frac{1}{-a}$, which are identical;

so that the equation of a straight line through (x', y'), and perp. to $y=ax+b$, is $y-y'=-\frac{1}{a}(x-x')$, agreeing with equa. [2] in last page.

PROB. VII. To find the expression for the distance of two given points (co-ordinates rectangular). Let the given points be $M(x', y')$, and $N(x'', y'')$, then if MP, parallel to AX, meet y'' in P, we shall have



$MP=x''-x'$, and $NP=y''-y'$, $\therefore$ the distance

$D=MN$ is $D=\sqrt{\{(x''-x')^2+(y''-y')^2\}}$;

or if one of the points (x'', y'') be at the origin, that is,

if $x''=0$, $y''-0$, then $D=\sqrt{(x'^2+y'^2)}$.

If the axes had been oblique, MN would have been

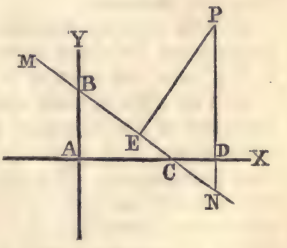
$MN=\sqrt{(MP^2+NP^2-2MP\,.\,NP\cos MPN)}$, art. (228),

or since $\cos MPN=-\cos M'PN=-\cos A$,

$\therefore D=\sqrt{\{(x''-x')^2+(y''-y')^2+2(x''-x')(y''-y')\cos A\}}$;

and if $x''=0$, $y''=0$, $D=\sqrt{(x'^2+y'^2+2x'y'\cos A)}$.

PROB. VIII. To find the expression for the distance of a point from a line (axes rectangular). Let $P(x', y')$ be the point, and $BC, y=ax+b$, the line. If PN be a parallel to the axis of y, the abscissa AD $(=x')$ will be equally that of P and N. Hence, putting x' for x in the equa. of BC, we have $DN=y=ax'+b$,



$\therefore PN=y'-ax'-b$. Now $PE=PN\cos P=\frac{PN}{\sec P}$;

but $P=180°-C$, and $\tan C=a$, $\therefore \sec^2 P=a^2+1$,

$\therefore PE=\frac{y'-ax'-b}{\sqrt{(a^2+1)}}$=Perp. ...[1].

As we know from the equation of the given line whether $\tan^{-1}a$ is acute or obtuse, that is, whether a is + or −, there cannot be any ambiguity here; for knowing whether the tangent is + or −, we know also whether the secant $\sqrt{(a^2+1)}$ is + or −. If the point be at the origin,

$$\text{the distance, or Perp.}=\frac{-b}{\sqrt{(a^2+1)}}\text{.....[2].}$$

If α be the angle of inclination of the given line, then $a=\tan\alpha$, and the expression [1] for the perp. may be written

$$P=\frac{y'-x'\tan\alpha-b}{\sec\alpha}\text{, that is,}$$

$$P=y'\cos\alpha-x'\sin\alpha-b\cos\alpha=(y'-b)\cos\alpha-x'\sin\alpha\text{.....[3].}$$

This form for the length of the perp. from a point (x', y'), upon the line $y = x \tan \alpha + b$, will be found useful hereafter. When the point is the origin (0, 0), the expression is

$$P = -b \cos \alpha \ldots\ldots [4].$$

NOTE.—As the preceding problem is to determine the absolute length of a straight line which, in general, is not in either of the co-ordinate directions, but is usually oblique to them, the sign of *direction* is an appendage which we may, in general, disregard, more especially as in our original conventions we have not expressly provided for the directive signs of lines not in the co-ordinate directions. Neither, in the TRIGONOMETRY, did we provide formally for the directive sign of the secant of an angle, yet we found that the secant had a determinate sign, necessarily claimed by it as a consequence of the sign which our conventions had expressly given to the cosine. So here, the sign which algebra supplies to any line oblique to the co-ordinates, whether we want it or not, will always be found to be a necessary consequence of the co-ordinate signs, and to be in strict consistency with the co-ordinate conventions. Thus, in reference to the expression [4] above, suppose BN to turn about the point B, till it comes into a position of parallelism with the axis of x, the perp. from A will then be AB, which of course, being in a co-ordinate direction, ought to take the sign $+$. This, however, it would not do but for the *minus* prefixed to the general expression for a perp. from A above. For the angle $\alpha = BCX$, then becomes 180°, the cosine of which is -1, and as [4] ought to apply to *every* value of the angle α, the necessity for the minus is obvious.

As many writers on this subject have felt a perplexity about the directive sign of a perpendicular from a point to a line, and have given erroneous interpretations of it, it may be well to add the following to the foregoing remarks.

1. The sign of the denom. in the expression marked [1] is never ambiguous; for that denom. is the secant of the fixed angle the line makes with the axis of x.

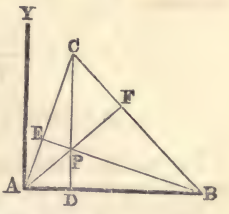
2. The sign of the numerator is $+$ or $-$, according to the relative positions of the line and point: there is no ambiguity of sign; one or other appears of necessity, as an inevitable consequence of the co-ordinate conventions. The perp. being drawn, if the line be moved parallel to itself, till the point coincides with the origin, the proper sign will be ascertained from [4] above; it will be the same sign that we should give to it if it were the *secant* of the angle which it makes with the axis of x; we thus see how improper it is to prefix the ambiguous sign to the expression [1], or to speak about *choosing* the sign, so as always to make the expression positive: there is no choice offered to us:—the sign is definitely fixed in all cases.

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349. **Problems requiring the Equation of the Straight Line.**—The following problems will show the geometrical application of what has been established respecting the straight line in the preceding articles.

PROB. I. To determine whether perpendiculars from the three vertices

to the opposite sides of a plane triangle all meet in one point. Let ABC be the triangle, and CD, AF, BE, the three perpendiculars. Assume the base AB for the axis of x, and the perp. AY for that of y. Put x', y', for the co-ordinates AD, DC, of the point C, also put x'' for AB, the abscissa of the point B, then the analytical representations of the three points A, B, C, will be $(0, 0)$, $(0, x'')$, (x', y'). And our object is to ascertain whether the abscissa of the intersection of AF, BE, is *also* the abscissa of C, that is, whether the x of this intersection is x'.



The lines concerned are

AC, passing through the origin and through the given point (x', y');

BC, through the two given points $(x'', 0)$, and (x', y');

BE, through the given point $(x'', 0)$, and perpendicular to AC;

AF, through the origin and perpendicular to BC.

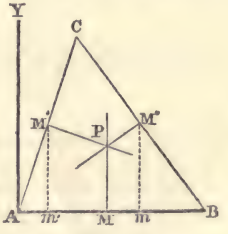
The equations of these four lines are, by the preceding articles,

Equa. of AC, $y=\frac{y'}{x'}x$; Equa. of BC, $y=\frac{y'}{x'-x''}(x-x'')$. (See p. 292).

Equa. of BE, $y=-\frac{x'}{y'}(x-x')$; Equa. of AF, $y=-\frac{x'-x''}{y'}x$. (See p. 294).

At the point where AF, BE, intersect, the ordinates must be the same for both lines: hence, at this point $\frac{x'}{y'}(x-x'')=\frac{x'-x''}{y'}x$, $\therefore x=x'$, that is, the x of the intersection is x': hence, AF, BE, intersect in CD.

PROB. II. To determine whether perpendiculars from the middle points of the sides of a plane triangle meet in a point. Taking the rectangular axes as before, let the middle points be M, M', M'', and represent the points C, B, by (x', y'), $(x'', 0)$, the point A being $(0, 0)$. Now M' being at the middle of AC, its co-ordinates are half those at the extremities A, C; and M'' being at the middle of BC, its co-ordinates are half the sums of those at the extremities B, C,



that is, the point M' is $\left(\frac{x'}{2}, \frac{y'}{2}\right)$, and the point M'' is $\left(\frac{x'+x''}{2}, \frac{}{2}\right)$.

And we have to ascertain whether the x of the intersection of perpendiculars from these points to AC, BC, respectively, is $\frac{x''}{2}$, the abscissa of M, or not.

Now (Prob. II., p. 292), Equa. of AC, $y=\frac{y'}{x'}x$. Equa. of BC, $y=\frac{y'}{x'-x''}(x-x'')$.

And [2] (p. 294), Equa. of Perp. from M', $y-\frac{y'}{2}=-\frac{x'}{y'}\left(x-\frac{x'}{2}\right)$.

,, ,, Perp. from M'', $y-\frac{y'}{2}=\frac{x''-x'}{y'}\left(x-\frac{x'+x''}{2}\right)$.

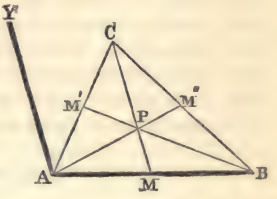
At the intersection of these perpendiculars their equations are simultaneous,

$$\therefore \frac{x'}{y'}\left(x-\frac{x'}{2}\right)=\frac{x'-x''}{y'}\left(x-\frac{x'+x''}{2}\right), \therefore 2x'x-x'^2=2x'x-2x''x-x'^2+x''^2$$

$$\therefore 2x''x=x''^2, \therefore x=\frac{x''}{2}, \text{ the abscissa of the intersection}=AM.$$

PROB. III. To determine whether straight lines from the vertices of a triangle to bisect the opposite sides meet in a point.

Let M, M', M'' be the middle points of the sides. Draw CM, and for axes take AB and AY parallel to MC. As before, let C be (x', y'), then B will be $(2x', 0)$: and as in last prob., M', M'' will be $\left(\frac{x'}{2}, \frac{y'}{2}\right), \left(x'+\frac{x'}{2}, \frac{y'}{2}\right)$. And we have to ascertain whether the line through A, and the second of these, and that through B, and the first, intersect in a point of which the abscissa is $x=x'$.



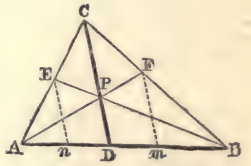
$$\text{Equa. of } AM'', y=\frac{\frac{1}{2}y'}{\frac{3}{2}x'}x. \quad \text{Equa. of } BM', y=\frac{-\frac{1}{2}y'}{2x'-\frac{1}{2}x'}(x-2x').$$

$$\text{At the intersection, these equations are simultaneous, } \therefore \frac{x}{\frac{3}{2}x'}=\frac{2x'-x}{\frac{3}{2}x'}, \therefore x=x'.$$

$$\text{The corresponding } y \text{ of the intersection is } y=\frac{\frac{1}{2}y'}{\frac{3}{2}x'}x'=\frac{1}{3}y' \text{ : hence } PM=\frac{1}{3}CM.$$

PROB. IV. To determine whether the straight lines bisecting the three angles of a plane triangle meet in a point.

Let AF, BE, bisect the two angles A, B: it is required to ascertain whether these bisectors cross CD, the bisector of the angle C, at the same point. Take DC, DB, for axes, and let AF cross DC in P, and let BE cross it, if possible, in some other point P', then (p. 290) the equations of AF, BE, are



$$AF, \frac{x}{-DA}+\frac{y}{DP}=1; \quad BE, \frac{x}{DB}+\frac{y}{DP'}=1.$$

The ordinates of the points where these cross DC are $y=DP$, and $y=DP'$, and the inquiry is—are these equal?

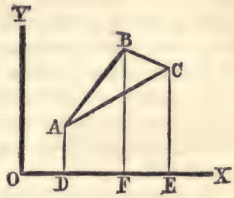
$$PD=AD\frac{\sin\frac{1}{2}A}{\sin\frac{1}{2}(A+C)}=\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}B}; \quad P'D=DB\frac{\sin\frac{1}{2}B}{\sin\frac{1}{2}(B+C)}=DB\frac{\sin\frac{1}{2}B}{\cos\frac{1}{2}A}.$$

Now (Euc. 3. VI.), $\frac{AD}{DB}=\frac{\sin B}{\sin A}$, $\therefore AD=\frac{\sin\frac{1}{2}B\cos\frac{1}{2}B}{\sin\frac{1}{2}A\cos\frac{1}{2}A}DB$,

which, substituted in the above value of PD, gives $PD=\frac{\sin\frac{1}{2}B}{\cos\frac{1}{2}A}DB=P'D$.

NOTE.—The preceding investigation can scarcely be considered as an exercise in co-ordinate geometry, for except the first two equations, which in reality are not needed, the equations of lines do not occur. The problem has been introduced merely for the sake of completing the set of problems about three lines, connected with a triangle, all intersecting in a point. It may be solved much more easily as follows: thus, disregarding the line from C, let the bisectors AF, BE cross in P: then the two angles at A being equal, perpendiculars p, p_1, from P, on AB, AC, must be equal. Again, the two angles at B being equal, perpendiculars p, p_2, from P, on BA, BC, must be equal, $\therefore p_1=p_2$. Hence the line CP must bisect the angle C.

PROB. V. To express the area of a plane triangle, in terms of the rectangular co-ordinates (in the same plane) of its three vertices.



Let ABC be the triangle, and let the rectangular co-ordinates of A, B, C, be (x_1, y_1), (x_2, y_2), (x_3, y_3). Then from the three trapezoids we have

Area ABC=($ABFD+BCEF-ACED$).

$$\therefore \text{Area}=\frac{1}{2}\{(y_2+y_1)(x_2-x_1)+(y_3+y_2)(x_3-x_2)-(y_3+y_1)(x_3-x_1)\}=$$

$$\frac{1}{2}\{y_1(x_2-x_3)+y_2(x_3-x_1)+y_3(x_1-x_2)\}.$$

If A be at the origin, then $x_1=0$, $y_1=0$, and the expression is Area$=\frac{y_2x_3-x_2y_3}{2}$.

PROB. VI. Any number of straight lines are given in length and position; it is required to determine the locus of a point P, such that wherever in that locus P be taken, and lines be drawn from it to the extremities of the given lines, the sum of the areas of the triangles thus formed may be constant.

Any rectangular axes being chosen, let the inclinations of the several given lines to the axis of x be α, α_1, α_2, &c., and their intercepts on the axis of y, b, b_1, b_2, &c.; then if from any point $P(x, y)$, in the required locus, perpendiculars be drawn to these lines, we shall have for their lengths, [3], (p. 295),

$$p=y\cos\alpha-x\sin\alpha-b\cos\alpha,\quad p_1=y\cos\alpha_1-x\sin\alpha_1-b_1\cos\alpha_1,$$

$$p_2=y\cos\alpha_2-x\sin\alpha_2-b_2\cos\alpha_2,\ \&c.$$

Hence, calling the lengths of the given semi-bases of the triangles l, l_1, l_2, &c., we have for the sum of their areas, lp, l_1p_1, l_2p_2, &c.,

$$(l\cos\alpha+l_1\cos\alpha_1+l_2\cos\alpha_2+\ldots)y-(l\sin\alpha+l_1\sin\alpha_1+l_2\sin\alpha_2+\ldots)x-$$

$$(lb\cos\alpha+l_1b_1\cos\alpha_1+l_2b_2\cos\alpha_2+\ldots)=\text{Constant},$$

which is the equation of a straight line, the locus required. It is plain that if the areas be united together by the signs + and − in any way, and the result is to be constant, the locus of the point which is the common vertex of all the triangles will still be a straight line. And the same is true if any multiples of the areas in like manner be united together, provided the result be constant.

Examples for Exercise.

(1) Determine the equation to the straight line which passes through a given point (x', y'), and makes equal angles with the given oblique axes.

(2) At what distance from the origin of the rectangular axes is the line whose equation is $2y-4x+1-0$?

(3) Find the general equation of all lines passing through the intersection of the lines $y=ax+b$, $y=a'x+b'$.

(4) Find the equation of that particular line in the last group that passes through the origin.

(5) A triangle is right angled at A, the sides are produced through A till the prolongation of the base is equal to the perpendicular, and the prolongation of the perp. equal to the base; the parallelogram, whose sides are these prolongations, is then completed. Prove that the line through A and the opposite vertex of this parallelogram is perp. to the hypotenuse.

(6) A line is drawn parallel to the base of a triangle, and from its extremities lines are drawn, crossing each other in P, to the extremities of the base. Prove that the locus of P is a straight line from the vertex to the middle of the base.

(7) Squares are described on the three sides of any triangle ABC, and outside the figure three lines are drawn connecting the corners of these squares. Prove that if perpendiculars to the three lines be drawn from A, B, C, they will all pass through the same point.

(8) Squares are described on the base and perpendicular of the triangle ABC, right-angled at A. From the vertices B, C, lines crossing each other are drawn to the opposite corners of the squares. Prove that these intersect on the perp. to the hypotenuse from A.

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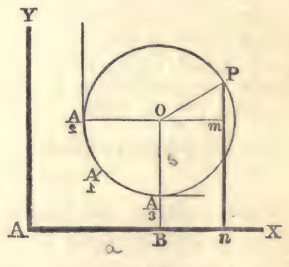
350. **The Circle—Rectangular Axes.**—Let r be the radius OP of the circle whose centre is O, and let AX, AY, be any rectangular axes in its plane. It is required to determine the equation in x and y, which shall be satisfied for every point $P(x, y)$, in the circumference, and for such points only.

Let the co-ordinates AB, BO, of the centre be a, b, and those of any point P in the circumference, $An=x$, $nP=y$; then drawing Om parallel to AX, we have $Om=x-a$, $Pm=y-b$, and therefore since

$$Om^2+Pm^2=OP^2,\ (x-a)^2+(y-b)^2=r^2,$$

$$\text{or } x^2+y^2-2(ax+by)+a^2+b^2=r^2\ldots\ldots[1],$$

which is the most general equation of the circle in reference to rectangular axes, and from which the following simpler forms are deduced:—



1. If the origin be at any point A_1 of the circumference, then $a^2+b^2=r^2$, and the equation becomes

$$x^2+y^2-2(ax+by)=0\ldots\ldots[2].$$

If the origin be at A_2, so that the axis of x passes through the centre, then $a=r$ and $b=0$, and the equa. is

$$x^2+y^2-2rx=0.....[3].$$

But if it be at A_3, so that the axis of y passes through the centre, then since $b=r$, and $a=0$, the equa. is

$$x^2+y^2-2ry=0.....[4].$$

2. If the origin be at the centre O, then since in this case $a=0$, and $b=0$, the equation is simply

$$x^2+y^2=r^2.....[5],$$

which is the form most frequently employed.

If in the general equation [1], we put $x=0$, we shall have for the corresponding values of y, that is, for the ordinates of the two points in which AY intersects the circumference (whenever the circumference *is* cut by that axis) $y=b\pm\sqrt{(r^2-a^2)}$. If $a=r$, these two values unite in one, namely, $y=b$, for the axis of y then touches without cutting the circle. If $a<r$, the two values are then imaginary: the axis of y being wholly without the circle.

351.—**Oblique Axes.**—If the angle A were oblique instead of right, the equation [1] would have been more complicated; for we should then have had (p. 295),

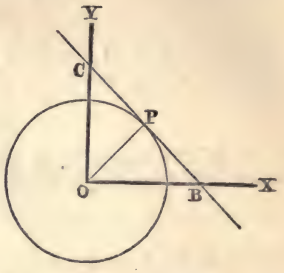
$$(x-a)^2+(y-b)^2+2(x-a)(y-b)\cos A=r^2.....[6],$$

and the other forms would have been similarly modified, the simplest, namely, that in which the oblique axes originate at the centre, being

$$x^2+y^2+2xy\cos A=r^2.....[7].$$

It will of course be understood that in the general equations [1], [6], the origin A is anywhere in the plane of the circle:—either without the circle or within it. And in the latter of these, since the angle A is equally unrestricted, we may deduce from it at once the property of the circle in Euclid 36, 37, Book III.: thus, let A be within the circle, then if in [6] we put $x=0$, and solve the resulting equation, the two values of y will be the two segments of Y within the circle, measured from A, and if in the same equation we put $y=0$, the two resulting values of x will be the two segments of X, within the circle measured from the same point A. In the former case the product of the two segments, or of the two roots of the quadratic, after putting $x=0$, in [6] is the term in that quadratic (r^2 being transposed) which is independent of y (68): hence when r^2 is transposed we get this product at once by putting in [6] first $x=0$, and then $y=0$. In like manner we get the product of the other two segments by putting first $y=0$, and then $x=0$, in [6], which product is of course the same as the former: hence if two chords of a circle intersect, the product of the segments of the one is equal to the product of the segments of the other. Similarly, if A be without the circle, the product of the segments of X between A and the circumference is equal to the product of the segments of Y between A and the circumference: when X or Y *touches* the circle, the two segments are equal, so that the case is then that of the 37th of B. III.

352. **Equation of Tangent at a Point.**—Taking the origin of the rectangular axes at the centre, the equation of OP, through the point P, (x', y'), is $y=\frac{y'}{x'}x$, and since to this line the tangent AB, through (x', y'), is perpendicular, its equation must be (p. 294),



$$y-y'=-\frac{x'}{y'}(x-x'), \text{ or } yy'+xx'=y'^2+x'^2,$$

that is, $yy'+xx'=r^2$.

This is the simplest way of deducing the equation of the tangent, availing ourselves of the geometrical property that the tangent is a line perp. to the radius drawn to the point of contact. The circle, however, is the only curve, having a centre, to which this property belongs, we shall therefore give another mode of investigation that is applicable generally. As before, let (x', y') be the point at which the tangent is to be applied, and first let a secant be drawn through (x', y') cutting the curve in some second point (x'', y'').

The equation of this secant is $y-y'=\frac{y'-y''}{x'-x''}(x-x')$.....[1],

and, for both points of intersection, we have $x'^2+y'^2=r^2$, $x''^2+y''^2=r^2$,

$\therefore$ by subtraction, $y'^2-y''^2=-(x'^2-x''^2)$ $\therefore \frac{y'-y''}{x'-x''}=-\frac{x'+x''}{y'+y''}$.

Consequently equa. [1] is $y-y'=-\frac{x'+x''}{y'+y''}(x-x')$.....[2].

Now imagine the cutting line to turn about (x', y') till (x'', y'') approaching (x', y') at length comes into coincidence with it, then the secant will become a tangent at (x', y'), and we shall have $x'=x''$, and $y'=y''$; $\therefore$ substituting these values in [2] we have for the equa. of the tangent $y-y'=-\frac{x'}{y'}(x-x')$ as before found. And from this we may infer, since the equation of a line through the point (x', y') from the origin is $y=\frac{y'}{x'}x$, that the tangent through any point of a circle is perp. to the radius at that point.

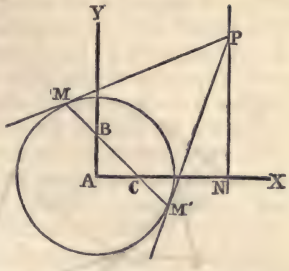
353. **Equation of Tangent from a Point.**—We have now to find the equation of a tangent from a point $P(a, b)$ without the circle. Put (x', y') for the point of contact, then

$$x'^2+y'^2=r^2.....[1],$$

and the equation of the tangent being $xx'+yy'=r^2$, on which (a, b) is a point,

$$\therefore ax'+by'=r^2.....[2].$$

In the equations [1], [2], x', y' are the only unknown quantities, these therefore may be determined from those equations, and since [1] is a quadratic, there will be two values for x' and two for y': hence *two* tangents may be drawn from $P(a, b)$ to the circle. Now without thus solving the pair of simultaneous equations referred to, we may determine the two points by a simple construction: for since equa. [2] must be satisfied by both values of x' and both values of y', it follows that the two points of contact are both on the straight line $ax+by=r^2$, which is therefore called the *equation of the chord of contact:* if this line be constructed, it must cut the circle in the two points required. Proceeding,



for this purpose, in the usual way, we have for $y=0$, $x=AC=\frac{r^2}{a}$, and for $x=0$, $y=AB=\frac{r^2}{b}$: then the line MM', drawn through C and B, will be the chord of contact, M, M' being the two points where the tangents from P touch the circle.

It is worthy of notice that since $AC=\frac{r^2}{a}$, an expression that remains the same for the same value of a, however b or PN may change, it follows that wherever on the line PN, P be taken, and tangents from P drawn, the several chords of contact will all pass through the same fixed point C: hence the following geometrical property of the circle, namely:—*If from each of any number of points in a straight line, pairs of tangents be drawn to a circle, the chords joining the points of contact of each pair will all intersect in the same point.* And it is a peculiarity of analytical geometry that unsought-for properties often spontaneously offer themselves to our notice in this way.

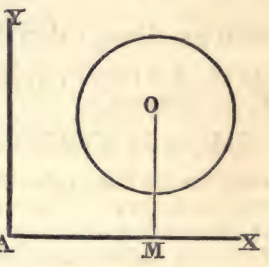
354. In the above diagram the proposed line PN lies wholly without the circle, and the intersection of the chords of contact is, in this case, always within the circle, since then $AC=\frac{r^2}{a}<r$; but if, on the contrary, PN cut the circle, then $\frac{r^2}{a}>r$, and the point of intersection is therefore without the circle. In both cases this point is called the *pole* of the line PN, and the line itself the *polar* of the point. It has been shown above, that wherever a point $P(a, b)$, without the circle, be taken, the equation $ax+by=r^2$ is the equation of the chord of contact in reference to tangents drawn from that point, provided the axes be any pair of rectangular axes whatever, originating at the centre of the circle. In like manner, P_1, P_2, P_3, &c., being a series of points in the same straight line as P, the annexed equations all have place simultaneously, since the lines they represent all pass through one point (α, β), the pole of the line of which $P\,P_1 \ldots P_n$ is the polar. Hence the equation $a\alpha+b\beta=r^2$ holds for every point (a, b) on this polar: in other words $\alpha x+\beta y=r^2$ is the equation of the polar of which (α, β) is the pole. Every

$$\begin{aligned} a\,x+b\,y&=r^2\\ a_1x+b_1y&=r^2\\ a_2x+b_2y&=r^2\\ &\vdots\\ a_nx+b_ny&=r^2 \end{aligned}$$

equation therefore of this form, the axes being rectangular, represents the polar of which (α, β) is the pole:—the chord of contact above, namely, $ax+by=r^2$, is thus the polar of which (a, b) is the pole. And generally, if any point (a, b) be on the polar of another point (α, β), then will the point (α, β) be on the polar of the first point (a, b).

[For a very masterly discussion of poles and polars, the inquiring student is referred to the writings of the late Professor Davies, of Woolwich, as contained in the second volume of his "Course of Mathematics," and in his various papers in "The Mathematician," and in the "Lady's and Gentleman's Diaries," 1850–1. The elaborate researches of this distinguished Geometer, on these and kindred topics, in the writings referred to, and the two volumes of Mr. Salmon on Conic Sections, and the Higher Curves, contain a body of valuable information on the subjects of the present short treatise, which is unequalled in the English language].*

355. **Locus of the Equation** $x^2+y^2+Ax+By+C=0$. We have seen that the general equation of a circle when referred to rectangular axes is of this form: it remains to be shown, conversely, that the locus of every equation of the above form is a circle referred to rectangular axes. Let AX, AY be any rectangular axes, and determine the point O of which the abscissa AM is $-\frac{A}{2}$, and the ordinate MO, $-\frac{B}{2}$. From O as a centre, and with radius $r=\sqrt{\left\{\frac{A^2+B^2}{4}-C\right\}}$, describe a circle: the circumference will be the locus of the proposed equation. For the equation of the circle just constructed is $(x-a)^2+(y-b)^2=r^2$. which, since



$$a=-\frac{A}{2},\ b=-\frac{B}{2},\ \text{and}\ r^2=\frac{A^2+B^2}{4}-C,\ \text{is the same as}$$

$$\left(x+\frac{A}{2}\right)^2+\left(y+\frac{B}{2}\right)^2=\frac{A^2+B^2}{4}-C\ldots\ldots[1],$$

and this when developed, is $x^2+y^2+Ax+By+C=0$: hence the circle, constructed as above, is the locus of the proposed equation of the second degree. The student must pay particular attention to the peculiar form of this equation: he is to observe that the squares of the variables enter with *equal* coefficients:—these may be each made unit by division: he is also to observe that the *product* of the variables is absent. With these features in any equation of the second degree containing two variables, he may always pronounce that equation to be the analytical representation of a circle referred to rectangular co-ordinates; but not when the equation is of any other form.

If the general equation marked [6] at page 301, be developed, its form will be $x^2+y^2+2xy\cos\omega+Ax+By+C=0$, where ω is the *oblique* angle

* The value of Professor Davies's volume, above referred to, is greatly enhanced by the beautifully-executed diagrams, all drawn by Mrs. Davies, now Principal of the Ladies' College, Southampton.

at which the axes are inclined; and it may be shown, as above, that every equation coming under this form represents a circle constructed in reference to axes of which the angle of inclination is ω. Here we see that the product of the variables *does* enter, but that the coef. of that product must never exceed 2, because cos ω can never exceed 1. The coefficients A, B, C, in the above equation, are, for brevity, put for the following expressions, involving constants only, namely,

$$A=-2(a+b\cos\omega),\ B=-2(b+a\cos\omega),\ C=a^2+b^2+2ab\cos\omega-r^2,$$

and since cos ω is given, from the equation, it being half the coef. of xy, these three conditions suffice for the determination of the centre (a, b) of the circle, and of its radius r.

When we say, as above, that every equation of the second degree which comes under a certain specified form represents a circle, we mean that it has no geometrical representation but a circle. The coefficients of the proposed equation may be so related that the equation may admit of no geometrical representation at all, in which case we should infer that the existence of what the equation may have been supposed to represent is impossible. Thus, in constructing the locus of $x^2+y^2+Ax+By+C=0$, we have seen that the radius of the circle conceived to be represented by it is $r=\sqrt{\left(\frac{A^2+B^2}{4}-C\right)}$. Now it may happen that $\frac{A^2+B^2}{4}=C$, and thence that $r=0$: the inference in such a case would be that what we had expected to be a circle turns out to be only a single point, the point (a, b) which would have been the centre if the circle had had any finite radius: for [1], $\left(x+\frac{A}{2}\right)^2+\left(y+\frac{B}{2}\right)^2=0$, and two squares cannot give 0 for their sum, unless each separately is 0, $\therefore\ x=-\frac{A}{2}$, $y=-\frac{B}{2}$, the co-ordinates of the centre. In the case here supposed, therefore, the equation does not imply any geometrical absurdity, it merely denotes not a circle but a single point. If, however, the relation among the coefficients be such that $\frac{A^2+B^2}{4}<C$, then r is imaginary, and there is no locus: and put whatever real value we may for x in the equation, the resulting value of y will always be imaginary. We shall illustrate what is here said by an example or two.

(1) To construct the locus of $2x^2+2y^2-3x+4y-1=0$, or of $x^2+y^2-\frac{3}{2}x+2y-\frac{1}{2}=0$. This is the form for a circle referred to rectangular co-ordinates, the centre (a, b), being $\left(\frac{3}{4}, -1\right)$, and the radius $r=\sqrt{\left\{\left(\frac{3}{4}\right)^2+1^2+\frac{1}{2}\right\}}$; and if with this given centre, and this given radius, a circle be described, it will be the locus of the proposed equation.

(2) Required the locus of $x^2+y^2-3x-2y+\frac{13}{4}=0$. This also has the prescribed form, the centre of the circle being $\left(\frac{3}{2}, 1\right)$; but as its radius is

$$r=\sqrt{\left\{\left(\frac{3}{2}\right)^2+1^2-\frac{13}{4}\right\}}=0,$$

the point $\left(\frac{3}{2}, 1\right)$ is the only point whose co-ordinates satisfy the proposed equation.

(3) Let the equation be $x^2+y^2-3x+2y+6=0$; then since $\frac{3^2+2^2}{4}-6$ is *negative*, we at once infer that no line or point can be represented by the equation: no real values of x, y can satisfy its conditions.

Suppose we take the equation and seek to obtain y in terms of x, thus:

$$y^2+2y=3x-x^2-6, \quad \therefore y^2+2y+1=3x-x^2-5,$$
$$\therefore y=-1\pm\sqrt{(3x-x^2-5)}=-1\pm\sqrt{\{-(x^2-3x+5)\}},$$

in which we see that the quantity under the radical is always *negative*, whatever real value be put for x (p. 107).

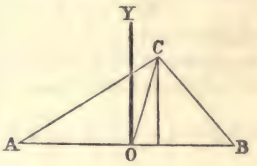
It will be observed that for the locus to be imaginary, as here, the absolute term of the equation must be *positive*.

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356. **Problems requiring the Equation of the Circle.**— We shall now give a few problems on circular loci.

PROB. I. Given the base, and the sum of the squares of the sides of a plane triangle, to determine the locus of the vertex.

Let AB be the given base, and put $AC^2+BC^2=m^2$. Let O, the middle of the base, be the origin of the rectangular axes OX, OY, and put a for OB, the semi-base; then (x, y) being any point C in the locus, we have $y^2+(a-x)^2=BC^2$, and $y^2+(a+x)^2=AC^2$. The sum of these is

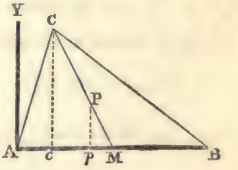


$$2y^2+2x^2+2a^2=m^2 \ldots\ldots [1],$$

which is the equation of a circle of which O is the centre, and the radius $\sqrt{\left(\frac{m^2}{2}-a^2\right)}$. If, therefore, with this radius and centre O, a circle be described, and lines CA, CB, be drawn from any point C in its circumference to the extremities A, B, of the given base, the triangle thus formed will always have the sum of the squares of its sides the same constant quantity. Since $2(y^2+x^2)=OC^2$, this constant sum, as shown by [1], is equal to twice the square of half the base, and twice the square of the line from the vertex to the middle of the base.

PROB. II. Given the base and vertical angle of a triangle, to find the locus of the intersection of straight lines drawn from the vertices to bisect the opposite sides.

Let AB be the fixed base, and C one position of the opposite vertex, and let CM bisect the base, P being a point in the required locus: then we know (p. 298) that $PM=\frac{1}{3}CM$: hence, taking the rectangular axes AB, AY, calling the co-ordinates of the variable point C, X, Y, and those of P, x, y, the given base being a, we shall have $Y=3y$, and



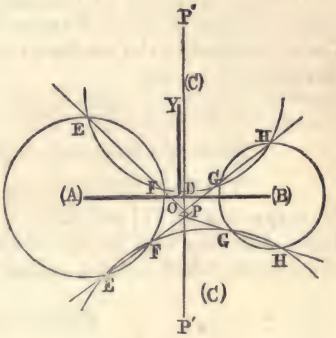
since $Mc=3Mp$, or $=\frac{1}{2}a-X=3\left(\frac{1}{2}a-x\right)$, $X=3x-a$. Now the locus of (X, Y) is the arc of a circle passing through A, B, and containing

the given angle: hence X, Y are connected together by the equation $X^2+Y^2-2(\alpha X+\beta Y)=0$, (α, β) being the centre of the circle (350). Substituting the above values for X and Y, we find that x, y are connected together by the equation $(3x-a)^2+(3y)^2-2(3\alpha x-\alpha a+3\beta y)=0$, or $9x^2+9y^2-6(a+\alpha)x-6\beta y+a^2+2\alpha a=0$, which is the equation of a circle. This problem is given chiefly for the sake of the principle illustrated in the solution of it: we see that when two loci are mutually dependent, how one may always be deduced from the other. The next problem, too, involves another principle of importance.

PROB. III. Two given circles (A), (B) are cut arbitrarily by a third circle (C), and the chords EF, HG, in (A), (B), joining the points of intersection, are prolonged till they meet in P: to prove that the locus of P is a straight line.

Let O, the point bisecting the distance between the two centres A, B, be the origin of the rectangular coordinates, the axis of x coinciding with OB. Put $OB=a$, and let r_1, r_2, be the radii of the two fixed circles and r that of the variable one, (α, β) being the centre: then we have for the equations of the three circles

$$(A)\ldots y^2+x^2+2ax=r_1^2-a^2,$$
$$(B)\ldots y^2+x^2-2ax=r_2^2-a^2,$$
$$(C)\ldots y^2+x^2-2\alpha x-2\beta y=r^2-\alpha^2-\beta^2.$$



Now at the points where either of the two fixed circles are cut by the third circle, the corresponding pair of equations are simultaneous, that is, they are each satisfied for the same values (and for no others) of x and y, which values equally satisfy every equation deduced from them by combining the two original equations together. Subtracting, then, (C) from (A) and (B), the resulting equations must be the equations of the lines EF, HG, that is,

$$EF\ldots\quad 2ax+2\alpha x+2\beta y=(r_1^2-a^2)-(r^2-\alpha^2-\beta^2)\ldots\ldots[1],$$
$$HG\ldots -2ax+2\alpha x+2\beta y=(r_2^2-a^2)-(r^2-\alpha^2-\beta^2)\ldots\ldots[2],$$

inasmuch as the first is satisfied for the two points E, F, and the second for the two points H, G; and, of course, every locus passes through all the points which satisfy its equation. These equations, again, exist simultaneously for the x, y of the point P, in which they intersect: hence, subtracting the lower equa. from the upper, the result $4ax=r_1^2-r_2^2$ has place for every intersection P of the variable lines EF, HG: the locus of P is therefore a straight line PP' perpendicular to OB, since the abscissa of every point in it is constant, namely,

$$x=\frac{r_1^2-r_2^2}{4a}=OD\ldots\ldots[3].$$

The fixed line PP' thus determined, is what is called the *radical axis* of the two fixed circles: its equation is found by simply subtracting the equa. of one circle from that of the other: for, subtracting (B) from (A), the result is $4ax=r_1^2-r_2^2$.

The principle adverted to above, and which this investigation illustrates, is the following, namely·—

If two loci, whether fixed or variable, intersect, the sum or difference of their equations (either multiplied by a constant factor or not) will be a third equation representing a single line straight or curved, or a group of lines, passing through the intersections. Thus the equation [1], or the equation [2], resulting from the combination of a fixed locus with a variable one, represents a group of straight lines, upon one or other of which all the intersections of the original loci lie, for α, β, are constant only for a single one of these lines. Again, the equation [3], resulting from the combination of [1] and [2], represents the line upon which all the intersections of the variable loci [1] and [2] lie, and it is found to be a straight line.

PROB. IV. To prove that the radical axes of three circles, taken two and two, all meet in the same point.

Let the equations of the three circles be as annexed. Then the radical axes of (r_1, r_2), (r_2, r_3), and (r_1, r_3)—denoting the circles by their radii—by what is shown above, are

$$(y-y_1)^2+(x-x_1)^2=r_1^2$$
$$(y-y_2)^2+(x-x_2)^2=r_2^2$$
$$(y-y_3)^2+(x-x_3)^2=r_3^2$$

$$2y(y_1-y_2)+2x(x_1-x_2)=(x_1^2+y_1^2)-(x_2^2+y_2^2)-(r_1^2-r_2^2)\ldots\ldots[1],$$
$$2y(y_2-y_3)+2x(x_2-x_3)=(x_2^2+y_2^2)-(x_3^2+y_3^2)-(r_2^2-r_3^2)\ldots\ldots[2],$$
$$2y(y_3-y_1)+2x(x_3-x_1)=(x_3^2+y_3^2)-(x_1^2+y_1^2)-(r_3^2-r_1^2)\ldots\ldots[3].$$

Now the sum of [1] and [2] is a line passing through the intersection of both: but this sum is [3]; hence the three radical axes all meet in the same point, which point is called the *radical centre* of the three circles.

EXAMPLES FOR EXERCISE.

(1) From a given point in the circumference of a circle chords are drawn: required the locus of their middle points.

(2) Required the equation to the straight line drawn from the point (1, 1) to touch the circle $3x^2+3y^2=4$.

(3) Prove that the locus of a point P, from which the tangents to two fixed circles are equal, is the radical axis of those circles.

(4) From a point A, without a given circle, two lines are drawn to cut it, and the points of intersection joined, so that an inscribed quadrilateral may be formed: produce these joining lines to meet in A', and draw the diagonals of the quadrilateral; then, if through the intersection of these diagonals a line be drawn from A', it will cut the circle in two points, to which lines drawn from A will be tangents.

(5) Prove that the loci of the vertices of parallelograms of constant area circumscribing a circle are also circles.

(6) Two fires, of which the intensities are as m to 1, are at a given distance apart; it is required to determine the path which a person must pursue in order that he may always be equally heated by both fires. [The heat received, at different distances from the source of it, varies inversely as the square of the distance.]

(7) If there be n given points (x_1, y_1), (x_2, y_2), $\ldots(x_n, y_n)$, and straight lines from each to a point P be such that the sum of the squares of all of them is a constant quantity c^2, prove that the locus of P is a circle whose equation is

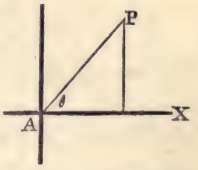
$$y^2+x^2-\frac{2}{3}(x_1+x_2+\ldots+x_n)x-\frac{2}{3}(y_1+y_2+\ldots+y_n)y=\frac{1}{3}\{c^2-(x_1^2+y_1^2+\ldots+x_n^2+y_n^2)\}.$$

(8) Given the equation of a circle $x^2+y^2+ax+by+c=0$, and that of a straight line $y=a_1x+b_1$: what relation must exist between a_1 and b_1 in order that the line may touch the circle?

(9) If from any point in the radical axis of two circles secants be drawn to each of the circles, the rectangle of the segments of the one will be equal to the rectangle of the segments of the other: required the proof.

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357. **Polar Co-ordinates.**—Hitherto a point has been algebraically represented by its two linear co-ordinates, or lines drawn from it parallel to and terminating in two rectilinear axes. But its position would be marked equally well by one fixed axis only, provided the distance of the point from the origin of that axis, and the inclination of this distance to it be given. The distance of a point P from the origin A of the fixed axis AX is called the *radius vector* of P, and the angle θ which AP makes with AX, is the angular distance of AP from AX. Calling AP, the radius vector, r, the point P, in reference to polar co-ordinates, is (r, θ).



358. **Change of Rectangular for Polar Co-ordinates.**—For the purpose of passing from one of these systems of co-ordinates to the other, we have the following obvious conditions, namely,

$$x=r\cos\theta,\ y=r\sin\theta,\ \therefore \frac{y}{x}=\tan\theta,\ x^2+y^2=r^2.$$

Take, for instance, the general equation of a straight line,

$$Ax+By+C=0.$$

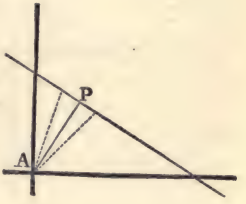
In polar co-ordinates it is

$$Ar\cos\theta+Br\sin\theta+C=0\ldots\ldots[1].$$

If p represent the perp. AP to the line from the origin, and θ' be the inclination of it to the fixed axis, the inclination of any other radius vector being θ, we shall have

$$r=p\sec(\theta\sim\theta')\ldots\ldots[2],$$

as is sufficiently obvious from the diagram, where the dotted lines are two radii vectores (r), one on each side of the perp. (p).



If a perp. from any point (x, y) of the proposed line, to a parallel to that line, through the origin, be drawn, it will of course be equal *in length* to AP, and (p. 295), the expression for it will be $y\cos\alpha-x\sin\alpha$. But since p, drawn from A to P, is opposite in direction, we have

$$p=x\sin\alpha-y\cos\alpha,\ \text{and since}\ \alpha=\theta'+90°,\ \therefore\ \sin\alpha=\cos\theta',\ \cos\alpha=-\sin\theta',$$

$$\therefore\ p=x\cos\theta'+y\sin\theta'\ldots\ldots[3].$$

The polar equation of a circle, the pole or origin being at any point, is easily written down, the centre (r_1, θ_1), and the radius R of the circle being given; for two radii vectores being drawn, one to the centre C and the other to any point P in the curve, the constant R^2 will be given by the equation (228),

$$r^2+r_1^2-2r_1r\cos(\theta\sim\theta_1)=R^2\ldots\ldots[4].$$

If the pole be on the curve, and the diameter from it be the fixed axis, then $r_1=R$, and $\theta_1=0$, and the equa. is $r=2R\cos\theta\ldots[5]$, and if the pole coincide with the centre, $r_1=0$, and the equation is $r=R$.

The two following examples will illustrate the application of some of the preceding equations. [NOTE.—Cos $(\theta-\theta')$ is the same as cos $(\theta\sim\theta')$.]

(1) To find the polar equation of a line passing through the two given points (r_1, θ_1), (r_2, θ_2). By equa. [2] we have for any point in the line, $p=r\cos(\theta\sim\theta)$, and $\therefore$ for the two given points,

$$p=r_1\cos(\theta_1\sim\theta'),\ p=r_2\cos(\theta_2\sim\theta').$$

Eliminating p from the first and second, and then from the second and third, we have

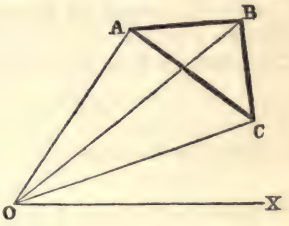
$$r\cos\theta-r_1\cos\theta_1+(r\sin\theta-r_1\sin\theta_1)\tan\theta'=0,$$

$$r\cos\theta-r_2\cos\theta_2+(r\sin\theta-r_2\sin\theta_2)\tan\theta'=0.$$

And now eliminating tan θ', there results for the equation sought

$$\frac{\sin(\theta-\theta_1)}{r_2}-\frac{\sin(\theta_2-\theta_1)}{r}-\frac{\sin(\theta-\theta_2)}{r_1}=0,\ \text{or}\ r\left\{\frac{\sin(\theta-\theta_1)}{r_2\sin(\theta_2-\theta_1)}-\frac{\sin(\theta-\theta_2)}{r_1\sin(\theta_2-\theta_1)}\right\}=1.$$

(2) To express the area of a plane triangle in terms of the polar co-ordinates of its three vertices.



Let the vertices A, B, C, be

$$(r_1, \theta_1), (r_2, \theta_2), (r_3, \theta_3),$$

the pole being at O, and OX the fixed axis, then the areas of the three triangles AOB, BOC, COA, are respectively as follows (308):

$$\frac{1}{2}AO\ .\ OB\sin AOA=\frac{1}{2}r_1r_2\sin(\theta_1-\theta_2),\ \frac{1}{2}BO\ .\ OC\sin BOC=\frac{1}{2}r_2r_3\sin(\theta_2-\theta_3),$$

$$\text{and}\ \frac{1}{2}CO\ .\ OA\sin COA=-\frac{1}{2}r_1r_3\sin(\theta_3-\theta_1),$$

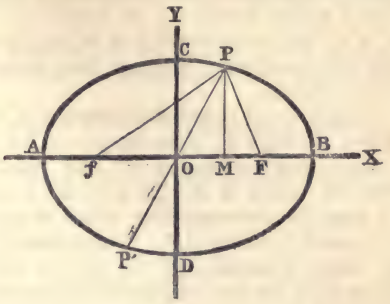
$$\therefore ABC=AOB+BOC-COA=\frac{1}{2}\{r_1r_2\sin(\theta_1-\theta_2)+r_2r_3\sin(\theta_2-\theta_3)+r_1r_3\sin(\theta_3-\theta_1)\}.$$

If the points A, B, C, are in the same straight line, the area is 0; hence the expression within the braces, when equated to 0, will be the equation of condition that the points (r_1, θ_1), (r_2, θ_2), (r_3, θ_3), may all lie in the same straight line.

ANALYTICAL GEOMETRY: PART II. THE CONIC SECTIONS.

359. Besides the circle there are three other curves—and three only—which may be represented by an equation of the second degree between two variables (x, y). These are called the *three Conic Sections*, because, by cutting a cone by a plane, in different directions, these three plane curves—the Ellipse, the Hyperbola, and the Parabola, as they are respectively called—will be the sections. They are, however, here to be discussed, independently of reference to the cone.

360. **The Ellipse.**—This is a curve, such that if from any point P in it straight lines be drawn to two certain fixed points F, f, called *foci*, the sum of the two lines will be always the same; hence, if to the two foci F, f, the extremities of a cord be fastened, and the cord be stretched into a loop by a pencil P, the motion of this pencil, thus confined by the stretched cord, will trace out an ellipse, seeing that the sum of the two *focal distances*, namely, $PF+Pf$, is always the same.



361. **Equation of the Ellipse.**—Take O the middle of F, f, for the origin of the rectangular axes OX, OY; put also $PF+Pf=2a$, and OF or $Of=c$, then representing, as usual, P by (x, y), we have the following conditions:—

$2a=PF+Pf$.....[1], $y^2+(x\sim c)^2=PF$.....[2], $y^2+(x+c)^2=Pf^2$.....[3].

From [2], [3], by adding and subtracting,

$$2y^2+2x^2+2c^2=PF^2+Pf^2\text{.....[4]},$$

$$4cx=Pf^2-PF^2,\ \therefore \text{ dividing by [1]},\ \frac{2cx}{a}=Pf-PF,\ \therefore [1],$$

$$Pf=a+\frac{cx}{a},\ PF=a-\frac{cx}{a},\ \therefore [4],\ y^2+x^2+c^2=a^2+\frac{c^2}{a^2}x^2,$$

$$\therefore a^2y^2+(a^2-c^2)x^2=a^2(a^2-c^2),\text{ the equation of the curve.}$$

To find the points B, C, where it cuts the axes, put first $y=0$, and then $x=0$ in this equation, and we have $OB=\pm a$, and $OC=\pm\sqrt{(a^2-c^2)}=\pm b$, showing that the curve cuts the axis of x at two points B, A, equidistant from O, and the axis of y at two points C, D, also equidistant from O. Since $b^2=OC^2=a^2-c^2$, we have, by substitution in the above equation,

$$a^2y^2+b^2x^2=a^2b^2,\text{ or } y^2=\frac{b^2}{a^2}(a^2-x^2)\text{.....[A]};$$

and this is the most simple form of the equation of the ellipse. If $b=a$, that is, if $OC=OB$, it becomes that of a circle of radius a, the origin being at the centre O. The circle, therefore, is only a particular case of the ellipse, the two foci F, f, then uniting in the centre O, for when $b=a$, $c^2=a^2-b^2=0$. From the general equation [A] we have

$$y=\pm\frac{b}{a}\sqrt{(a^2-x^2)},\text{ and } x=\pm\frac{a}{b}\sqrt{(b^2-y^2)}\text{.....[B]};$$

hence, for the same value of x there are two equal and opposite values for y; in like manner for the same value of y there are two equal and opposite values for x; the chord AB, therefore, bisects all the chords drawn parallel to CD, and the chord CD bisects all those drawn parallel to AB. If we put $x=a$, or $x=-a$, the corresponding value of y is 0; hence parallels to CD drawn through B and A never meet the curve again, that is, they are tangents at B and A. In like manner for $y=\pm b$, the corresponding values of x are each 0, therefore parallels to AB through

C, D, are tangents to the curve at those points. The curve is entirely comprehended within the rectangle formed by these parallels, for if x be taken either positively or negatively to exceed OB or OA, that is, if $x^2>a^2$, the corresponding y is imaginary; and if $y^2>b^2$, the corresponding x is imaginary, showing that there is no point of the curve without the rectangle.

The point O is called the centre of the ellipse, for it bisects all chords drawn through it; thus, let $y=mx$ be the equa. of a line through O, then substituting mx for y in [A], the result is a pure quadratic in x, so that the two values of x—the x of the intersections—are equal and opposite in sign, the corresponding values of y are therefore equal and opposite in sign, and $\therefore$ $OP'=OP$. Every line PP' thus drawn through the centre is called a diameter of the curve: the greatest of these is AB, and the least CD; for the length OP of any semidiameter is

$$D^2=x^2+y^2, \text{ but } y^2=b^2-\frac{b^2}{a^2}x^2 \text{ by [A]}, \therefore D^2=b^2+\frac{a^2-b^2}{a^2}x^2,$$

which since a^2-b^2 is positive being $=c^2$ is the greatest possible when x is, that is, when $x=a$, the semidiameter D being then $=a$: and it is the least possible when $x=0$, D being then $=b$. For these reasons AB is called the *major diameter*, and CD the *minor diameter* of the ellipse; when spoken of together they are called the *principal diameters*, or the *principal axes* of the ellipse; hence the equation

$$a^2y^2+b^2x^2=a^2b^2, \text{ or} \left(\frac{y}{b}\right)^2+\left(\frac{x}{a}\right)^2=1$$

is the equation of the ellipse related to its principal diameters, the origin being at the centre of the ellipse. Still retaining the major diameter for axis of x, if the origin be placed at its extremity A instead of at its middle O; then although every y will remain the same, the new x of any point will exceed the old x by a, so that if the changed x be written X, we shall always have $X=x+a$, or $x=X-a$: hence, to get the equation of the ellipse when the origin is removed to A, we have only to substitute $x-a$ for x in the original equation [A], and we thus have

$$a^2y^2+b^2x^2-2ab^2x=0, \text{ or } y^2=\frac{b^2}{a^2}(2ax-x^2)\text{.....[C]},$$

for the equation of the ellipse when the origin is at the vertex of the major diameter taken for axis of x.

There is also another form of the equation in reference to rectangular axes which is sometimes employed: it is deduced immediately from the equation [A], the origin and axes remaining the same, but e^2 substituted for $\frac{a^2-b^2}{a^2}$, or, which is the same thing, e for $\frac{c}{a}$; thus, since $b^2=a^2-c^2$, equa. [A] is the same as

$$y^2=\left(1-\frac{c^2}{a^2}\right)(a^2-x^2), \text{ or } y^2=(1-e^2)(a^2-x^2)\text{.....[E]}.$$

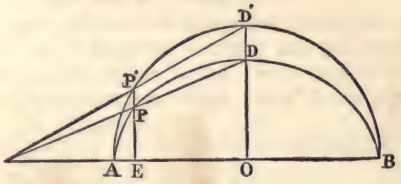
The fraction $e=\sqrt{\frac{a^2-b^2}{a^2}}=\frac{c}{a}$, is called the *eccentricity* of the ellipse, and the equation E is the equation of the ellipse as a function of the eccentricity.

362. **The Ellipse in relation to its Principal Diameters.**—The following are the chief properties of the ellipse, derivable from the foregoing equations:—

THEOREM I. In an ellipse, the squares of the ordinates are as the products of the segments into which they divide the major diameter.

By [A], $\frac{y^2}{(a+x)(a-x)}=\frac{b^2}{a^2}$, $\therefore y^2:(a+x)(a-x)::b^2:a^2$, that is, y^2 and the product $(a+x)(a-x)$ of the parts into which y divides $2a$, are always in the same constant ratio.

Suppose a semicircle to be described on the major diameter AB; then calling Y the ordinate of a point P' in the circle having the same abscissa x as the point P in the ellipse, we have



$$\frac{Y^2}{(a+x)(a-x)}=\frac{a^2}{a^2}=1,$$

$$\therefore, \frac{Y^2}{(a+x)(a-x)}=\frac{a^2}{b^2}\frac{y^2}{(a+x)(a-x)}, \therefore \frac{Y}{y}=\frac{a}{b}, \text{ that is, } P'E:PE::D'O:DO$$

hence the chords DP, $D'P'$, if produced, will meet in the prolongation of BA; and because $\frac{Y}{y}$ is a constant ratio, the same is true wherever D, D', and P, P' may be.

To find the length of the ordinate through the focus F, we have only to put $x=c$ in the equation [E]; we then have $ay=a^2-c^2=b^2$, so that $2y=\frac{2b^2}{a}=\frac{(2b)^2}{2a}$; that is, the double ordinate through the focus is a third proportional to the major and minor diameters: this double ordinate is called the *right parameter* (p), or the *latus rectum*. Introducing it into the equations [A], they become

$$y^2=\frac{p}{2a}(a^2-x^2), \text{ and } y^2=px-\frac{p}{2a}x^2 \ldots\ldots [1].$$

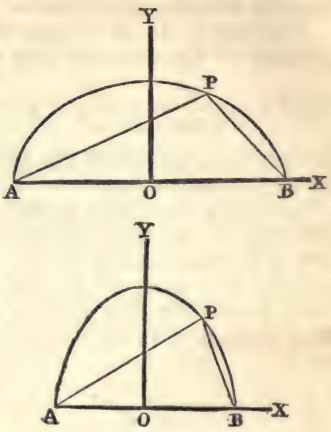
The semiparameter $\frac{1}{2}p$ is always greater than once, but less than twice the focal distance of the nearer vertex B; for in the first of [1] put $y=\frac{1}{2}p$, then $\frac{1}{2}p=\frac{a+x}{a}(a-x)$, which is greater than $a-x$, because $\frac{a+x}{a}>1$, but it is less than $2(a-x)$, because $\frac{a+x}{a}<2$, since $x<a$.

THEOREM II. All the angles in an ellipse, subtended by the major diameter, are obtuse: all subtended by the minor diameter are acute: the greatest of the former is at the vertex of the minor diameter, the least of the latter, at the vertex of the major diameter, and these greatest and least angles are supplements of each other. And the product of the trig. tangents of the inclinations to the principal diameter,

of two lines from the extremities of that diameter to meet in the curve, is constant. The equation of a line AP through the point $(-a, 0)$, is AP,*

$y=m(x+a), \therefore m=\frac{y}{x+a}$. Also for BP,

$y=m'(x-a), \therefore m'=\frac{y}{x-a}$. The tangent of the angle P in which these meet is (242), $\tan P=\frac{m'-m}{1+mm'}=\frac{2ay}{y^2+x^2-a^2}$, by putting the above expressions for m, m'. Now, for the point of meeting P, to be on the curve $a^2y^2+b^2x^2=a^2b^2$, all these equations must exist simultaneously: from the last we get $x^2-a^2=-\frac{a^2}{b^2}y^2$...[1], which substituted in the expression for $\tan P$, gives



$$\tan P=2ay\div\left(y^2-\frac{a^2}{b^2}y^2\right), \text{ that is, } \tan P=-\frac{2ab^2}{(a^2-b^2)y}\text{...[2].}$$

As y remains the same whether x be positive or negative, it follows that there are two points in the semiellipse at which the angles subtended by the opposite diameter are equal, the points (x, y), and $(-x, y)$. If the angle be subtended by the major diameter, as in the first figure, that is if $a>b$, $\tan P$ is negative, $\therefore P$ is *obtuse*. The numerical value of $\tan P$ is the least possible when y is the greatest, that is when $y=b$; but the smaller the numerical value of the tangent of an *obtuse* angle, the greater that angle is: hence the angle at P is greatest when P is at the extremity of the minor diameter. The expression for the maximum angle is

$$\tan P=\frac{-2ab}{a^2-b^2}\text{...[3].}$$

The investigation from which [1] above is deduced will equally apply to the second diagram, where the angle P is subtended by the minor diameter, if we write the equation of the curve $a^2x^2+b^2y^2=a^2b^2$, instead of $a^2y^2+b^2x^2=a^2b^2$; for it is plain that by so doing we merely interchange the axes of x and y, or, which is the same thing, turn the curve round into the position it takes in the second diagram. But the change spoken of is equally effected by leaving the x and y in the equation untouched, and merely interchanging a, b: hence, thus adapting [1] to the case where the minor diameter is the axis of x, as in the second diagram, we have for $\tan P$ in that diagram

$$\tan P=\frac{-2ba^2}{(b^2-a^2)y}=\frac{2ba^2}{(a^2-b^2)y},$$

which is always *positive:* the angle P, therefore, is always *acute;* and since an acute angle diminishes as the tangent of it diminishes, P is

* The student may confine his attention to the first diagram till he arrives at the end of the investigation.

the least possible when y is the greatest possible, that is, when $y=a$. The expression for this minimum angle is therefore tan $P=\frac{2ab}{a^2-b^2}$; and since this differs from tan P in the first diagram only in its sign, we infer that the two angles together make 180°. [This is obvious, too, from the simplest geometrical considerations, for the figure formed by joining the extremities of the axes is a parallelogram (a rhombus), and therefore the two angles to which any side is interjacent make together two right angles.]

From the circumstance that all angles in an ellipse subtended by the major diameter are obtuse, and all subtended by the minor diameter acute, it follows that if upon the major diameter there be described a semicircle or any greater arc, it will lie wholly without the semiellipse: but if a semicircle or any less arc be described upon the minor diameter, it will lie wholly within it: for in the first case the angles in the circular segment will all be *less* than those in the semiellipse, and in the latter case all *greater*.

If the expressions for m, m', (page 314) be multiplied together, we shall have $mm'=\frac{y^2}{x^2-a^2}$, or [1], $mm'=-\frac{b^2}{a^2}$, showing that the product of the trig. tangents of the two angles formed by lines, from the extremities of a principal diameter, to meet in the curve, is constantly equal to $-\frac{b^2}{a^2}$, where a is the semidiameter subtending the angles. If we multiply each of these trig. tangents by $2a$, we shall have the linear tangents of the same angles to radius AB: hence the following property:—

If from the extremities of either principal diameter perpendiculars to that diameter be drawn, terminating in two lines from the same extremities, and crossing each other on the curve, the rectangle of these perpendiculars will always be the same, namely, the square of the diameter to which they are parallel. [The perps. are *tangents* (p. 312).]

The foregoing properties have been derived from considering the ellipse in reference to its principal diameters as axes: by viewing it in relation to other pairs of diameters, additional properties may be obtained. But to get the equation of the curve when the axes coincide with oblique diameters, it is necessary that we know how to pass from rectangular to oblique axes having the same origin: the following article explains what changes are necessary for this purpose.

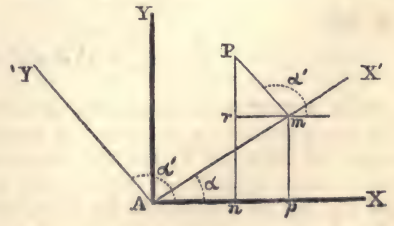
363. **Transformation of Co-ordinates from Rectangular to Oblique.**—Let AX, AY, be the original rectangular axes to which any line is referred, and let AX', AY' be the new pair of axes of reference. Then P being any point on the line, the original co-ordinates of it are

$$x=An,\ y=nP,$$

and the new co-ordinates are

$$x'=Am,\ y'=mP,$$

so that between the old and new co-ordinates we have these relations, namely—



$$\left.\begin{array}{l} x=Ap-rm, \text{ that is, } x=x'\cos\alpha+y'\cos\alpha' \\ y=pm+rP, \text{ that is, } y=x'\sin\alpha+y'\sin\alpha' \end{array}\right\}\ldots\ldots[\text{A}].$$

Hence, in order to transform the equation of a curve, in reference to rectangular axes, into another referring it to any different pair of axes originating at the same point, we have only to write for x, y, in the original equation, the expressions for them in [A], and it is plain that the accents over the new co-ordinates need not be retained.

If the new axes are to have a different origin from the primitive, then, calling the primitive co-ordinates of the new origin x_1 and y_1, the equations [A] would be

$$x=x_1+x'\cos\alpha+y'\cos\alpha',\ y=y_1+x'\sin\alpha+y'\sin\alpha'.$$

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364. **Ellipse related to Conjugate Diameters.** — What conjugate diameters are will be seen presently: our object now is to deduce the equation of the curve when referred to any axes of co-ordinates whatever, originating at the centre. Substituting the expressions [A] for x and y, omitting the accents over the new co-ordinates, in the primitive equation $a^2y^2+b^2x^2=a^2b^2$, it becomes

$$\begin{array}{l|l|l|l} a^2\sin^2\alpha' \\ b^2\cos^2\alpha' \end{array}\left|\begin{array}{l} y^2+2a^2\sin\alpha\sin\alpha' \\ \qquad 2b^2\cos\alpha\cos\alpha' \end{array}\right|\begin{array}{l} xy+a^2\sin^2\alpha \\ \qquad b^2\cos^2\alpha \end{array}\left|\begin{array}{l} \\ x^2=a^2b^2\ldots\ldots[1], \end{array}\right.$$

in which equation the axes of x, y make any proposed angles α, α' with the primitive rectangular axes, that is with the semi-major and semi-minor diameters of the curve, and we may now choose these angles so as that certain prescribed conditions may be satisfied. Thus, without giving them any fixed values, we may establish such a relation between the two that in virtue of that relation the term in xy shall vanish, and thus the form of the equation [1], become analogous to the form of the primitive. For this purpose we have only to assume

$$a^2\sin\alpha\sin\alpha'+b^2\cos\alpha\cos\alpha'=0,\ \therefore a^2\tan\alpha\tan\alpha'+b^2=0,\ \therefore \tan\alpha'=\frac{-b^2}{a^2\tan\alpha}\ldots[2].$$

Hence xy will be absent from the equation of the curve at whatever angle α the diameter taken for the axis of x is inclined to the major diameter, provided only that the inclination α', of the diameter taken for the axis of y, to the same major diameter, fulfil the condition [2]. Subject, therefore, to the condition [2], the equation [1] is

$$(a^2\sin^2\alpha'+b^2\cos^2\alpha')y^2+(a^2\sin^2\alpha+b^2\cos^2\alpha)x^2=a^2b^2\ldots\ldots[3].$$

Now divide both sides of it by a^2b^2, and for brevity put in the result $\frac{1}{b'^2}$ for the coef. of y^2, and $\frac{1}{a'^2}$ for the coef. of x^2, the form of the equation is then

$$\frac{1}{b'^2}y^2+\frac{1}{a'^2}x^2=1\ \therefore a'^2y^2+b'^2x^2=a'^2b'^2\ldots\ldots[4],$$

which is exactly the form of the primitive. And we shall find that just as in the primitive, a, b, were the semidiameters taken for axes of reference, so here a', b', are the semidiameters taken for the new axes of reference: for if in [4] we put first $x=0$, and then $y=0$ for the purpose of finding the expressions for the semidiameters which lie upon the axes of y and x respectively, we find $y=\pm b'$, and $x=\pm a'$. Hence the diameters

coincident with the oblique axes are $2a'$, $2b'$, and it is shown exactly as at (361), that each of these bisects all the chords drawn parallel to the other. Pairs of diameters having this property are called *conjugate diameters;* and [4] is thus the general equation of the ellipse referred to any pair of conjugate diameters as axes: of these, the principal diameters is the only pair of conjugates which are at right angles to one another; for, that the two diameters may form a right angle we must have $\tan\alpha \tan\alpha' = -1$ (p. 295), whereas [1], $\tan\alpha \tan\alpha' = -\frac{b^2}{a^2}$ which is -1 only when $a=b$, or when the curve is a circle. It further appears from [1], that when one of the angles α is acute the other α' must be obtuse, or when one of the two tangents is positive the other must be negative: consequently, according as the axis of x is above or below the major diameter, the axis of y must be to the left or to the right of the minor diameter. The major and minor themselves are called the principal conjugates, or the principal axes, the greater is the *transverse diameter*, the other its conjugate.

365. Since the properties established in (362) were all derived from the equation $a^2y^2+b^2x^2=a^2b^2$, where a, b, are the principal semiconjugates, similar properties are, of course, in like manner, deducible from the equation $a'^2y^2+b'^2x^2=a'^2b'^2$, where a', b', are any semiconjugates whatever. Hence

1. Straight lines at the extremities of any diameter, and parallel to its conjugate, are tangents to the curve.

2. The squares of chords parallel to one of two conjugates are as the rectangles of the parts into which they divide the other.

3. If from the extremities of any diameter $2a'$ straight lines $y=m(x+a')$ and $y=m'(x-a')$ be drawn to meet in the curve, then $mm'=-\frac{b'^2}{a'^2}$, where m, m', are the coefficients of inclination of the lines, referred to the conjugates $2a'$, $2b'$ as axes. These coefficients no longer denote the *tangents* of the inclinations α, α', of the chords to the axis of x, that is to a', seeing that the axes are now oblique; the property, however, at page 315, still has place, that is if linear tangents be drawn at the extremities $2a'$ of any diameter, and these be limited by chords, from the same extremities, prolonged beyond their point of intersection on the curve, the rectangle of these tangents will be equal to $(2b')^2$, the square of the diameter to which they are parallel. For putting t, t', for the tangents, it is plain that $m=\frac{t}{2a'}$, and $m'=-\frac{t'}{2a'}$, so that $mm'=-\frac{b'^2}{a'^2}=-\frac{tt'}{4a'^2}$, $\therefore tt'=4b'^2$. Every pair of chords thus drawn from the extremities of a diameter to meet in the curve is called a pair of *supplemental chords.* The consideration of supplemental chords suggests the following two additional properties.

THEOREM I. If supplemental chords be drawn from the extremities of a principal diameter, the diameters parallel to these will be conjugate; and conversely, chords from the extremities of a principal diameter, drawn parallel to a pair of conjugates meet in the curve.

For from the condition [2] of a pair of conjugates, $\tan\alpha \tan\alpha' = -\frac{b^2}{a^2}$, and it was shown at p. 315, that for a pair of supplemental chords from

the ends of a principal diameter, $mm' = -\frac{b^2}{a^2}$, where m, m', are the tangents of the inclinations of those chords: hence if $m = \tan \alpha$, then must $m' = \tan \alpha'$, and conversely; that is, the inclinations which satisfy a pair of sup. chords from the ends of a principal diameter also satisfy a pair of conjugates, and conversely.

THEOREM II. Of all pairs of conjugates those parallel to the *equal* supplemental chords from the ends of the major diameter contain the greatest angle, and the conjugates themselves are *equal conjugates*. They contain the greatest angle because the chords to which they are parallel do, and they are equal because they make equal angles with the major diameter, or rather with $+a$ and $-a$: the tangents of the angles they make with $+a$, are $\frac{b}{a}$, and $-\frac{b}{a}$ by [2]. When these equal conjugates are taken for axes of reference, the equation of the ellipse is $y^2 + x^2 = a^2$, which would characterize a *circle*, were it not that here the axes are necessarily oblique.

We shall conclude this article with a theorem which is independent of the foregoing deductions, and which is sufficiently interesting to deserve notice.

THEOREM III. If a_1, b_1, be any two semidiameters of an ellipse at right angles to one another, then $\frac{1}{a_1^2} + \frac{1}{b_1^2} = \frac{1}{a^2} + \frac{1}{b^2}$, that is the sum of the squares of the reciprocals of two rectangular diameters is constant. The general equation [1] is that of an ellipse related to any axes whatever originating at its centre, the angles α, α', denoting the inclinations of these axes to the semimajor diameter a. Leaving α, α', arbitrary for the present, if we put $y = 0$, and then $x = 0$, in [1], we shall have for the squares of the semidiameters coincident with the axes

$$x^2 = \frac{a^2 b^2}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}, \quad y^2 = \frac{a^2 b^2}{a^2 \sin^2 \alpha' + b^2 \cos^2 \alpha'}.$$

Now let α, α', be subjected to the condition that these semidiameters contain a right angle; then $\alpha = \alpha' - 90° = -(90° - \alpha')$,

$$\therefore \sin^2 \alpha = \cos^2 \alpha', \text{ and } \cos^2 \alpha = \sin^2 \alpha' \text{ ; hence}$$

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{a_1^2} + \frac{1}{b_1^2} = \frac{a^2(\sin^2 \alpha + \sin^2 \alpha') + b^2(\cos^2 \alpha + \cos^2 \alpha')}{a^2 b^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

We shall now show how by returning from the oblique conjugates as axes, to the original rectangular conjugates, other interesting properties will present themselves.

---

366. **Transformation of Co-ordinates from Oblique to Rectangular.**—In order to pass from oblique to rectangular co-ordinates having the same origin, we have only to find the expressions for x', y', in terms of x, y, from the conditions [A], and then to substitute these values for x, y, in [4]. Now from the equations referred to we have

$$x \sin \alpha' - y \cos \alpha' = x'(\sin \alpha' \cos \alpha - \cos \alpha' \sin \alpha),$$

$$y \cos \alpha - x \sin \alpha = y'(\sin \alpha' \cos \alpha - \cos \alpha' \sin \alpha),$$

therefore representing $\alpha'-\alpha$, the angle between a', b', by $[a', b']$, we have

$$x'=\frac{x\sin\alpha'-y\cos\alpha'}{\sin[a', b']},\quad y'=\frac{y\cos\alpha-x\sin\alpha}{\sin[a', b']}.$$

Substituting these for x, y, in [4], the transformed equation is

$$\begin{array}{l|l|l|l} a'^2\cos^2\alpha & y^2-2a'^2\sin\alpha\cos\alpha & xy+a'^2\sin^2\alpha & x^2=a'^2b'^2\sin^2[a', b'] \\ b'^2\cos^2\alpha' & -2b'^2\sin\alpha'\cos\alpha' & b'^2\sin^2\alpha' & \end{array}$$

Now in order that this equation may be identical with the original equation $a^2y^2+b^2x^2=a^2b^2$, the coefficients must satisfy the following conditions, namely,—

$$a'^2\cos^2\alpha+b'^2\cos^2\alpha'=a^2\ldots\ldots[1],\qquad a'^2\sin^2\alpha+b'^2\sin^2\alpha'=b^2\ldots\ldots[2],$$

$$a'^2\sin\alpha\cos\alpha+b'^2\sin\alpha'\cos\alpha'=0\ldots\ldots[3],\qquad a'^2b'^2\sin^2[a', b']=a^2b^2\ldots\ldots[4].$$

The sum of [1] and [2] gives the remarkable property $a'^2+b'^2=a^2+b^2$, that is the sum of the squares of the diameters forming any pair of conjugates is always the same, namely, the sum of the squares of the principal diameters.

The condition [4] too furnishes another property equally deserving of notice, for it shows that $4a'b'\sin[a', b']=4ab$, that is, that parallelograms circumscribing an ellipse, and having their sides parallel to a pair of conjugates are all equal in surface (308) each being equal to the rectangle of the principal diameters (see diagram, p. 321).

Problem. The principal diameters of an ellipse and the vertex of any other diameter being given, to find the length of that diameter and of its conjugate.

Referring the curve to the principal diameters as axes, the distance of the vertex (x', y'), of the semidiameter a', from the centre is $a'^2=x'^2+y'^2$;

$$\text{and since } y'^2=b^2-\frac{b^2}{a^2}x'^2,\ \therefore\ a'^2=b^2+\frac{a^2-b^2}{a^2}x'^2=b^2+e^2x'^2.$$

$$\text{But, as shown above, } a'^2+b'^2=a^2+b^2,\ \therefore\ b'^2=a^2-e^2x'^2,$$

$$\therefore\ a'=\sqrt{(b^2+e^2x'^2)},\ b'=\sqrt{(a^2-e^2x'^2)}\ldots\ldots[1].$$

Cor. Since (361) the product of the radii vectors of (x', y'), is $fP.FP=a^2-e^2x'^2$, it follows that

$$fP.FP=b'^2\ldots\ldots[2].$$

---

367. **Tangent to the Ellipse.**—In order to find the equation of the tangent at a point (x', y'), of an ellipse, let us first consider a secant through (x', y'), cutting the curve again in (x'', y''), as in the circle (352). We thus have the three equations,

$$y-y'=\frac{y'-y''}{x'-x''}(x-x'),\ a'^2y'^2+b'^2x'^2=a^2b^2,\ a'^2y''^2+b'^2x''^2=a'^2b'^2.$$

Subtracting the third from the second,

$$a'^2(y'^2+y'')(y'-y'')=-b'^2(x'+x'')(x'-x''),\ \therefore\ \frac{y'-y''}{x'-x''}=-\frac{b'^2}{a'^2}\cdot\frac{x'+x''}{y'+y''};$$

hence $y-y'=-\frac{b'^2}{a'^2}\cdot\frac{x'+x''}{y'+y''}(x-x')$, the equa. of the secant through (x', y'), and (x'', y'').

If these two points unite, then $x''=x'$, and $y''=y'$, and the line becomes a tangent at (x', y'), of which the equation is

$$y-y'=-\frac{b'^2}{a'^2}\cdot\frac{x'}{y'}(x-x'), \text{ or } a'^2y'y+b'^2x'x=a'^2b'^2, \text{ or } \frac{y'y}{b'^2}+\frac{x'x}{a'^2}=1.....[1],$$

the axes coinciding with any pair of conjugate diameters. But another form of the equation is sometimes useful, in which the point of contact (x', y'), is not given, but m, the coef. of inclination of the touching line, instead; from the first of the preceding forms we see that this coef. is

$$m=-\frac{b'^2}{a'^2}\cdot\frac{x'}{y'}, \text{ so that } \frac{x'}{y'}=-\frac{a'^2}{b'^2}m;$$

also from the equations of the curve and tangent, we have

$$1+\frac{b'^2}{a'^2}\cdot\frac{x'^2}{y'^2}=\frac{b'^2}{y'^2}, \text{ and } y+\frac{b'^2}{a'^2}x\cdot\frac{x'}{y'}=\frac{b'^2}{y'}.$$

Substituting in these the value of $\frac{x'}{y'}$, just deduced, they become

$$1+\frac{a'^2}{b'^2}m^2=\frac{b'^2}{y'^2}, \text{ and } y-mx=\frac{b'^2}{y'}.$$

Multiplying the first of these by b'^2, and squaring the second, to eliminate y', there results

$$y-mx=\pm\sqrt{(m^2a'^2+b'^2)}.....[2].$$

This is the form of equation which a line whose coef. of inclination is m, in reference to any conjugate diameters for axes, must have in order that it may touch the ellipse. We see, on account of the double sign, that such lines occur in parallel pairs, m being the same for both lines of each pair.

THEOREM I. If pairs of tangents to an ellipse intersect at right angles the locus of the points of intersection will be a circle. In the equation [2], the axes coincide with a pair of conjugate diameters: suppose now that these are the principal conjugates, and let us regard m (which is then the tangent of inclination) as open to any arbitrary condition. Removing the radical and arranging the terms according to the powers of m, we have

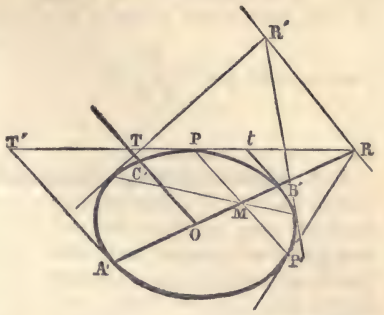
$$(x^2-a^2)m^2-2xym+y^2-b^2=0.....[3],$$

where we see that for any point (x, y) on the tangent, m has two values, that is, [3] has two roots, m, m', so that *two* tangents pass through the point. Let the roots be so related that their product $mm'=-1$: then since by the theory of equations (142) this product is also

$$\frac{y^2-b^2}{x^2-a^2}, \therefore y^2-b^2=a^2-x^2, \text{ or } y^2+x^2=a^2+b^2;$$

hence the locus of the point (x, y) common to the two tangents (these being by the condition $mm'=-1$ at right angles) is a circle of which the centre is that of the ellipse, and the radius $\sqrt{(a^2+b^2)}$.

THEOREM II. Any pair of conjugates being taken for axes, the rectangle of the ordinate at the point of contact, and of that at the point in the tangent where it meets the axis of ordinates, is equal to the square of that semidiameter (b') which is taken for the axis of ordinates. And the rectangle of the abscissa of the point of contact and that of the point where the tangent meets the axis of abscissas is equal to the square of the semidiameter (a') taken for axis of abscissas. Thus, in the annexed diagram, OB', OC', being any semiconjugates, $OT.MP=OC'^2$, and $OR.OM=OB'^2$. For in the second of the equations [1], put first $x=0$, and then $y=0$, and we shall have $y'y=b'^2$, and $x'x=a'^2$, which establishes the theorem.



THEOREM III. If at the extremities of any diameter lines parallel to its conjugate be drawn terminating in any tangent, their rectangle will be equal to the square of the semidiameter (b') to which they are parallel.

Thus, in the preceding diagram, $A'T'.B't=OC'^2$. For in the same equation of the tangent put first $x=a'$, and then $x=-a'$,

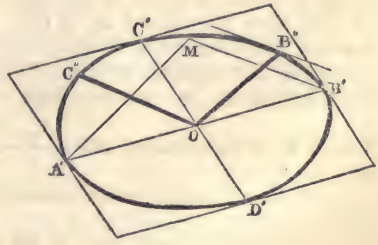
and we get $y=B't=\frac{b'^2(a'-x')}{a'y}$, and $y=A'T'=\frac{b'^2(a'+x')}{a'y'}$.

The product of these is $A'T' \,.\, B't=\frac{b'^4(a'^2-x'^2)}{a'^2y'^2}$;

but from the equation of the ellipse, $y'^2=\frac{b'^2(a'^2-x'^2)}{a'^2}$, $\therefore A'T' \,.\, B't=b'^2=OC'^2$.

THEOREM IV. Diameters parallel to *any* pair of sup. chords are conjugate. Let the conjugates $A'B'$, $C'D'$, be $2a'$, $2b'$, these being the axes of reference, then $A'M$, $B'M$, being two sup. chords, their coefficients of inclination m, m', are such that

$$mm'=-\frac{b'^2}{a'^2} \quad (365).$$

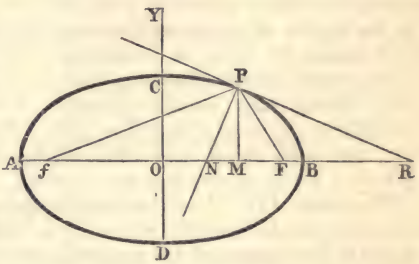


Let a tangent at B'' (x', y') be parallel to $B'M$, then m' will equally belong to this tangent and to $B'M$: but from the equation of the tangent, $m'=-\frac{b'^2x'}{a'^2y'}$, $\therefore m=\frac{y'}{x'}$, and since $\frac{y'}{x'}$ is the coef. of inclination of OB'', therefore OB'' is parallel to $A'M$ and since OC'', parallel to the tangent at B'', is conjugate to it (365), therefore OB'', OC'', parallel to the sup. chords $A'M$, $B'M$, are semiconjugates

The equation of a pair of conjugates, referred to any other pair $2a'$, $2b'$, as axes, are $y=mx$, $y=m'x$, in which $mm'=-\frac{b'^2}{a'^2}$, that is, the equations are $y=mx$, and $y=-\frac{b'^2}{a'^2 m}x$.

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368. **Subtangent, Normal, and Subnormal.**—The part MR of the axis of x intercepted between the ordinate PM of the point of contact and the point R, where the tangent meets the axis, is called the *subtangent;* the perpendicular PN to the tangent, from the point of contact to the axis of x, is the *normal;* and NM, the distance between the foot of the normal and the foot of the ordinate, is the *subnormal:* thus the tangent and subtangent lie on one side of the ordinate MP, and the normal and subnormal on the other side. The axes of reference being the principal diameters of the ellipse, the equation of the normal, since it is perp. to the tangent at the point of contact $(x', y',)$ is (p. 294), $y-y'=\frac{a^2y'}{b^2x'}(x-x')$, or, clearing fractions, and, as at (361), putting c^2 for a^2-b^2, the equation of the normal is



$$b^2x'y-a^2y'x+c^2x'y'=0, \text{ or it is } b^2\frac{y}{y'}-a^2\frac{x}{x'}+c^2=0.$$

To get the expression for the *length* of the subtangent, we have only to put $y=0$ in the equation of the tangent, which will give for x the length OR, and then to subtract x', or OM, from it. In like manner, for the length of the subnormal, we have only to subtract ON, which is the x of the normal corresponding to $y=0$ in its equation, and to subtract that length from $OM=x'$: we thus have

Length T_1 of the subtangent $MR=\frac{a^2-x'^2}{x'}$. Length N_1 of the subnormal $MN=\frac{b^2}{a^2}x'$.

For the length N of the normal, we have $PN=\sqrt{(MN^2+PM^2)}$,

which, since by the equa. of the curve, $PM^2=y'^2=\frac{b^2}{a^2}(a^2-x'^2)$, is

$$\text{Length } N \text{ of normal } PN=\sqrt{\left\{\frac{b^4}{a^4}x'^2+\frac{b^2}{a^2}(a^2-x'^2)\right\}}=$$

$$\frac{b}{a}\sqrt{\left\{\frac{b^2-a^2}{a^2}x'^2+a^2\right\}}=\frac{b}{a}\sqrt{(a^2-e^2x'^2)}.$$

At the vertex B, where $x'=a$, the normal coincides with BO in direction, but its length is still definite: putting $x'=a$, in the second of the

above expressions, the length of the normal at the vertex B is $\frac{b^2}{a}$, the semi-parameter of the principal diameter.

As to the length of the tangent $PR = \sqrt{(\text{subtan}^2 + y'^2)}$, it is

$$\text{Length } T \text{ of Tangent } PR = \sqrt{\left\{\left(\frac{a^2-x'^2}{x'}\right)^2 + \frac{b^2}{a^2}(a^2-x'^2)\right\}}.$$

THEOREM I. From any external point (R) two tangents may be drawn to an ellipse, and the chord of contact will be parallel to the conjugate to that diameter which passes through the point (see fig. p. 321). From the way in which the expression for the subtangent has been deduced, it is plain that if oblique conjugates (a', b') had been employed for axes of reference instead of the rectangular pair, that expression would have had the same form, namely $T_1 = \frac{a'^2 - x'^2}{x'}$, which we see is independent of the *sign* of y', showing that whether the point of contact be (x', y') or $(x', -y')$, the tangent at the point meets the axis of x in the same point, the distance $x' + T_1$ of the centre from that point being the same: two tangents therefore may be drawn from the external point, and since the two points of contact are $(x', y'), (x', -y')$ the straight line PP' passing through them is parallel to the axis of y.

It is further obvious, since T_1 is independent of b, that though innumerable ellipses, all having a principal diameter $2a$ in common, be described, all the points having the same abscissa x', will have the same subtangent, a truth which might indeed have been inferred from the property at (362).

The equation of the line joining the contacts (x', y'), (x'', y'') of two tangents, from any external point (x_1, y_1), is

$$\frac{y_1 y}{b'^2} + \frac{x_1 x}{a'^2} = 1 \ldots\ldots [1].$$

For the equations of the two tangents themselves are

$$\frac{y'y}{b'^2} + \frac{x'x}{a'^2} = 1, \text{ and } \frac{y''y}{b'^2} + \frac{x''x}{a'^2} = 1,$$

and (x_1, y_1) being a point on both, both equations are satisfied for $x = x_1$, and $y = y_1$; therefore [1] is satisfied for both $x = x'$, $y = y'$, and for $x = x''$, $y = y''$, so that [1] passes through both (x', y'), and (x'', y''), that is, it is the equation of the chord of contact.

THEOREM II. If from any number of points in a straight line pairs of tangents be drawn to an ellipse, the several chords of contact all pass through the same point. (See diagram to Theor. II.)

Let the axis of y (b') be parallel to the straight line RR', and (x_1, y_1) any point on that line: the chord of contact [1] cuts the axis of x (a') in the point $x = \frac{a'^2}{x_1}$, which point we see remains the same whatever be y_1; in other words, $x = x_1$ is the equation of a line from any point of which, if tangents be drawn, the several chords of contact will all intersect in the point $\left(\frac{a'^2}{x_1}, 0\right)$, which point (M) is called the *pole*, and the line $x = x_1$ the *polar* of it. And we see that the polar is always parallel to the diameter conjugate to that (or that produced) which passes through the pole: the

pole can never be at the centre, because then the chords of contact would be diameters, and tangents at the extremities of a *diameter* are parallel.

369. The general equations, in reference to any conjugate axes, of the tangent, the chord of contact, and the polar of a point, being all alike in form, it begets confusion when the *constants* in these equations, which are of course all different, are represented by the same symbols, as is frequently done: it may therefore save perplexity to the student to have these three distinct equations put thus:

Eq. of tangent through the point of contact (x', y'), $\frac{y'y}{b'^2}+\frac{x'x}{a'^2}=1$.....[1].

Eq. of chord of contact of two tangents from (x_1, y_1), $\frac{y_1y}{b'^2}+\frac{x_1x}{a'^2}=1$.....[2].

Eq. of polar RR' of any point M (α,β), $\frac{\beta y}{b'^2}+\frac{\alpha x}{a'^2}=1$.....[3].

That this is the equation of the polar is obvious, since (α, β) being a point on the chord of contact, we have [2], $\frac{y_1\beta}{b'^2}+\frac{x_1\alpha}{a'^2}=1$, therefore [3] is the locus of (x_1, y_1). [Observe: (α, β) may be *without* the circle as well as *within it*].

Theorem III. If any point (x_1, y_1) be taken on the polar of (α, β), then the polar of the point (x_1, y_1) passes through the point (α, β).

For the polar of the point (x_1, y_1) by [3] is $\frac{y_1y}{b'^2}+\frac{x_1x}{a'^2}=1$, which is a chord of contact passing through (α, β).

Theorem IV. If the focus be the pole, the polar will be a perpendicular to the principal diameter at the distance $\frac{a}{e}$ from the centre.

For taking the principal diameters for axes, the distance of the pole from the centre is $c=\frac{a^2}{x_1}$, $\therefore x_1=\frac{a^2}{c}$, and $c=ae$ (p. 312), $\therefore x_1=\frac{a}{e}$: hence $x=\frac{a}{e}$ is the equation of the polar of the focus, $2a$, $2b$, being the axes of co-ordinates. The polar of the focus is called the *directrix* of the ellipse: as there are two foci, there are two directrices.

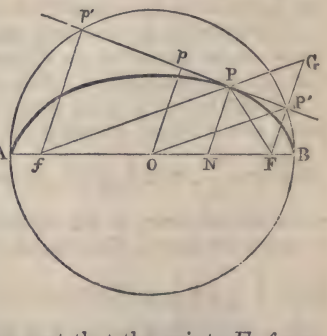
Theorem V. The normal at any point bisects the angle between the *radii vectores* of that point.

Putting $y=0$ in the equation of the normal PN, we have

$$ON=e^2x',\ \therefore fN=(a+ex')e,$$
$$\therefore FN=2ae-fN=(a-ex')e,$$

therefore (361), $fN : FN :: Pf : PF$,

$\therefore$ (Euc. 3. VI.) PN bisects the angle P, and consequently if fP be prolonged, the tangent PP' must bisect the angle FPG.



[Because FP, fP, make equal angles with the perpendicular PN to the curve at P, rays of light or heat issuing from F, and striking the curve, will all, after reflexion, unite again at f. It is on this account that the points F, f, are called foci, or burning points.]

THEOREM VI. The perpendicular from either focus on any tangent to the curve will meet the tangent in a point P', the locus of which is a circle described on the diameter AB.

For if PG be made $=PF$, the line PP' bisecting the angle P must also bisect FG. As, therefore, P' is the middle of FG, and O the middle of fF, the line OP' will be the half of fG, or of $fP+PF=AB$: hence OP' is constant, and $=OB$: the locus of P' is therefore a circle on the diameter AB.

Otherwise. The general equation of the tangent [2], page 320, is

$$y=mx+\sqrt{(m^2a^2+b^2)}, \text{ or } y-mx=\sqrt{(m^2a^2+b^2)}\ldots\ldots[1];$$

and the equa. of a line through $(\pm c, 0)$, that is, through $(\sqrt{a^2-b^2}, 0)$, perp. to this, is

$$y=-\frac{1}{m}(x-\sqrt{a^2-b^2}), \text{ or } my+x=\sqrt{(a^2-b^2)}\ldots\ldots[2].$$

Squaring [1], [2] to eliminate m, we have, by adding the results,

$$(y^2+x^2)(m^2+1)=a^2(m^2+1), \therefore y^2+x^2=a^2:$$

hence, the locus of the intersections of [1], [2], is a circle of radius a.

THEOREM VII. The rectangle of perpendiculars from the foci upon any tangent is equal to the square of the semiminor diameter.

The length of the perp. from either focus $(\sqrt{a^2-b^2}, 0)$, upon the line [1] above, is (p. 295)

$$\frac{\pm m\sqrt{(a^2-b^2)}-\sqrt{(m^2a^2+b^2)}}{\sqrt{(m^2+1)}}, \text{ and the product of these two expressions is}$$

$$\frac{-\{m^2(a^2-b^2)-(m^2a^2+b^2)\}}{m^2+1}=\frac{(m^2+1)b^2}{m^2+1}=b^2.$$

NOTE.—It will be observed here that the double sign is not given to the radical in [1], because we are dealing with a single tangent; but the point $(\sqrt{a^2-b^2}, 0)$ being either of the two points $(\pm c, 0)$, the radical takes the double sign.

COR. The two perpendiculars FP', fp', from the foci on the tangent, evidently form, with the two radii vectores, two similar right-angled triangles, so that

$$\frac{FP'}{FP}=\frac{fp'}{fP}, \therefore \left(\frac{FP'}{FP}\right)^2=\frac{FP'.fp'}{FP.fP}=\left(\frac{b}{b'}\right)^2, \therefore \frac{FP'}{FP}=\frac{b}{b'},$$

a property that will be found useful hereafter. Also $FP'^2=\left(\frac{b}{b'}FP\right)^2=\frac{b^2}{b'^2}(a-ex')^2$:

but (p. 319), $b'^2=a^2-e^2x'^2, \therefore FP'^2=b^2\frac{a-ex'}{a+ex'}$, $\therefore$ from the theorem, $fp'^2=b^2\frac{a+ex'}{a-ex'}$.

The perp. Op from the centre being $\frac{1}{2}(FP'+fp')$, we have

$$Op=\frac{1}{2}b\left\{\sqrt{\frac{a-ex'}{a+ex'}}+\sqrt{\frac{a+ex'}{a-ex'}}\right\}=\frac{1}{2}b\left\{\frac{2a}{\sqrt{(a^2-e^2x'^2)}}\right\}=\frac{ab}{\sqrt{(a^2-e^2x'^2)}}.$$

EXAMPLES FOR EXERCISE.

(1) Of all pairs of conjugates the sum of those which are equal is the greatest, and of those which are rectangular the least.

(2) The rectangle of the subtangent and abscissa of the point of contact is equal to the rectangle of the parts into which the diameter taken for axis of abscissas is divided by the ordinate.

(3) The rectangle of the radii vectores of any point is to the square of the normal, as the square of the major diameter is to the square of the minor.

(4) If a tangent be drawn at any point of an ellipse, the square of the semidiameter conjugate to that from the point of contact, will be equal to the rectangle of the two parts of the tangent intercepted between the point of contact, and any pair of conjugates whatever.

(5) The tangent at the extremity of the latus rectum cuts from the tangent at the vertex of the major diameter a part equal to the distance of the focus from that vertex.

(6) If from the foot of the normal at any point P a perp. be drawn to either radius vector of the point, the part of that radius intercepted between P and the perp. will always be equal to the principal semiparameter $\frac{b^2}{a}$.

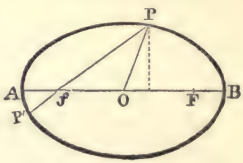
(7) A circle is described on the major diameter of an ellipse: normals are drawn from those points P, P', in the two curves which have the same abscissa: prove that the locus of the intersections of these normals is a circle.

(8) Every chord drawn through the focus is perpendicular to the line joining the pole of that chord with the focus.

(9) If any rectangular semidiameters of an ellipse be drawn, and a perpendicular from the centre to the chord joining their extremities cut that chord in P, the locus of P will be a circle.

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370. **Polar Equation of the Ellipse.**—First let the centre O be the pole, OB being the fixed axis; then for any point P, $OP=r$, and $POB=\theta$; and substituting $y=r\sin\theta$, and $x=r\cos\theta$ in $a^2y^2+b^2x^2=a^2b^2$, we have



$$r^2(a^2\sin^2\theta+b^2\cos^2\theta)=a^2b^2.$$

Now $b^2=a^2(1-e^2)$, $\therefore\ a^2b^2=a^4(1-e^2)$:

hence $r^2a^2\{(1-\cos^2\theta)+(1-e^2)\cos^2\theta\}=a^4(1-e^2)$,

$$\therefore\ r^2(1-e^2\cos\theta)=a^2(1-e^2),\ \therefore\ r^2=\frac{a^2(1-e^2)}{1-e^2\cos^2\theta},$$

which is the polar equation when the centre is the pole.

Next, let the focus f be the pole: then $fP=r=a+ex$, and $x=r\cos\theta-ae$,

$$\therefore\ r=a+re\cos\theta-ae,\ \therefore\ r=\frac{a(1-e^2)}{1-e\cos\theta}$$ is the polar equation for pole f.

If F had been the pole, x would have been $=r\cos\theta+ae$, so that

$$r=\frac{a(1-e^2)}{1+e\cos\theta}$$ is the polar equation for pole F.

If we put for the constant $a(1-e^2)$, the semiparameter $\frac{1}{2}p$, to which it is equal (p. 313), we may write the polar equation thus, $\frac{1}{r}=2p(1\mp\cos\theta)$, the upper sign applying when f is the pole, and the lower when F is the pole. Also, whichever sign applies, the opposite to it must be taken for the opposite radius vector fP'. If r be prolonged, completing the focal chord PP', we shall have $\frac{1}{r}+\frac{1}{r'}=4p$, or $\frac{r+r'}{rr'}=4p$: hence focal chords are to each other as the rectangles of the parts into which they are divided by the focus.

371. **Curvature of the Ellipse.**—The circle is the only plane curve of which the curvature is uniform throughout; and since a circle may be described of any length of radius, we may get a circle of any degree of curvature we please. It is thus well adapted to exhibit throughout its entire circumference the curvature that any other plane curve may have only at a particular point. The circle whose curvature is thus the same as the curvature of another curve at a particular point, is called *the circle of curvature* at that point. If P mark the point at which the circle of curvature is required, we may conceive that circle arrived at thus: let two other points on the curve, in the vicinity of P, be taken, and let a circle be described through all three: this circle will the more nearly approach in curvature to the curvature at any one of the three points, the closer those points are together; and if by causing the three points to approach nearer and nearer till they at length all coalesce in P, and by watching the continuously-varying radius of the circle passing through them, we could detect its exact length and position when the coalescence occurred, we should know the magnitude of the circle of curvature at the point P. Or the centre of this circle would be discovered thus: let the equation of the normal at P be found, and combine this with that of another normal in its vicinity; we shall thus get the point where the two intersect, and we have then only to find whereabouts on the fixed normal at P this intersection settles, when the second normal, by continuously approaching nearer and nearer to the first, coalesces with it. It was somewhat in this manner that the equation of a tangent was deduced from that of a secant.

372. **Centre and Radius of Curvature.**—Let (x', y') be the proposed point on the ellipse, and (x'', y'') any other point in the curve, then if the two normals at these points intersect in the point (α, β), we shall have for their equations (p. 322),

$$b^2x'\beta - a^2y'\alpha + c^2x'y' = 0,\ b^2x''\beta - a^2y''\alpha + c^2x''y'' = 0,$$

which solved as a pair of simultaneous equations, we get for α, and β,

$$\alpha = \frac{c^2x'x''(y'-y'')}{a^2(x''y'-x'y'')},\ \beta = -\frac{c^2y'y''(x'-x'')}{b^2(y''x'-y'x'')} \ldots\ldots [1].$$

Now $x''y'-x'y''=x'(y'-y'')-y'(x'-x'')$, $\therefore$ dividing num. and den. of α by $y'-y''$,

$$\alpha = \frac{c^2}{a^2}\frac{x'x''}{x'-y'\dfrac{x'-x''}{y'-y''}} = \frac{c^2}{a^2}\frac{x'x''}{x'+\dfrac{a^2}{b^2}y'\dfrac{y'+y''}{x'+x''}}.$$ (See page 302.)

Let now the points (x', y'), (x'', y''), on the ellipse unite in one, then

$$\alpha = \frac{c^2}{a^2}\frac{x'^2}{x'+\dfrac{a^2y'^2}{b^2x'}} = \frac{c^2}{a^2}\frac{b^2x'^3}{b^2x'^2+a^2y'^2} = \frac{c^2}{a^2}\frac{b^2x'^3}{a^2b^2} = \frac{c^2x'^3}{a^4}.$$

And since β in [1] differs from α only in having b for a, and x, y, for y, x, it follows that the co-ordinates of the centre of curvature are

$$\alpha = \frac{c^2x'^3}{a^4},\ \beta = -\frac{c^2y'^3}{b^4} \ldots\ldots [2],$$

and the distance r between the points (α, β), (x', y'), that is, the radius of curvature, is

$$r=\sqrt{\left\{\left(x'-\frac{c^2x'^3}{a^4}\right)^2+\left(y'+\frac{c^2y'^3}{b^4}\right)^2\right\}}\ldots\ldots[3].$$

At the vertex of the major diameter $x'=a, y'=0$; hence, at this point, the distance of the centre of curvature from the centre of the ellipse is $\frac{c^2}{a}$, and the length of the radius is $\frac{b^2}{a}$. At the vertex of the minor diameter, where $x'=0$, $y'=b$, the distance of the centre of the ellipse from the centre of curvature is $-\frac{c^2}{b}$, and the length of the radius $\frac{a^2}{b}$; the rectangle of these two radii is therefore equal to that of the principal semi-diameters.

Another form for r.

$$x'-\frac{c^2}{a^4}x'^3=\frac{x'}{a^4}\{a^4-(a^2-b^2)x'^2\}=\frac{x'}{a^4b^2}\{a^2(a^2b^2-b^2x'^2)+b^4x'^2\}=\frac{x'}{a^4b^2}(a^4y'^2+b^4x'^2).$$

The second term of [3] becomes the first when b is changed into a, y' into x', and c^2 into $-c^2$,

$$\therefore y'+\frac{c^2}{b^4}y'^3=\frac{y'}{b^4a^4}(b^4x'^2+a^4y'^2),$$

$$\therefore r^2=\left(\frac{x'^2}{a^8b^4}+\frac{y'^2}{b^8a^4}\right)(a^4y'^2+b^4x'^2)=\frac{b^4x'^2+a^4y'^2}{a^8b^8}(a^4y'^2+b^4x'^2)^2=\frac{(a^4y'^2+b^4x'^2)^3}{a^8b^8},$$

$$\therefore r=\frac{(a^4y'^2+b^4x'^2)^{\frac{3}{2}}}{a^4b^4}\ldots\ldots[4].$$

Third form for r.

Put $\frac{b^2}{a^2}(a^2-x'^2)$ for y'^2 in [4], then

$$r=\frac{\{a^2(a^2-x'^2)+b^2x'^2\}^{\frac{3}{2}}}{a^4b}=\frac{\{a^4-(a^2-b^2)x'^2\}^{\frac{3}{2}}}{a^4b}=\frac{(a^2-ex'^2)^{\frac{3}{2}}}{ab}\ldots\ldots[5].$$

Fourth form in terms of the Normal.

From the expression for the length of the normal at page 322, we have

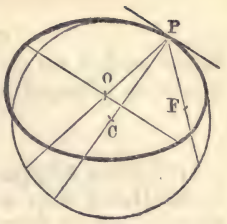
$$N^3=\frac{b^3}{a^3}(a^2-ex'^2)^{\frac{3}{2}},\ \therefore r=\frac{a^2}{b^4}N^3\ldots\ldots[6],$$

or since $a^2-ex'^2=PF\,.\,Pf$ (p. 319)$=b'^2$, $\therefore$ [5], $r=\frac{b'^3}{ab}=\frac{b'^3}{a'b'\sin[a',b']}$ (p. 319),

that is, $r=\frac{b'^2}{a'\sin[a',b']}$, $\therefore r\sin[a',b']=\frac{b'^2}{a'}=$ the semiparameter of $2a'$, of the diameter of the ellipse at the point (x', y').

373. **Chord of Curvature.**—The chord of the circle of curvature through the point of contact (x', y'), and which passes through the

centre O, of the ellipse, is called the *central chord of curvature*, and that from the same point through the focus is called the *focal chord of curvature*.



The angle $[a', b']$, above is that between the semidiameter a' at the point of contact, and its semiconjugate b'; and since the tangent at the vertex of a' is parallel to b', sin $[a', b']$, is the same as the sine of the angle between a' and the tangent, or as the cosine of the angle between a' and the normal, and this cosine multiplied by r is half the chord (of the circle) upon which a' is situated; hence,

$$\text{The central Chord of Curvature} = 2\frac{b'^2}{a'} = \frac{2(a^2 - ex'^2)}{a'} \ldots\ldots [1], \text{ (page 319).}$$

In like manner any other chord of contact is found by multiplying $2r$ by the cosine of the angle that chord makes with r. If the chord pass through the focus F, its length is

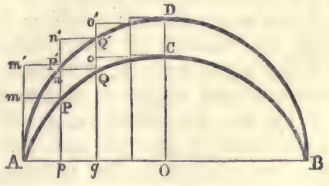
$2r \cos NPF$ (see fig. at p. 324), but $NPF = PFP'$,

and $\cos PFP' = \frac{FP'}{FP} = \frac{b}{b'}$, (p. 325), and as shown above, $r = \frac{b'^3}{ab}$.

$$\text{Consequently the focal chord of curvature} = 2\frac{b'^2}{a} = \frac{2(a^2 - e^2x'^2)}{a}.$$

374. **Area of the Ellipse.**—Let AB be the major diameter of a semiellipse ACB, upon which diameter let a semicircle be described.

Divide the semidiameter AO into any number of equal parts, at the extremities of which let ordinates be drawn both to the ellipse and to the circle, and complete the rectangles as in the figure. Then, since rectangles of the same base are to each other as their altitudes, we have



rect. Qp : rect. $Q'p$: : Qq : $Q'q$: : b : a (p. 313); hence the rectangles PA, $P'A$, the rectangles Qp, $Q'p$, &c., are always to each other in the constant ratio $b : a$. And since one antecedent is to its consequent, as the sum of all the antecedents to the sum of all the consequents, it follows that the irregular polygon

$$Amno \ldots CO : Am'n'o' \ldots DO : b : a.$$

And this is true, however numerous the subdivisions Ap, pq, &c., may be, that is, however small be the bases of the rectangles; but the smaller these bases are, the smaller is the difference between the surface of each polygon and the curve connected with it; and since there is no limit to the smallness of Ap, pq, &c., so is there no limit to the smallness of this difference; hence the curvilinear spaces must also be to each other as the polygons, that is, as b to a. Consequently the semiellipse ACB is to the semicircle ADB as b to a,

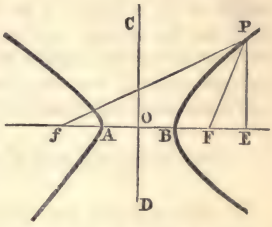
$$\therefore \text{Semiellipse} = \frac{b}{a} \text{ times the semicircle} = \frac{b}{a} \times \frac{1}{2} a^2\pi = \frac{1}{2} ab\pi,$$

$$\therefore \text{Area of ellipse} = ab\pi = 2a \times 2b \times \cdot 7854.$$

Hence, to find the area of an ellipse we have only to multiply the product of its principal diameters by ·7854.

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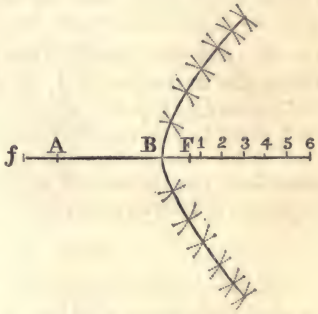
375. **The Hyperbola.**—This curve is such that if from any point P in it two straight lines be drawn to two fixed points F, f, called the *foci*, their *difference* will always be the same. As in the ellipse, the lines PF, Pf, from any point P to the foci, are called the *radii vectores*, or focal distances of that point.



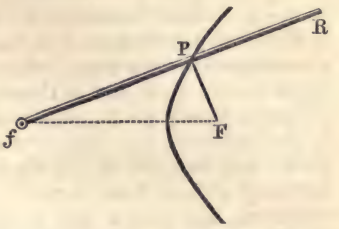
376. The definition of the curve suggests the mode of determining any number of points in it, and of thus discovering its form by observing the track of the series of points. The foci F, f, being given in position, and the constant length of the difference, between the focal distances of any point, being known, let a circle of any radius be described from F as centre: then let the radius be lengthened by the given difference, and with this increased radius, let another circle be described, from centre f. The two circles will intersect in two points: one as at P, above fF, and the other at an equal distance below fF, provided the first radius be not assumed too small to render intersection possible.

By repeating this operation, with two new radii, the second still exceeding the first by the constant difference, another pair of points on the curve will be determined: and this determination of pairs of points may be continued till the track of the curve becomes sufficiently apparent.

In the above, the smaller circles are considered as described from F, and the larger from f; but if on the contrary the smaller be described from f, and the larger from F, it is plain that a distinct series of points will be marked out by the intersections, so that the curves passing through each series will be entirely detached; but as both satisfy the definition, the two are regarded as merely branches of the same curve. The annexed diagram will sufficiently illustrate this method, by points, of describing either branch of the hyperbola. The line fF is drawn, joining the two given foci: if the branch is to pass through an assigned point B on this line, then from Bf cut off a part BA, equal to the proposed constant difference, and in BF prolonged, take any number of points 1, 2, 3, &c., the first being either at or beyond F. Then with F and f as centres, and with the respective radii $B1$, $A1$; $B2$, $A2$, &c., let the intersecting arcs be described as in the diagram, and the branch of the hyperbola may be traced through them.



Or the branch may be described by continuous motion thus: Let a ruler fR, be fixed to one focus f, and be at liberty to turn round that point, in the plane whereon the curve is to be described: then having assumed, or having given, the other focus F, connect it by means of a cord, shorter than the ruler by the given difference, to the extremity R. A pencil P, keeping this cord always stretched and pressed closely to the edge of the ruler, will, as the ruler revolves round f, describe an arc of the hyperbola of which f, F, are the *foci*; for $Rf-RPF=Pf-PF$, the constant difference.



377. **Equation of the Hyperbola.**—Let O, the middle of fF (first fig. above) be taken for the origin of the rectangular axes OB, OC, and for the constant difference $Pf-PF$, of any point (x, y), put $2a$, as also c for OF, or Of: then we shall have the three following conditions, namely,

$$Pf-PF=2a\ldots[1], \qquad y^2+(x+c)^2=Pf^2\ldots[2], \qquad y^2+(x-c)^2=PF^2\ldots[3],$$

and from these equations we have to eliminate Pf, PF. In order to this, take the sum and difference of [2], [3], and we have

$$2(y^2+x^2+c^2)=Pf^2+PF^2\ldots[4], \qquad 4cx=(Pf+PF)(Pf-PF), \quad \therefore\ Pf+PF=\frac{2cx}{a}.$$

Hence [1], $Pf=\frac{c}{a}x+a$, and $PF=\frac{c}{a}x-a$; and substituting these in [4],

$$y^2+x^2+c^2=\frac{c^2}{a^2}x^2+a^2, \text{ or } a^2y^2+(a^2-c^2)x^2=a^2(a^2-c^2)\ldots[5],$$

which being the general relation of the x and y of any point P, is the equation of the curve.

If in this equation we put $y=0$, the resulting values of x will be the abscissas of the points where the curve crosses fF: these are $x=\pm a$. Hence the curve intersects fF in two points B, A, equidistant from O. But if in the same equation we put $x=0$, the corresponding y will be

$$y=\sqrt{(a^2-c^2)}=\sqrt{(c^2-a^2)}\sqrt{-1}, \text{ an imaginary quantity}\ldots[6].$$

There can be no doubt about this being imaginary, for since $2c$ is the base fF, and $2a$ the difference $Pf-PF$, of the sides of a triangle, c must exceed a, so that (c^2-a^2) is necessarily positive. We thus see that the axis of ordinates never meets the curve: if, however, we mark on this axis two points C, D, each at the distance $\sqrt{(c^2-a^2)}$ from O, and call this distance b, the equation [5] will then assume a form analogous to that for the ellipse, namely,

$$a^2y^2-b^2x^2=-a^2b^2, \text{ or } \frac{1}{b^2}y^2-\frac{1}{a^2}x^2=-1, \text{ or } y^2=\frac{b^2}{a^2}(x^2-a^2)\ldots[7],$$

$$\therefore\ y=\pm\frac{b}{a}\sqrt{(x^2-a^2)}, \quad x=\pm\frac{a}{b}\sqrt{(y^2+b^2)}\ldots[8].$$

From the first of these it appears that ordinates between the limits $x=a$, $x=-a$, never meet the curve, because between these values for x, y is imaginary: but without these limits, however great the value of x may be, there are always two real and equal values of y corresponding to it, one marking the point (x, y) on the curve, above the axis of x, and the other the opposite point $(x, -y)$, below that axis. In like manner, what-

ever value be given to $-x$, there are always two real and equal values of y corresponding to it, one marking the point $(-x, y)$, and the other the point $(-x,-y)$. Each branch of the curve, therefore, is of unlimited extent.

The two parallels to CD, through B and A, that is, the two parallels whose equations are $x=a$, $x=-a$, touch the two branches of the curve in the points B, A, and in those points only; because for these abscissas, [8] gives merely $y=0$. Between these two parallels, therefore, no point in the hyperbola can exist:—the unlimited branches lie wholly without them; and because, as just shown, to equal and opposite abscissas, terminating without these parallels, there correspond a pair of equal positive ordinates, and also a pair of equal negative ordinates, it follows that chords parallel to AB, whether drawn on one side of AB, or on the other, are all bisected by CD or its prolongation. And that chords parallel to CD are all bisected by the prolongation of AB.

378. It thus appears that the curve is entirely dissimilar to the ellipse: the ellipse is a *closed* curve, limited in extent: the hyperbola is an *open* curve, unlimited in extent: the ellipse consists of a single continuous curve line, the hyperbola consists of two curve lines, detached from each other. But notwithstanding this marked dissimilarity, the two curves have many properties in common, or, at least, the properties of one curve naturally suggest corresponding properties of the other. This results from the comparatively slight dissimilarity between their *equations;* for if we only change $-b^2$ in [7], into b^2, the equation of the hyperbola becomes converted into that of the ellipse: and since the chief properties of the ellipse were derived from its equation, if, in the algebraic expression of those properties, we change b^2 into $-b^2$, or b into $b\sqrt{-1}$, corresponding properties of the hyperbola will be expressed; and this is one advantage of discussing these curves through their equations. For instance, the expression D, at p. 312, for the distance of any point (x, y), or $(-x,-y)$ from O, becomes adapted to the hyperbola by putting $-b^2$ for b^2, which change leaves undisturbed the conclusion that all chords, uniting the two branches of the curve, and passing through O, are bisected at that point. The point O is, therefore, with propriety called *the centre* of the hyperbola, and the chords drawn through it, *diameters*. The chord AB, or $2a$, joining the vertices of the two branches, is called the *principal transverse diameter*, and the line $CD=2\sqrt{(c^2-a^2)}$, or $2b$, though unconnected with the curve, is called the principal *second diameter:* together, they are the *principal axes* of the curve. We could not designate these the major and minor diameters, as in the ellipse, because b may *exceed* a, since c may be of any length greater than a in $\sqrt{(c^2-a^2)}$.

379. It thus appears that [7] is the equation of the hyperbola in reference to its principal diameters $2a$, $2b$. If these are *equal*, the curve is called an *equilateral hyperbola:* its equation is $y^2-x^2=-a^2$; so that the modification which converts an ellipse into a circle, changes a common hyperbola into an equilateral hyperbola. As in the ellipse, the ratio $\frac{c}{a}=e$ is called the *excentricity*; and since in the hyperbola, $c^2=a^2+b^2$,

$\therefore 1-e^2=-\frac{b^2}{a^2}$, instead of $\frac{b^2}{a^2}$, as in the ellipse: hence the change of b^2 into $-b^2$, is actually implied in substituting the hyperbolic e for the elliptic e in [E], at page 312; so that the equations of the two curves, in terms of the excentricity proper to each, is precisely the same: but it

is usual, for the hyperbola, to change the signs of each of the two factors, and to write the equation thus:—

$$y^2=(e^2-1)(x^2-a^2)\text{...[9], the equa. in terms of the excentricity,}$$

which of course is the better form, because *here* x^2 is always greater than a^2: in the ellipse, it is always less; moreover, e^2 is always greater than 1: in the ellipse, it is always less.

The ordinate from the focus where, $x=c=ae$, as given by this equation, is $y=(e^2-1)a=\frac{b^2}{a}$, as at (362); hence the double of this ordinate, or the *parameter* p, is $p=\frac{2b^2}{a}$; so that $\frac{p}{2a}=\frac{b^2}{a^2}$, and therefore [7],

$$y^2=\frac{p}{2a}(x^2-a^2)\text{ is the equation in terms of the parameter...[10].}$$

In the hyperbola, the semiparameter exceeds twice the distance of the focus from the vertex, since for $y=\frac{1}{2}p$ in [10], we have

$$\frac{1}{2}p=\frac{x+a}{a}(x-a),\text{ where since } x>a,\ \therefore x+a>2a,\ \therefore \frac{1}{2}p>2(x-a).$$

In the equilateral hyperbola, where $b=a$, $p=2a=$the transverse diameter. By removing the origin of the rectangular axes from O to B, that is, substituting $a+x$ for x, [7] becomes

$$a^2y^2-b^2x^2-2ab^2x=0,\text{ or } y^2=\frac{b^2}{a^2}(x^2+2ax)\text{...[11].}$$

which [C], at p. 312, also becomes when b^2 is changed into $-b^2$, and a put for $-a$. In terms of the parameter, the equation under this change of origin, is

$$y^2=\frac{p}{2a}x^2+px\text{.....[12].}$$

380. The student will observe that there is no departure from analogy in placing the origin at B in the hyperbola, and at A in the ellipse: in both cases the curve, for positive abscissas, proceeds from the origin towards the right. He will also do well further to notice, as instanced above, that in passing from one curve to the other, the changing of b^2 into $-b^2$, is equivalent to changing $1-e^2$, and a^2-x^2, into e^2-1, and x^2-a^2. And wherever b^2 is left unchanged in sign, $x-a$ must be written for $a-x$ (as in next article), and $ex-a$ for $a-ex$, as at p. 336. It is necessary to observe these particulars, in passing from the ellipse to the hyperbola, because in many of the expressions deduced from the former curve, b^2 does not explicitly enter.

---

381. **Properties of the Hyperbola related to its Principal Diameters.**—1. Changing b^2 into $-b^2$ in (362), or leaving b^2 unchanged, and putting $x-a$ for $a-x$, we have

$$\frac{y^2}{(x+a)(x-a)}=\frac{b^2}{a^2},\ \therefore\ y^2:(x+a)(x-a)::b^2:a^2,$$

that is, the squares of the ordinates are as the products of the parts of the prolonged principal diameter, included between those ordinates and the two vertices of the curve.

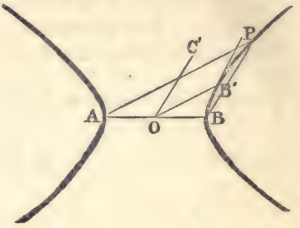
If $b=a$, then $y^2=(x+a)(x-a)$; so that in the equilateral hyperbola,

the square of any ordinate is equal to the product of the parts of the principal diameter (prolonged) between the ordinate and the vertices of the curve; a property analogous to that of the circle.

2. It was shown at p. 314, that an angle P at any point of an ellipse, and subtended by the diameter $2a$, had for its tangent the value—

$$\text{In the Ellipse, } \tan P=\frac{-2ab^2}{(a^2-b^2)y},$$

$$\therefore \text{ in the hyperbola, } \tan P=\frac{2ab^2}{(a^2+b^2)y},$$



which expression, being independent of the sign of x, must be the same for the point $(-x, y)$ as for (x, y). And since it is always positive for every point P above the axis of x, it follows that all the angles at the curve, subtended by the transverse axis, are acute: they diminish as y increases, becoming 0 when y is infinite: they thus range between 0° and 90°, or rather between 90° and 0°. The lines from A, B, meeting in P, as in the ellipse, are called *supplemental chords:* their tangents of inclination to the axis of x being m, m', we have, (p. 315), $mm'=\frac{b^2}{a^2}$, which being always positive, it follows that the angles, which two supplemental chords make with the axis of x, are either both acute, or both obtuse: it is plain that they are acute when the point is on the right-hand branch of the curve, and obtuse when it is on the left-hand branch.

$$\text{If } b=a, \text{ then } mm'=1, \ \therefore \ m'=\frac{1}{m}:$$

hence, if the hyperbola is equilateral, the angles which a pair of sup. chords make with the principal axis, are together equal to a right angle.

---

382. **The Hyperbola related to Conjugate Diameters as Axes.**—We have seen, at p. 315, that by a transformation of the axes of reference from rectangular to oblique, the equation of the ellipse preserves its original form, provided that, in the transformed equation, we put

$$a'^2 \text{ for } \frac{a^2b^2}{a^2\sin^2\alpha+b^2\cos^2\alpha}, \text{ and } b'^2 \text{ for } \frac{a^2b^2}{a^2\sin^2\alpha'+b^2\cos^2\alpha'}.$$

Hence, changing b^2 into $-b^2$, and putting

$$a'^2 \text{ for } \frac{-a^2b^2}{a^2\sin^2\alpha-b^2\cos^2\alpha}, \text{ and } -b'^2 \text{ for } \frac{-a^2b^2}{a^2\sin^2\alpha'-b^2\cos^2\alpha'},$$

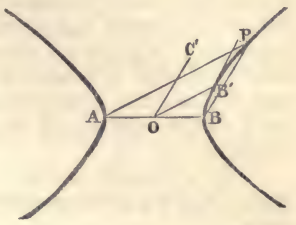
the equation of the hyperbola becomes the same as that at p. 331, when b^2 is changed into b'^2; that is, the equation, in reference to the new axes, is

$$-\frac{1}{b'^2}y^2+\frac{1}{a'^2}x^2=1, \text{ or } \frac{1}{b'^2}y^2-\frac{1}{a'^2}x^2=-1, \text{ or } a'^2y^2-b'^2x^2=-a'^2b'^2 \ldots [1],$$

whenever α, α', at which the new axes of x and y are inclined to the primitive axis of x, namely to the diameter $2a$, are such that $\tan\alpha'=\frac{b^2}{a^2\tan\alpha}$.

In [1], for $x=0$, we have $y=\sqrt{-b'^2}=\pm b'\sqrt{-1}$; and for $y=0$, we have $x=\pm a'$.

It is impossible, therefore, for the curve to meet the new axis of y; but it crosses the new axis of x, at the extremities of the diameter $2a'$. Although the new axis of y never meets the curve, any more than the primitive axis of y, yet, as in the former case, a length of this axis, equal to $2b'$, the centre O being at the middle of this length, is marked off for a *second diameter* to the corresponding transverse $2a'$; so that the semidiameters a', b', enter the new equation exactly as the semidiameters a, b, enter the primitive: and therefore, as in the ellipse, analogous properties are deducible from each. Thus from the equation [1], as at p. 317, we infer the following particulars:—



1. Each diameter $2a'$, $2b'$, bisects the chords parallel to the other: such diameters, as in the ellipse, are called *conjugate* diameters,* so that

$$a'^2y^2-b'^2x^2=-a'^2b'^2, \text{ or } \frac{1}{b'^2}y^2-\frac{1}{a'^2}x^2=-1.....[2]$$

is the equation of the hyperbola related to a pair of conjugate diameters. And, of every pair, one only is a transverse diameter, that is, one only has its extremities in the curve: the other is a *second diameter*, that is, it never meets the curve however far it be prolonged.

2. Straight lines, at the extremities of any transverse diameter, and parallel to its conjugate, are tangents to the curve (p. 332).

3. The parts of a prolonged transverse diameter, between the ordinate at any point, and the vertices of that diameter, are such that their product is to the square of that ordinate as the square of the transverse to the square of the conjugate (p. 333).

4. Since $\tan\alpha \tan\alpha'=\frac{b^2}{a^2}$, and that, as already proved (p. 334), $mm'=\frac{b^2}{a^2}$, it follows, as at (365), that diameters parallel to a pair of supplemental chords, from the ends of the principal transverse, are conjugate; and conversely. Therefore (381), the angle included by a pair of conjugates may be of any magnitude from 90° to 0°.

5. Again: from the above relation we also see that if $y=mx$ represent any diameter of an hyperbola referred to its principal axes, then will $y=\frac{b^2}{a^2m}x$ represent the conjugate to that diameter. And moreover, as at (365), when *any* pair of conjugate diameters $2a'$, $2b'$, are taken for axes, still $mm'=\frac{b'^2}{a'^2}$; where m, m', are the coefficients of inclination, to the diameter $2a'$, of the pair of conjugates referred to ($2a'$, $2b'$), as axes.

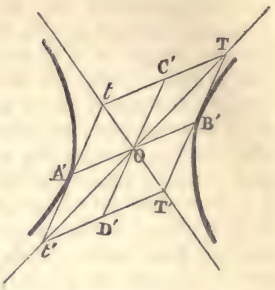
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* In the figure, OB', OC', are a pair of conjugate semidiameters.

6. Still referring to the former investigations, the equations at p. 319 upon changing b^2, b'^2, for $-b^2$, $-b'^2$, become

$$a'^2b'^2\sin^2[a', b'] = a^2b^2, \text{ and } a'^2 \sim b'^2 = a^2 \sim b^2,$$

so that the parallelogram constructed on any pair of conjugates is equivalent to the rectangle on the principal axes of the hyperbola; and the difference of the squares of any pair of conjugates is equal to the difference of the squares of the principal diameters.



7. In a similar manner, changing b^2, b'^2, into $-b^2$, $-b'^2$, in the problem at p. 319, we have

$$a' = \sqrt{(e^2x'^2 - b^2)}, \text{ and } b' = \sqrt{(e^2x'^2 - a^2)}, \text{ and } \therefore b'^2 = (ex' + a)(ex' - a) = Pf \,.\, PF,$$

where b' is the semiconjugate to the diameter at P. This property we might at once have inferred from the corresponding one of the ellipse, by merely writing $e^2x'^2 - a^2$ for $a^2 - ex'^2$ (p. 319).

383.—**Tangent to the Hyperbola.** — The tangent to the ellipse at any point (x', y'), being

$$\text{For the ellipse, } y - y' = -\frac{b'^2}{a'^2} \cdot \frac{x'}{y'}(x - x'), \text{ or } a'^2y'y + b'^2x'x, \text{ or } y - mx = \sqrt{(a'^2m^2 + b'^2)},$$

$$\therefore \text{ For the hyperbola, } y - y' = \frac{b'^2}{a'^2} \cdot \frac{x'}{y'}(x - x'), \text{ or } a'^2y'y - b'^2x'x,$$

$$\text{or } y - mx = \sqrt{(a'^2m^2 - b'^2)}.$$

$$\text{And the equa. of the normal, therefore, is } y - y' = -\frac{b^2}{a^2}(x - x').$$

We might now proceed to give the expressions for the lengths of the tangent, subtangent, &c., as at p. 322, and thence to deduce properties analogous to those already established for the ellipse. But enough has been done to convince the student how easily the conclusions arrived at, in reference to the ellipse, may be adapted to the hyperbola; we shall, therefore, pass on to the consideration of certain properties peculiar to the curve now under consideration.

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384. **The Asymptotes of the Hyperbola.**—It has already been seen that certain diameters of the hyperbola never meet the curve at any finite distance from the centre, or at any finite values for x, y. Of every pair of conjugate diameters, one of that pair—the *second* diameter, as it has been called—is always in this predicament. It has been seen, too, that the conjugate to the principal transverse makes with that transverse an angle of 90°; and that as the angle which an oblique transverse makes with the principal increases, the angle it makes with its conjugate diminishes, till at length, conceiving the transverse to have arrived at its ultimate position, meeting the two branches of the curve at an infinite distance, the conjugate, which all along has been approaching, actually

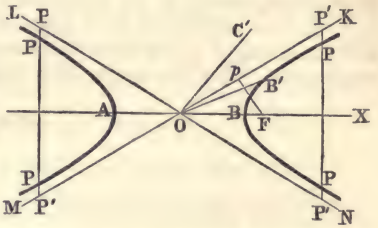
unites with it, and the two coalesce, the angle between them being then 0 (p. 335).

These peculiarities constitute a marked difference between the hyperbola and the ellipse, and imply that, in reference to these peculiarities, the former curve must have properties entirely distinct from those which are common to both: we have now to investigate these.

From the general equation of the hyperbola, in reference to its principal diameters,

$$y^2=\frac{b^2}{a^2}(x^2-a^2), \text{ we have } y=\pm\frac{b}{a}\sqrt{(x^2-a^2)}, \text{ or } y=\pm\frac{b}{a}x\sqrt{\left(1-\frac{a^2}{x^2}\right)};$$

and since as x increases, the fraction $\frac{a^2}{x^2}$ diminishes, it follows that for these increasing values of x, the corresponding values of y go on continuously approaching towards $\pm\frac{b}{a}x$, which extreme value, however, is never actually reached till x becomes infinite, rendering the fraction zero.



If then through the origin O two straight lines KM, LN, be drawn, making angles KOX, LOX, with OX, whose tangents are respectively $+\frac{b}{a}$, and $-\frac{b}{a}$, these two lines will continually approach nearer and nearer to the curve the further they are prolonged, and yet can never meet the curve within any finite distance from the origin, or for any finite value of x. For the equations of these two lines, or

the values of y for any value of x being $y=\frac{b}{a}x$, and $y=-\frac{b}{a}x$; and

for the same x the y of the curve being $y=\frac{b}{a}x\sqrt{\left(1-\frac{a^2}{x^2}\right)}$, $y=-\frac{b}{a}x\sqrt{\left(1-\frac{a^2}{x^2}\right)}$,

it is plain that the difference (PP') diminishes continually as x increases, and vanishes altogether when $x=\infty$, but not till then.

These two lines, KM, LN, are called the *asymptotes* to the hyperbola; and it is plain that they separate all the transverse diameters from their conjugates; the former being all situated within the angles KON, MOL, and the latter within the angles KOL, NOM: and moreover, that the asymptote is the limiting or ultimate position of a tangent to the curve; that is, it is coincident with the tangent when the point of contact is infinitely remote. In fact,

if we substitute for y' in the equation of the tangent, its value $y'=\frac{b}{a}x'\sqrt{\left(1-\frac{a^2}{x'^2}\right)}$,

that equation, namely, $a^2y'y-b^2x'x=-a^2b^2$, becomes $a^2y\frac{b}{a}x'\sqrt{\left(1-\frac{a^2}{x'^2}\right)}-b^2x'x=-a^2b^2$,

or, $ay\sqrt{\left(1-\frac{a^2}{x'^2}\right)}-bx=-\frac{a^2b}{x'}$, which, when $x'=\infty$, is $y\sqrt{1}=\frac{b}{a}x$,

$\therefore\ y=\pm\frac{b}{a}x$, the equation of the asymptotes.

Two tangents to a branch PBP of an hyperbola, always meet within the angle KON of the asymptotes which embrace that branch: for if they could meet without that angle, as within the angle LOK, then might a tangent be drawn at less inclination to OB than the limiting or ultimate tangent, though the inclination of this is the least possible; for in the equation of the tangent just written, the general expression for the tangent of inclination of the line is

$\frac{b}{a}\div\sqrt{\left(1-\frac{a^2}{x'^2}\right)}$, which is necessarily greater than $\frac{b}{a}$ for all finite values of x'.

From any point, therefore, within the upper angle LOK, only one tangent can be drawn to each branch; and, in like manner, can only one to each branch be drawn from a point within the lower angle MON.

The following two properties of the hyperbola are also readily deduced from the equations of the asymptotes:—

1. If from the focus F, a perp. Fp be drawn to the asymptote, then will Op be equal to the semitransverse $OB=a$, and Fp equal to the semiconjugate b. For the angle pFO, being the complement of pOF,

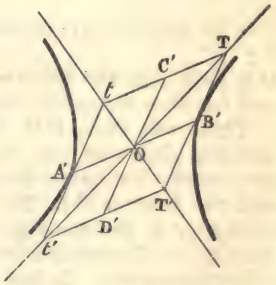
its tangent is $\frac{a}{b}$, and its secant $\sqrt{\left(1+\frac{a^2}{b^2}\right)}$.

Now, OF divided by this secant is Fp, and since $OF^2=c^2=a^2+b^2$,

$\therefore\ Fp^2=(a^2+b^2)\div\left(1+\frac{a^2}{b^2}\right)=b^2$, $\therefore\ Fp=b$. Also $Op=Fp\div\tan O=b\div\frac{b}{a}=a$.

2. All parallelograms inscribed in an hyperbola,* and having their sides parallel and equal to pairs of conjugates, have their vertices on the asymptotes, and every tangent between the asymptotes is bisected at the point of contact.

Although, in what is done above, the asymptotes have been referred to the principal conjugates as axes, yet it is plain, from the manner in which the equations of those lines have been derived, that if any other conjugates had been taken for axes of reference, the equations would still have been of the same form, namely, $y=\pm\frac{b'}{a'}x$.



If in these we put $x=\pm a'$, we shall have for y, the length of the tangent, at each extremity of $2a'$, the value $y=b'$, or rather the two values $y=b'$, $y=-b'$. Hence the length of each of the tangents TT', tt', is $2b'$; and since $2b'$ is the diameter $C'D'$ conjugate to $A'B'$, it follows that the ver-

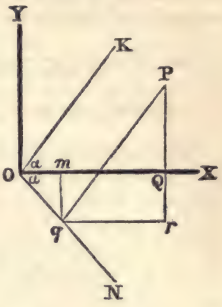
* As the sides *touch* the curve, the parallelograms may be said, with propriety, to be *circumscribed*.

tices T, t, T', t', of the inscribed parallelogram whose sides are parallel and equal to $A'B'$, $C'D'$, are all on the asymptotes, and that TT' is bisected in B'.

We thus see, too, when any transverse $A'B'$ is given, together with the position of the asymptotes, how readily we may draw the conjugate to that transverse, and exhibit its actual length; for it will be parallel and equal to the tangent TT', at the vertex of the transverse, limited by the asymptotes.

---

385. **The Hyperbola related to its Asymptotes as Axes.**—Let OX, OY, be the primitive axes of reference coincident with the principal diameters of the hyperbola, and let the new axes be the asymptotes ON, OK, the angle between which is bisected by OX. Then P being any point on the curve, the new co-ordinates are $x_1=Oq$, $y_1=qP$, the primitive co-ordinates being $x=OQ$, $y=QP$; and, drawing the parallels as in the diagram, it is obvious that



$$\left.\begin{aligned} Om &= Oq\cos\alpha \\ qr &= qP\cos\alpha \end{aligned}\right\} \therefore\ OQ=x=(y_1+x_1)\cos\alpha.$$

$$\left.\begin{aligned} qm &= Oq\sin\alpha \\ rP &= qP\sin\alpha \end{aligned}\right\} \therefore\ QP=y=(y_1-x_1)\sin\alpha.$$

Hence, in the primitive equation of the curve, if we substitute

$$(y+x)\cos\alpha \text{ for } x, \text{ and } (y-x)\sin\alpha \text{ for } y,$$

the transformed equation will be that of the hyperbola referred to its asymptotes as axes.

Putting, then, these values for x and y in [7], p. 331, the transformed equation is

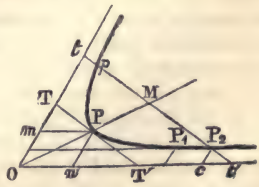
$$a^2(y-x)^2\sin^2\alpha-b^2(y+x)^2\cos^2\alpha=-a^2b^2, \text{ but } \tan\alpha=\frac{b}{a},$$

$$\therefore \cos^2\alpha=\frac{1}{1+\tan^2\alpha}=1\div\frac{a^2+b^2}{a^2}=\frac{a^2}{a^2+b^2} \therefore \sin^2\alpha=\frac{b^2}{a^2+b^2};$$

hence the equa. is $(y-x)^2-(y+x)^2=-(a^2+b^2)$, or $xy=\dfrac{a^2+b^2}{4}$.....[1].

It thus appears that of any point (x, y), in the hyperbola referred to its asymptotes, the rectangle xy, of the ordinate and abscissa, is constant; and since the angle O, between the asymptotes, is fixed, we have these remarkable properties, namely:—

1. All the parallelograms OP, OP_1, OP_2, &c., having O for a common vertex, and a point in the curve for the opposite vertex—the sides from that point being parallel to the asymptotes—are equal in area.



$$\text{For } xy\sin O=\frac{1}{4}(a^2+b^2)\sin O.$$

2. Moreover, the triangle OTT', cut off by any tangent TT' to the curve, is always constant in area. For this tangent is bisected by OP (p. 338), so that OT is bisected by Pm, and OT' by Pn; hence the triangle is double of the parallelogram mn, and therefore

constant in area. [This property might have been inferred from the property 6, at p. 336.]

3. We may also further notice that, since OP bisects not only TT', but also every chord pP_2 parallel to it, as well as the whole line tt', it follows that the intercepts pt, P_2t', between the curve and the asymptotes, are always equal to each other, in whatever direction the line tt' be drawn.

This line cannot meet *both* branches, and yet be parallel to a tangent, but the two intercepts of a line meeting both branches will still be equal; for OC' (fig. p. 338) bisects tT, and it equally bisects the complete chord (p. 335).

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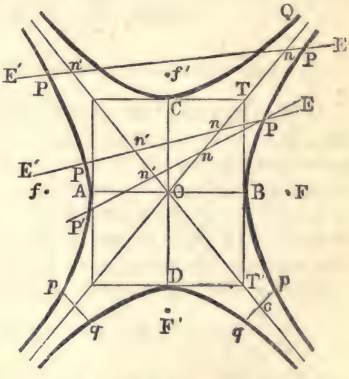
386. **Conjugate Hyperbolas.**—The expression for the square of any semiconjugate diameter b', whose inclination to the principal transverse is α', has been found at page 334, to be

$$b'^2=\frac{a^2b^2}{a^2\sin^2\alpha'-b^2\cos^2\alpha'},$$ where, putting (x, y) for the extremity of b',

$$\sin\alpha'=\frac{y}{b'},\ \cos\alpha'=\frac{x}{b'},\ \therefore \text{ by substitution, } 1=\frac{a^2b^2}{x^2y^2-b^2x^2},\ \text{or } b^2x^2-a^2y^2=-a^2b^2.$$

This equation, then, is that of the locus of the extremities of the conjugate diameters of an hyperbola; and it evidently is itself an hyperbola, differing from the original only in this, that what is the axis of x in the one, is the axis of y in the other; and what is the principal transverse a, of the original, is the principal conjugate of the other. It is, therefore, situated, with reference to the original, as in the diagram below. In virtue of this connection the transverse diameters of one hyperbola are all second diameters of the other, and the two hyperbolas have common asymptotes. For the tangent of the inclination of the asymptotes of the original hyperbola is $\pm\frac{b}{a}$, and the tangent of the inclination of the asymptotes of the conjugate hyberbola, to its principal transverse $2b$, is $\pm\frac{a}{b}$.

But these two principal diameters are at right angles to each other, therefore the two inclinations make together 90°, and consequently the two pair of asymptotes coalesce. It is further obvious that a straight line, through the extremity of any second diameter of one hyperbola and parallel to the conjugate of that diameter, is a tangent to the conjugate hyperbola at that point. This second diameter prolonged must, therefore, bisect all the chords parallel to its conjugate, or to the tangent referred to; and the inscribed, or rather the *circumscribed*, parallelogram at p. 338, has all its four sides in contact with the conjugate hyperbolas.



It is plain that the eight triangles about O, in the annexed diagram, are all equal; and it was shown (at p. 339) that OBT', which is $=ODT'$, is always equal to the parallelogram, whose sides are Oc, cp, or that whose sides are Oc, cq, where pq is parallel to OT; hence, a parallel pq, to one

asymptote, drawn between a pair of conjugate branches, is always bisected by the other asymptote, that is, $pc=qc$.

Now it was shown at p. 340 that if any straight line be drawn intersecting in two points either one branch or the two branches of an hyperbola, the two intercepts of it between the curve and asymptotes will be equal; hence, if the asymptotes be assigned, and a single point P only in one branch be given, we may thence determine as many points in all the four branches as we please. For in whatever direction through P across the asymptotes we draw a straight line EE', if we only mark off $n'P'=Pn$, a second point P' either in the same branch as P is, or in the opposite branch, will be determined. And, as just proved, any point P in one hyperbola suggests a corresponding point Q in the conjugate hyperbola, the parallel PQ to OT' being bisected by OT; hence we may determine as many points as we please in the four branches.

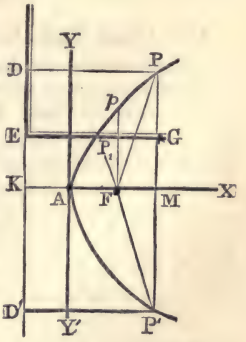
We can notice here but one more particular, namely, that the four foci, F, f, F', f', are all equidistant from O, for the common distance is

$$c=\sqrt{\frac{a^2+b^2}{a^2}},$$

so that a circle from O, with radius OF, will pass through all the foci.

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387. **The Parabola.**—This, like the ellipse, is a single continuous curve; but it is not a closed curve; for instead of returning to itself, it may be prolonged indefinitely. It is such a curve that, whatever point P in it be taken, the distance of P, from a certain fixed point F, called the *focus*, is always the same as its distance from a certain fixed straight line, called the *directrix*. Thus, in the annexed diagram, if F be any assumed fixed point, and DD' any assumed fixed straight line, then if P be determined so that $PF=PD$, the locus of P will be a parabola. The distance FP is called the *radius vector*, or *focal distance*, of the point P; and the point A on the curve, where the perp. FK from the focus to the directrix cuts it, the *principal vertex*:—it is plain that A bisects FK. This vertex, the focus, and the directrix, being given, any limited portion of the parabola to which these belong, may be described by continuous motion, thus:—To the directrix DD', apply one of the two sides of a *square* DEG; and to the points G, F, fasten the ends of a chord, equal in length to EG: then with a pencil P_1, stretch this chord, so that the part of it P_1G may always lie along the side EG of the square, while the other side DE is moved along the directrix DD': the pencil P_1 will then describe a portion of the parabola, for we shall always have $P_1F+P_1G=P_1E+P_1G$, and therefore always $P_1F=P_1E$.



388. **Equation of the Parabola.**—Let the perpendicular AX to the directrix DD' (figure above), and the parallel AY to the same line, be taken for axes of reference. Put $AF(=AK)=m$: then, P being any point (x, y) on the curve, we must have the following condition, namely:—

$FP^2=PD^2=y^2+(x-m)^2$, but $PD^2=(KA+AM)^2=(m+x)^2$,

$\therefore\ y^2+(x-m)^2=(x+m)^2$, $\therefore\ y^2=4mx$.....[1], the equation of the parabola.

From this equation we have, $x=\frac{y^2}{4m}$, $y=\pm\sqrt{4mx}=\pm 2\sqrt{mx}$.

The first expression shows that the curve lies entirely to the right of the axis of y, since x is always positive: this axis is also a tangent to the curve, since for $x=0$, we have $y=0$, and no other value; showing that (0, 0) is the only point common to the axis and curve. The second equation shows that for each abscissa there are two equal ordinates, on opposite sides of AX; so that AX bisects all the chords parallel to AY: it moreover shows that y is always possible, so long as x is positive, however long x may be; so that the curve extends without limit to the right of AY, and both above and below AX, which line is called the *principal axis* of the parabola; and AY is the *principal second axis*. These axes are both unlimited in length: but the former, since it bisects all chords parallel to the latter, is called a *diameter*—the *principal* diameter.

As in the other curves, the double ordinate through the focus is called the *latus rectum*, or *principal parameter* of the curve: the equation of the parabola, in terms of this parameter p, is easily obtained from [1]; for we have only to put m for x in that equation, and we have

$y^2=4m^2$, or $y=2m=\frac{1}{2}p$, $\therefore\ y^2=px$.....[2] is the equa. in terms of the parameter.

And since $2m=\frac{1}{2}p$, we see that the semiparameter Fp is equal to the distance FK of the focus from the directrix, as is indeed also evident from the construction of the curve.

The equation [2] shows that the abscissas of different points of the curve are to one another as the squares of the ordinates.

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389. **The Parabola referred to Conjugate Axes.**—As in the other curves, we shall now inquire what must be the peculiarities of those oblique axes, in reference to which the equation of the parabola preserves the same form. In order to this, we must substitute, in the equation [2], the expressions at p. 316, and thus transform that equation from rectangular to oblique co-ordinates; and we shall, at the same time, remove the origin from (0, 0), to a point (a, b), on the curve; that is, we shall put

$$a+x\cos\alpha+y\cos\alpha' \text{ for } x, \text{ and } b+x\sin\alpha+y\sin\alpha' \text{ for } y,$$

in [2] above, and we shall thus get, after transposing, the equation

$$\left.\begin{array}{l} y^2\sin^2\alpha'+2xy\sin\alpha\sin\alpha'+x^2\sin^2\alpha+b^2-ap+ \\ (2b\sin\alpha'-p\cos\alpha')y+(2b\sin\alpha-p\cos\alpha)x \end{array}\right\}=0.....[1].$$

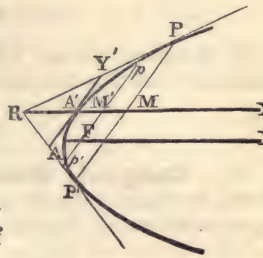
And that this may have the desired form $y^2=p'x$, we must have

$$\sin\alpha\sin\alpha'=0,\ \sin^2\alpha=0,\ 2b\sin\alpha'-p\cos\alpha'=0,\ b^2-ap=0.....[2],$$

the last implying merely that the new origin (a, b) must be on the curve; but this point, not being restricted to any particular position, may be *any* point (A') on the curve whatever.

The second condition necessitates the first, and shows that since $\alpha=0$ the new axis of x is parallel to the primitive AX; and from the third, we learn that the inclination α', of the new axes to each other, is

$\tan\alpha'=\frac{p}{2b}$: moreover, since $\sin\alpha=0$, $\therefore\cos\alpha=1$.



Under these restrictions, the equation [1] is

$$y^2=\frac{p}{\sin^2\alpha'}x, \text{ or } y^2=p'x\ldots\ldots[3];$$

and since, as already noticed, the origin may be anywhere on the curve, and therefore b of any value, positive or negative, and consequently, since $\tan\alpha'=p\div 2b$, the axes may have any inclination whatever, it follows that there are innumerable systems of co-ordinates, in reference to which the equation of the parabola is of the form [3]. And we further see that, since $\tan\alpha'$ diminishes as b increases, the more remote the origin is placed from A, the less becomes the angle α' at which the axis of y is inclined to that of x, or to the principal diameter AX.

Equation [3] being of the same form as [2], at p. 342, we draw from it similar inferences, thus:—

1. The axis of x bisects all chords parallel to the axis of y, which latter axis is always a tangent to the curve at the origin. Lines thus bisecting parallel chords are always called *diameters:* hence all parallels to the principal diameter are themselves diameters. The coefficient p' in equa. [3], which is of course fixed for any particular diameter taken for axis of x, is called the *parameter* of that diameter. And any diameter, with the tangent at its vertex, form a pair of conjugate *axes.*

2. The abscissas, measured on any diameter, are as the squares of the corresponding ordinates, measured parallel to the tangent at the vertex of that diameter.

In order to find the value of the parameter $\frac{p}{\sin^2\alpha'}=p'$, independent of trig. quantities,

we have only to observe that, since $\tan^2\alpha'=\frac{p^2}{4b^2}$, $\therefore\sec^2\alpha'=1+\frac{p^2}{4b^2}$,

$$\therefore\frac{1}{\sin^2\alpha'}=\frac{\sec^2\alpha'}{\tan^2\alpha'}=\frac{4b^2+p^2}{p^2}=\frac{4a+p}{p}\text{ by [2], }\therefore\frac{p}{\sin^2\alpha'}=4a+p=4\left(a+\frac{p}{4}\right)=4(a+m).$$

Now by the definition, $a+m$ is the distance of the point A' in the curve, of which the abscissa is a, from the focus: hence, putting for this distance, r, the radius vector of the origin of co-ordinates, the general equation of the parabola, in reference to any conjugate axes whatever, is

$$y^2=4rx\ldots\ldots[4].$$

The parameter p' of any diameter is thus always equal to the double ordinate corresponding to the abscissa $x=r$. We shall now show that this double ordinate necessarily passes through the focus.

Let $A'M'=r$, and let pp' be a chord through M', parallel to the tangent $A'Y'$. Refer the curve to the principal axes originating at A: then the abscissa of A' being a, and its ordinate b, we have for the abscissa of M', $x=a+r=2a+m$, and for its ordinate b. Also for the tangent of the inclination α', of pp', we have

$$\tan\alpha'=\frac{p}{2b}\text{: hence the equa. of } pp' \text{ is } y-b=\frac{p}{2b}(x-2a-m).$$

At the point where pp' crosses AX, $y=0$, for which value the equa. gives

$$x=2a+m-\frac{2b^2}{p}=2a+m-2a=m=AF.$$

Thus the chord through the focus is always equal to the parameter of the diameter bisecting that chord.

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390. **Tangent, Normal, &c.**—To find the equations of these lines, we proceed exactly as we did in reference to the former curves: the equa. of a straight line through two points (x', y'), (x'', y''), on the curve, is

$$y-y'=\frac{y'-y''}{x'-x''}(x-x'), \text{ where } y'^2=4rx', \text{ and } y''^2=4rx'',$$

$$\therefore (y'+y'')(y'-y'')=4r(x'-x''), \therefore \frac{y'-y''}{x'-x''}=\frac{4r}{y'+y''}; \text{ hence the equa. is}$$

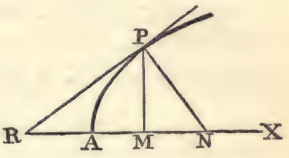
$$y-y'=\frac{4r}{y'+y''}(x-x'), \text{ which, when the points coincide, becomes}$$

$$y-y'=\frac{2r}{y'}(x-x'), \text{ or } yy'=2r(x+x').....[5],$$

the equation of the tangent at the point (x', y'). The second form is got from the first by multiplying by y', and then putting for y'^2 its equal $4rx'$. As the normal is a straight line through the same point perpendicular to this, we have for its equation, in reference to the rectangular conjugate axes,

$$y-y'=-\frac{y'}{2m}(x-x')...[6], \text{ Equa. of the Normal.}$$

Putting $y=0$ in [5], the resulting value of x, namely AR, is $x=AR=-x'=-AM$, $\therefore$ the subtangent $MR=2AM$, whatever diameter AX may be. The subtangent MR at any point P is therefore twice the abscissa of that point. In like manner, putting $y=0$ in [6], we have



$x-x'=2m=MN$, the subnormal.

For the length PR of the tangent, we have $PR^2=MP^2+MR^2=y'^2+4x'^2=4(mx'+x'^2)$,

and for the length of the normal, $PN^2=MP^2+MN^2=y'^2+4m^2=4(mx'+m^2)$.

At the vertex A, where $x'=0$, the length of the normal is $2m$; so that, as in the former two curves, the length of the normal at the vertex of the principal diameter, is equal to that of the principal semiparameter. The equation of the tangent [5] may be put in another form, from which the co-ordinates x', y', of the point of contact, shall be eliminated, and in which the coefficient (n), of the inclination of it to the diameter, shall appear instead. For, writing that equation thus,

$$y=\frac{2r}{y'}x+\frac{2rx'}{y'}, \text{ where } \frac{2r}{y'} \text{ is now to be replaced by } n, \text{ we shall have,}$$

$$\text{from the equation of the curve, } \frac{2r}{y'}=\frac{y'}{2x'}=n,$$

$$\therefore \frac{2rx'}{y'}=\frac{r}{n}, \therefore y=nx+\frac{r}{n} \text{ is the equa. of the tangent [7].}$$

From the foregoing expressions the following properties are at once deducible :—

1. The expression for the subtangent being independent of the constant r, it follows that for all parabolas, having a pair of conjugate axes in common, the subtangent, measured on the common diameter, is always the same for the same abscissa.

2. And, in like manner, from the expression for the subnormal, it follows that, for every point in the curve, the subnormal is constant, and equal to the distance of the focus from the directrix.

3. As in the ellipse and hyperbola, two tangents from P, P', the extremities of any chord, meet in the same point R of the diameter bisecting that chord; for the expression for MR remains the same, whether y' be plus or minus. [See fig. p. 343.]

4. The tangent at A', the vertex of the diameter, being parallel to the chord PP', the part of it, intercepted between the two tangents at P, P', is bisected at the point of contact A', the same as in the other curves.

We shall now proceed to develope some properties requiring a little additional investigation.

391. **Properties of the Parabola connected with Tangents.**—The last form [7], given to the equation of the tangent, enables us very readily to solve problems concerning the parabola analagous to those already solved for the ellipse, when a similar modification of the common form for the tangent is employed; thus,

PROB. I. To determine the equation of the tangent to a parabola drawn from an external point (x_1, y_1).

The point (x_1, y_1) being on the tangent, we have from [7] above,

$$y_1 = nx_1 + \frac{r}{n}, \quad \therefore n^2 - \frac{y_1}{x_1}n + \frac{r}{x_1} = 0, \text{ giving two values for } n.$$

Hence, through the same point two tangents may be drawn to the curve, their equations being $y = nx + \frac{r}{n}$, and $y = n'x + \frac{r}{n'}$, where $(n+n') = \frac{y_1}{x_1}$, and $nn' = \frac{r}{x_1}$. If the two tangents intersect at a right angle, then, regarding the axes of reference as rectangular, and therefore putting m for r, we must have the condition $nn' = \frac{m}{x_1} = -1$, $\therefore x_1 = -m$; hence the locus of the point, from which a pair of tangents are always at right angles, is the directrix of the parabola.

PROB. II. To find the equation of the chord joining the two points of contact of a pair of tangents from (x_1, y_1).

Taking the other equation [5] of the tangent, the points of contact being (x', y'), and (x'', y''), we have, since (x_1, y_1), is a point on each tangent,

$$y_1 y' = 2r(x_1 + x'), \text{ and } y_1 y'' = 2r(x_1 + x'').$$

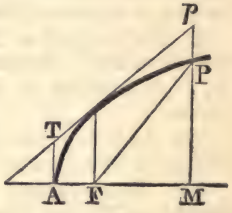
Hence the two points (x', y'), (x'', y''), must lie upon the straight line

$$y_1 y = 2r(x_1 + x), \text{ the equation of the chord of contact.}$$

This line (as in the ellipse) is the polar of the point (x_1, y_1), and it may be shown, in precisely the same manner as at p. 323, that if from any point (x_1, y_1), in a straight line, tangents be drawn to the parabola, the several chords of contact intersect in a point, which is the *pole* of that straight line, the line itself being the *polar* of the point. The polar of the focus, for which $x=m$, and $y=0$, has for its equation $x=-m$, the directrix, and we have just seen that the pairs of tangents from *this* always intersect at right angles.

The tangent, through an extremity of the latus rectum, is called the *focal tangent.*

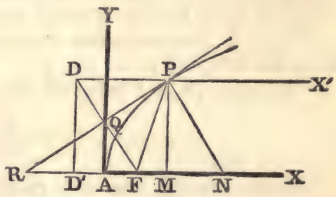
THEO. I. Any rectangular ordinate to the focal tangent is equal to the radius vector of the point, where that ordinate cuts the curve. Referring the curve to its principal axes, when the abscissa of the point of contact is $x'=m$, we see, from the expressions at (390), that the length of the tangent is equal to that of the normal, and that the subtangent is always equal to the subnormal. The equa. of the tangent is



$$y-2m=\frac{2m}{2m}(x-m), \text{ or } y=x+m, \text{ that is, } Mp=FP.$$

Let $x=0$, then $y=m$, that is, $AT=AF$.

THEO. II. The radius vector, and the diameter at any point of the parabola, are equally inclined to the tangent at that point.



To $MN=2m$, add $FM=x'-m$,

$\therefore FN=x'+m=FP, \therefore FPN=FNP=NPX'.$

[Rays of light or heat, issuing from the point F, will all be reflected in a direction parallel to the principal axis AX, since incident and reflected rays always make equal angles with the normal at the point on which they impinge. Also rays, striking the curve in directions parallel to XA, will all be reflected to and unite in the point F: it is on this account that F is called the *focus*, or burning point.]

COR. Since (p. 344), $FR=x'+m$, $\therefore FR$, FN, FP, are all equal.

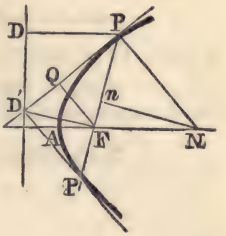
THEO. III. A perpendicular from the focus on any tangent always meets that tangent on the principal second axis, or tangent at the principal vertex.

Let FQD be the perp. to the tangent PR; then since $PF=PD$, Q is at the middle of FD, and A is at the middle of FD', $\therefore Q$ is on the parallel AY, to $D'D$.

COR. Hence $FQ^2=FR \cdot FA=FP \cdot FA$, and FA is constant, therefore the perp. from the focus on any tangent varies as the square root of the radius vector of the point of contact.

THEO. IV. The part of the tangent, between the point of contact and the directrix, always subtends a right angle at the focus: in other words, the perp. to a focal chord from the focus always meets the tangents at the extremities of that chord at their point of intersection.

Let FD' be drawn to the point where the tangent at P crosses the directrix; we shall then have the two sides PD, PD', and the included angle, equal respectively to PF, PD', and the included angle, $\therefore$ the angles PDD', PFD', are equal, $\therefore$ PD' subtends a right angle at F.



Otherwise. The pole of PFP', that is, the point of concourse of the two tangents from P, P', is some point D' on the directrix (391); the co-ordinates of it will therefore be $(-m, y_1)$. The equation of PP' is $y_1y=2m(x-m)$,

$$\text{or } y=\frac{2m}{y_1}(x-m),$$

the axes being rectangular, and the equation of a perp. to PP' from F, or $(m, 0)$, is $y=-\frac{y_1}{2m}(x-m)$. To find where this meets the directrix, put $x=-m$, then $y=y_1$; hence it meets it in the point $(-m, y_1)$, that is, in D'; therefore a perp. from the focus to a focal chord meets the tangent on the directrix, the point of meeting being the pole of that chord.

THEO. V. A perpendicular to any radius vector, from the foot of the normal, cuts off from that radius a part, measured from P, equal to the principal semiparameter.

The part Pn cut off will evidently be $PN\cos FPN=PN\sin FPQ=PN\frac{FQ}{FP}$.

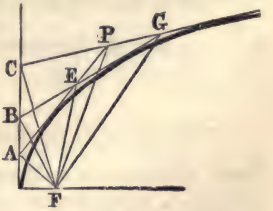
But (390), $PN=2\sqrt{(mx'+m^2)}$, and (p. 346), $\frac{FQ^2}{FP^2}=\frac{FA}{FP}=\frac{m}{x'+m}$;

hence the part cut off is $Pn=2\sqrt{(mx'+m^2)}\sqrt{\frac{m}{x'+m}}=2m=$the semiparameter,

the same as in the central curves before discussed.

THEO. VI. If two tangents be drawn from any external point, and a third tangent be intercepted between them, then the circle, circumscribing the triangle thus formed, will always pass through the focus.

Let PEG be the triangle formed by the intersections of the three tangents. The perpendiculars on these tangents from F will meet them in points A, B, C, all situated on the principal second diameter; and it is plain, since FG subtends each of the right angles B, C, that a circle may be described through F, B, C, G; hence the angles CGB, CFB, in the same segment are equal. In like manner a circle will pass through F, A, B, E, $\therefore$ $FBA=FEA$. Now $FBA=FCB+CFB=FGB+CGB=FGC=FEA$; but $FEA+FEP=180°$, $\therefore$ $FGC+FEP=180°$; hence the points F, E, P, G, all lie on the circumference of a circle.



COR. From whatever point in a tangent to a parabola two lines be drawn, one to the focus, and one to touch the curve, the two will always include the same angle, for, as here shown, $FGC=FEA=FBA$.

392. **Polar Equation of the Parabola.**—Taking the focus for the pole, and the principal diameter for the fixed axis, we have

$$r=m+x, \text{ and } x=m+r\cos\theta, \therefore r=2m+r\cos\theta, \therefore r=\frac{2m}{1-\cos\theta}$$

is the polar equation of the curve.

By referring to the value of p, the principal parameter, in each of the three curves, we find that

$$\text{In the Ellipse, } \frac{1}{2}p=a(1-e^2). \quad \text{In the Hyperbola, } \frac{1}{2}p=a(e^2-1).$$

$$\text{In the Parabola, } \frac{1}{2}p=2m.$$

Hence, by substituting $\frac{1}{2}p$ for these values, in the polar equations of the respective curves, the focus being the pole, all become included in a single equation, namely,

$$r=\frac{\frac{1}{2}p}{1-e\cos\theta}\ldots\ldots[1],$$

where, for the parabola, $e=1$. That this is really the value of e in the parabola will appear from considering the meaning of the symbol in the other curves. The distance of the focus from the centre, or point in which all the diameters intersect, is denoted by c, and $\frac{c}{a}$ is e. Now in the parabola the diameters, being parallel, can be said to meet only at an infinite distance from the focus; hence in *this* curve, $c=\infty$, and $a=c+m$ is also infinite,

$$\therefore \frac{c}{a}=e=\frac{c}{c+m}=\frac{1}{1+\frac{m}{c}}=\frac{1}{1+0}=1.$$

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393. The three curves may all be comprehended in a single equation, not only when polar co-ordinates are employed, but also when rectilinear co-ordinates are used, their origin being at the vertex of a diameter. For in both the ellipse and hyperbola the semiparameter of any diameter a, is $\frac{b^2}{a}$; so that all the three curves are included in the equation,

$$y^2=px+qx^2\ldots\ldots[2],$$

where $q=-\frac{p}{2a}$ in the ellipse, $+\frac{p}{2a}$ in the hyperbola, and 0 in the parabola.

These names seem to have been given to the curves either because in the first, $e<1$, in the second, $e>1$, and in the third, $e=1$, or because in the first the distance of any point in the curve from the focus is *less* than its distance from the directrix, in the second *greater*, and in the third the distances are *equal*.

It appears, from what is here said, that the properties of the three conic sections might be investigated by discussing either or both of the general equations [1], [2], giving to e, p, and q, the values proper to each

curve in the results. Or, just as in the preceding articles, much that concerned the hyperbola was deduced by slightly modifying the conclusions arrived at for the ellipse, so might the principal properties of the parabola have been derived, in like manner, from those of the same curve, a parabola being regarded as an extreme ellipse, or as one whose opposite vertices are infinitely distant from each other. Thus, it was shown in the ellipse that perpendiculars, from a focus to a series of tangents, had all their intersections with those tangents in the circumference of a circle, the diameter of which was the principal diameter of the ellipse. Had the ellipse degenerated into a parabola, by the removal of the centre to infinity, the radius of the circle spoken of would have become infinite; but a circular arc of infinite radius must be a straight line, perp. to that radius at its extremity: and hence the property of the parabola at p. 346.

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394. **Area of the Parabola.**—The parabola being referred to any system of conjugate axes, let Ax, xy, be the abscissa and ordinate of any point in the curve. Divide xy into any number n of equal parts, from the extremities of which let parallels be drawn to the curve, as in the figure; then Pp will be equal to one of these parts (k), and the ordinates Pp, Qq, &c., of the points P, Q, &c., will be

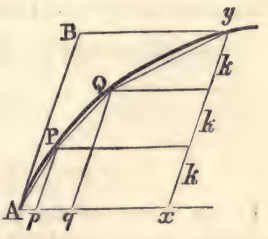
$$k,\ 2k,\ 3k,\ \&c.\ldots\ldots[1],$$

and since $y^2=4rx$, the abscissas of the same points will be

$$\frac{k^2}{4r},\ \frac{4k^2}{4r},\ \frac{9k^2}{4r},\ \&c.,$$

and consequently the distances Ap, pq, &c., will be

$$\frac{k^2}{4r},\ \frac{3k^2}{4r},\ \frac{5k^2}{4r},\ \&c.\ldots\ldots[2].$$



Now the chords AP, PQ, &c., being drawn, after the triangle on the base $Ap=\frac{k^2}{4r}$, there will be a series of trapezoids on the bases [2], and the area of each trapezoid will be found by multiplying half the sum of the two parallel sides by the base, and by the sine of the angle α, at which the axes are inclined, for the product of the base by this sine evidently gives the perp. distance between the parallel sides. Hence, commencing with the second of the ordinates [1], and adding each to the preceding, we have for the sum of the trapezoids (including the triangle APp) the expression

$$\text{Area of ins. polygon} = \frac{k^3}{8r}\{1+3^2+5^2+\ldots+(2n-1)^2\}\sin\alpha, \text{ where } k=\frac{y}{n},$$

$$\text{and } \therefore \frac{k^3}{8r}=\frac{y^3}{8r\,.\,n^3}=\frac{xy}{2n^3}, \text{ since } y^2=4rx.$$

$$\text{Now by (198), or (199), } 1+3^2+5^2+\ldots+(2n-1)^2=\frac{4n^3-n}{3},$$

$$\therefore \text{Area}=\left(\frac{2}{3}xy-\frac{xy}{6n^2}\right)\sin\alpha.$$

The greater the number n, the closer does the polygonal area approach

to that of the curvilinear space Ayx, and the two coalesce when $n=\infty$. Hence the parabolic area is $\frac{2}{3}xy \sin \alpha$, or $\frac{2}{3}$ of the parallelogram Bx; therefore any segment of a parabola is two-thirds of its circumscribing parallelogram.

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395. **Radius of Curvature.**—The point (α, β), where normals at (x', y'), (x'', y''), two points on the curve, intersect, will be determined from the equations of those normals, when α, β, are put for x, y, that is, from the equations

$$2m\beta+y'\alpha-2my'-x'y'=0, \text{ and } 2m\beta+y''\alpha-2my''-x''y''=0.$$

$$\text{By subtraction, } \alpha=\frac{2m(y'-y'')+x'y'-x''y''}{y'-y''}=2m+\frac{x'y'-x''y''}{y'-y''}.$$

$$\text{But } x'y'-x''y''=y'(x'-x'')+x''(y'-y''), \therefore \alpha=2m+x''+y'\frac{x'-x''}{y'-y''}; \text{ but (390),}$$

$$\frac{x'-x''}{y'-y''}=\frac{y'+y''}{4m}: \text{ hence when } x'=x'', \text{ and } y'=y'', \alpha=2m+x'+\frac{y'^2}{2m}=2m+x'+\frac{4mx'}{2m},$$

$$\text{that is, } \alpha=2m+3x', \therefore \text{ substituting in the first equation, } \beta=-\frac{x'y'}{m}.$$

These are the co-ordinates of the centre of curvature of the parabola at the point (x', y'): and for the radius of curvature, we have

$$r^2=(2m+2x')^2+\left(y'+\frac{x'y'}{m}\right)^2=4(m+x')^2+\frac{y'^2}{m^2}(m+x')^2=\left(4+\frac{y'^2}{m^2}\right)(m+x')^2=$$

$$\frac{4m^2+4mx'}{m^2}(m+x')^2=\frac{4}{m}(m+x')^3,$$

$$\therefore r=\frac{2}{\sqrt{m}}(m+x')^{\frac{3}{2}}=\frac{2\sqrt{m}}{m}(m+x')^{\frac{3}{2}}=\frac{(\text{Normal})^3}{4m^2} \text{ (390).}$$

At the vertex of the principal diameter, where $x'=0$, $y'=0$, we have $r=2m$, the semiparameter, as in the ellipse. And generally, the circle of curvature at any point, cuts from the diameter of the curve at that point, a part equal to the parameter of that diameter.

For the perp. from the centre (α, β) of the circle, to this diameter of the curve, must bisect the portion of it cut off by the circle, since this portion is a chord of the circle. The semichord will therefore be the radius of the circle multiplied by the cosine of its inclination to the diameter of the curve referred to, which cosine is the sine of the angle between the normal and the ordinate of the point, and is therefore expressed by the ratio $\frac{MN}{PN}$. (See fig. at p. 344.)

$$\text{But } \frac{MN}{PN}=\frac{2m}{2\sqrt{(mx'+m^2)}}=\frac{\sqrt{m}}{\sqrt{(x'+m)}} \text{ (p. 344);}$$

and this, multiplied by the above expression for r, gives

$2(m+x')$=the semiparameter of the diameter at (x', y') [p. 343]:

hence the diametral (we cannot call it the *central*) chord of curvature is always equal to the parameter.

The semichord of curvature through the focus is, in like manner the product of the radius by the cosine of the angle between the normal and

radius vector of the point (x', y'), or by the cosine of that between the radius vector and the perp. on the tangent, which cosine is expressed by

the ratio $\frac{FQ}{FP}$ (p. 347).

But $\frac{FQ}{FP}=\frac{\sqrt{m}}{\sqrt{(x'+m)}}$ (p. 347),

and this, multiplied by r, gives the same expression as before, so that the diametral chord of curvature is always equal to the focal chord of curvature. In the parabola, therefore,

$4FP$=the focal chord of curvature.

Since $\left(\frac{FP}{FQ}\right)^3=\left(\frac{x'+m}{m}\right)^{\frac{3}{2}}$, $\therefore r=\left(\frac{FP}{FQ}\right)^3 2m=\left(\frac{FP}{FQ}\right)^3\cdot\frac{1}{2}p$,

as in the other curves: and since in the parabola $FQ^2=FP\,.\,FA$, $\therefore r=2\frac{FP^2}{FQ}$.

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396. **Locus of the Equation of the Second Degree.**— We have now to show that every equation of the second degree with two variables is the algebraic representation of a conic section, whenever it is any locus at all.

In the more simple cases, where the equation proposed is in one or other of the forms

$$My^2+Nx^2=P\text{.....[1], or } y^2=Qx\text{.....[2],}$$

it is easily shown that the locus of the first is either an ellipse or an hyperbola, and that the locus of the second is a parabola. For let a^2 be written for $\frac{P}{N}$, and b^2 for $\frac{P}{M}$; then we shall have $N=\frac{P}{a^2}$, and $M=\frac{P}{b^2}$, so that [1] is converted into

$$\frac{P}{b^2}y^2+\frac{P}{a^2}x^2=P,\ \therefore\ \frac{1}{b^2}y^2+\frac{1}{a^2}x^2=1,$$

which we know to be the equation either of an ellipse or of an hyperbola: the former when the constants a^2, b^2, are both positive, and the latter when one of them is positive and the other negative; that which is negative applying to the second diameter of the other, which is the corresponding transverse. Either may of course be coincident with the axis of x; in other words, the hyperbola may take either of two conjugate positions.

If, however, P alone is negative, then [1] has no geometrical interpretation: the locus is imaginary, for since, in this case,

$$y=\sqrt{\left(-\frac{N}{M}x^2-\frac{P}{M}\right)},$$

whatever real value be given to x, the corresponding y will be imaginary, showing that no real locus exists. But if P is zero, and M, N, both positive, [1] represents merely a point, which, with the circle $(M=N)$, are *varieties* of the ellipse. If, P being zero, one of the other two coefficients is negative, N for instance, then [1] gives

$$y=\pm\sqrt{\frac{N}{M}}\,.\,x,\text{ or } y=\pm\frac{b}{a}x,$$

the asymptotes of the hyperbola $\frac{1}{a^2}x^2-\frac{1}{b^2}y^2=1$.

Thus the two straight lines passing through the origin, and of which the equations are

$$\sqrt{M} \cdot y + \sqrt{N} \cdot x = 0, \quad \sqrt{M} \cdot y - \sqrt{N} \cdot x = 0,$$

denote a variety of the hyperbola [1], N being negative and $P=0$.

The form of the equation [2] shows at once that the locus is a parabola, the parameter of the curve being the constant Q; we should regard this parameter as positive, even though Q were negative; for the equation would then merely imply that the curve is situated wholly to the left, instead of to the right of the axis of y; the abscissa of each point in it being negative.

The smaller Q becomes, the nearer do the two opposite portions of the curve approach towards each other; and they actually coalesce when $Q=0$, and become confounded with the axis of x: hence a single straight line—the axis of x, is a variety of the parabola. There is also another variety of this curve: suppose the origin of the axes to be removed towards the right, or, which is the same thing, imagine the vertex of the curve—still remaining on the axis of x—to recede towards the left, to a distance k from its original position: the equation of the curve will then be $y^2=Q(x+k)$, which we may write $y^2=Q\left(x+\frac{K}{Q}\right)$, where Q and K are arbitrary constants. Let now $Q=0$, then we shall have $y^2=K$, $\therefore y=\pm K$, which denote two straight lines, parallel to and equidistant from, the axis of x.

397. Let us now take the equation of the second degree in its most general form, namely,

$$Ay^2+Bxy+Cx^2+Dy+Ex+F=0 \ldots\ldots [1].$$

And first we may remark that, whatever this equation may be, the form of the locus cannot be altered by altering the direction of the axes to which it is referred: we may consider at the outset that these axes are rectangular, for if they were originally oblique we could change them for rectangular by a process similar to that at page 318, and it is plain that the transformed equation could not be *more* general than [1], since the form of this is the most general possible.

Now to convert this equation into another, referring the locus to a new system of rectangular axes, we have but for the primitive x and y to put (p. 316),

$$x = x \cos \alpha - y \sin \alpha, \quad y = x \sin \alpha + y \cos \alpha,$$

and we shall find, for the coef. of xy, the expression,

$$2A \sin \alpha \cos \alpha + B \cos^2 \alpha - B \sin^2 \alpha - 2C \sin \alpha \cos \alpha;$$

consequently we have only so to determine α that this may become zero, in order that the term in xy may disappear from the equation of the locus; that is, we have only to put

$$2(A-C) \sin \alpha \cos \alpha + B(\cos^2 \alpha - \sin^2 \alpha) = 0, \text{ or } (A-C) \sin 2\alpha + B \cos 2\alpha = 0,$$

$$\text{and we get the condition, } \tan 2\alpha = -\frac{B}{A-C} \ldots\ldots [2],$$

which being fulfilled for α, the angle which the new axis of x makes with the primitive, the resulting equation of the locus must be of the form

$$My^2+Nx^2+Ry+Sx+F=0 \ldots\ldots [3].$$

Hence the term in [1], containing the product of the variables, becomes removed by merely turning the axes round through an angle α.

398. We shall now see that, by simply shifting the origin of these new axes, without disturbing their directions, we may next remove the terms containing only the first powers of the variables.

In order to shift the origin from the point (0, 0) to the point (a, b), we have merely to put in [3], $x=x+a$, and $y=y+b$, when that equation will become $My^2+Nx^2+(2Mb+R)y+(2Na+S)x-P=0$, where $-P$ is written for the quantity independent of the variables. Now, that the first powers may disappear, we have only to determine the constants a, b, so as to fulfil the conditions

$$2Mb+R=0, \text{ and } 2Na+S=0, \therefore b=-\frac{R}{2M}, \text{ and } a=-\frac{S}{2N}.$$

Hence, by fixing the values of the otherwise arbitrary constants α, a, b, so as to conform to the three conditions,

$$\tan 2\alpha=\frac{B}{C-A}, \quad a=-\frac{S}{2N}, \quad b=-\frac{R}{2M},$$

we see that the equation of the locus [1] becomes reduced to the form

$$My^2+Nx^2=P.$$

399. In the foregoing reasoning, it is assumed that neither My^2 nor Nx^2 has disappeared from the equation [3]. Should, however, either of these, as Nx^2, be absent, then the coef. of x, in the subsequent transformed equation, would be simply S, the same as in [3] itself; and this, of course, we could not equate to zero. If, therefore, the term Sx be not absent from [3], as well as Nx^2, it cannot be removed from the transformed equation: we may then determine a so that the *final* term, $-P$, shall disappear: this final term is $Mb^2+Rb+Sa+F$, which, equated to zero, fixes for a the value $a=-\dfrac{Mb^2+Rb+F}{S}$; and with this value for a, and the above value $-\dfrac{R}{2M}$, for b, the equation of the locus takes the form

$$My^2+Sx=0, \text{ or the form } y^2=Qx, \text{ where } Q \text{ is put for } -\frac{S}{M}.$$

And, in like manner, if My^2 had been supposed absent from [3], instead of Nx^2, the reduced form would have been, $Nx^2+Ry=0$, or

$$x^2=Q'y, \text{ where } Q' \text{ is put for } -\frac{R}{N}.$$

If M and N were both zero in [3], the equation would no longer be one of the second degree: but if the terms containing the first and second powers of the same variable, as, for instance, the terms Sx, Nx^2, be absent from [3], then the equation, though of the second degree, will contain but one variable; so that the locus will be a pair of parallel straight lines:

$$\text{for, } My^2+Ry+F=0, \text{ gives } y=-\frac{R}{M}\pm\frac{1}{2M}\sqrt{(R^2-4FM)},$$

and these can exist only so long as $R^2>4FM$. If D and E be both absent from [1], R and S must both be absent from [3]: the axes $\therefore$ must then originate at the centre, if the curve have a centre (396).

It thus appears that the locus of the general equation [1] is always a *conic section*, or else a variety of one or other of the curves thus called.

It remains for us now to inquire how we may find out which of the three curves any given equation represents.

400. **Form of the Locus.**—Writing the general equation [1], p. 352, thus,

$$y^2+\frac{Bx+D}{A}y+\left(\frac{C}{A}x^2+\frac{E}{A}x+\frac{F}{A}\right)=0,$$

and solving it as a quadratic in y, we have for y the expression

$$y=-\frac{Bx+D}{2A}\pm\frac{1}{2A}\sqrt{\{(B^2-4AC)x^2+2(BD-2AE)x+D^2-4AF\}}\ldots\ldots[1],$$

which will be real or imaginary, for real values of x, according as the quantity under the radical is positive or negative. It will not interfere with the sign of this quantity if we divide it by x^2, since x^2 is positive, whether x be + or −. Hence the sign of the irrational part of [1] will be the same as the sign of

$$(B^2-4AC)+2(BD-2AE)\frac{1}{x}+(D^2-4AF)\frac{1}{x^2}.$$

Now (194) a value so small may be given to $\frac{1}{x}$ as to cause the sign of this expression to be the same as the sign of its first term (B^2-4AC), for this small value, and for all smaller values of $\frac{1}{x}$, whether positive or negative. In other words, there exists some value for x such that for it, and for all numerically greater values, positive and negative, the value of y in [1] will be real if B^2-4AC is *positive*, and imaginary if it be *negative*. But the only one of the three curves—one or other of which, as already shown, must be the locus of [1]—of which, after a certain pair of values positive and negative for x, the ordinates are all *real*, however great $\pm x$ may be, is the *Hyperbola*. And the only curve of which, after a certain pair of values positive and negative for x, the ordinates are all *imaginary*, is the *Ellipse*.

Hence, if $(B^2-4AC)<0$, the locus is an Ellipse.
if $(B^2-4AC)>0$, ,, ,, ,, Hyperbola.
And ∴, if $(B^2-4AC)=0$, ,, ,, a Parabola,

inasmuch as, in this latter case, it can be neither of the other curves. If, however, here $BD-2AE$ vanish, as well as B^2-4AC, then [1] will represent, not a *curve*, but two parallel straight lines, namely, the straight lines

$$y=-\frac{Bx+D}{2A}+\sqrt{(D^2-4AF)},\text{ and }y=-\frac{Bx+D}{2A}-\sqrt{(D^2-4AF)},$$

which are parallel because they have the same coef. of inclination. And we know that a pair of parallels is a variety of the parabola: the two parallels merge into one straight line if $D^2=4AF$.

There is also a variety of the hyperbola, namely, two *intersecting* straight lines (396), and these the locus will be, whenever the roots of the irrational part of [1], equated to zero, are equal: for, calling each root a_1, [1] will then be

$$y=-\frac{Bx+D}{2A}\pm\frac{\sqrt{(B^2-4AC)}}{2A}(x-a_1),$$

representing two straight lines which intersect, because the coefficients of x are unequal. Hence the form of the locus may always be ascertained

by examining the given coefficients of the equation. If either Ay^2, or Cx^2, be absent from the equation, Bxy being present, the locus is always an hyperbola, as shown above: it is equally so if *both* be absent, Bxy being still present. The form of the equation in this latter case being

$$Bxy+Dy+Ex+F=0, \quad \therefore \ y=-\frac{Ex+F}{Bx+D},$$

which shows that y is real for every real value of x; and that, for every such value, there corresponds but one value of y: hence the axes, to which the hyperbola is referred, must be parallel to the asymptotes.

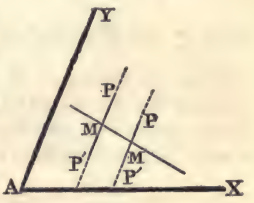
If we had solved the equation [1] for x, instead of for y, the only difference in the result would obviously have been that x and y would have been interchanged, as also A and C, and D and E; that is, we should have had

$$x=-\frac{By+E}{2C}\pm\frac{1}{2C}\sqrt{\{(B^2-4AC)y^2+2(BE-2CD)y+E^2-4CF\}}\ldots\ldots[2].$$

And it is plain, omitting the irrational part of both [1] and [2], that

$$y=-\frac{Bx+D}{2A}, \text{ and } x=-\frac{By+E}{2C}\ldots\ldots[3],$$

represent two diameters of the locus: for the first of these lines bisects all the chords parallel to the axis of y, and the second all the chords parallel to the axis of x. We say this is plain, because, calling the omitted irrational parts X and Y, respectively, if any point M, in the first of these lines, be taken, and we prolong the ordinate through M, and then cut off parts MP, MP', each equal to Y, the points P, P', must be in the locus of [1]; so that as the line thus bisects every chord PP', parallel to the axis of y, that line is a diameter. And in like manner is it shown that the other line is also a diameter. The centre of the curve, therefore—when it *has* a centre—must be at the intersection of the lines [3]: solving then [3], as two simultaneous equations, we have for the co-ordinates (x_1, y_1) of the centre, in the central curves,



$$x_1=\frac{2AE-BD}{B^2-4AC}, \quad y_1=\frac{2CD-BE}{B^2-4AC},$$

and these each become infinite when $B^2-4AC=0$, that is, when the curve is a parabola.

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401. **Determination of particular Loci.**—We shall now give an example or two of the practical application of the preceding theory.

(1) To determine the locus of the equation $y^2-2xy+3x^2+2y-4x-3=0$.

Since here $B^2-4AC<0$, the locus must be an ellipse: its position is determined as follows:—Solving the equation for each of the two variables,

$$y=x-1\pm\sqrt{-(2x^2-2x-4)}\ldots[1], \qquad x=\frac{1}{3}(y+2)\pm\frac{1}{3}\sqrt{-(2y^2+2y-13)}\ldots[2];$$

or, replacing by their factors the irrational parts of these,

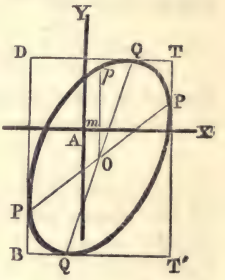
$$y=(x-1)\pm\sqrt{-(x-2)(x+1)}\ldots\ldots[1],$$

$$x=\frac{1}{3}(y+2)\pm\frac{1}{3}\sqrt{-\left(y+\frac{1+\sqrt{27}}{2}\right)\left(y+\frac{1-\sqrt{27}}{2}\right)}\ldots\ldots[2].$$

From [1] it appears that the locus is wholly comprised within the limits $x=-1$, and $x=2$; for it is only within this range that [1] is real. From [2] it appears that the locus is also wholly comprised within the limits

$$y=-\frac{1+\sqrt{27}}{2}, \text{ and } y=\frac{-1+\sqrt{27}}{2};$$

for it is only within this range that [2] is real. Consequently the ellipse is circumscribed by the parallelogram BT formed by parallels to the axes AX, AY, these parallels being distant from the origin by the values of x and y just deduced. The four points of contact are at the extremities of the diameters $y=x-1$, and $x=\frac{1}{3}(y+2)$, since the x, y, of each point causes the radical in [1] or [2] to vanish. The first of these diameters cuts the axes in the points $x=1$, $y=-1$, so that if through these the line PP be drawn, it will be a diameter; and its extremities P, P, will be two of the points of contact. The second cuts the axes in the points $x=\frac{2}{3}$, $y=-2$, so that the line QQ through these will be the diameter, at the extremities of which the other two points of contact will be situated; and the intersection O of these diameters will be the centre of the curve; that is, it will be at the middle of each: its co-ordinates, therefore, are



$$x=\frac{-1+2}{2},\ y=\frac{1}{2}\left(-\frac{1+\sqrt{27}}{2}+\frac{-1+\sqrt{27}}{2}\right), \text{ that is, } x=\frac{1}{2},\ y=-\frac{1}{2}.$$

Or the y of the centre may be found by putting $\frac{1}{2}$ for x in $y=x-1$.

A parallel from the centre O to the tangent TT', that is, to the axis AY, will be in the direction of the semiconjugate to OP: the length, Op, of this semiconjugate will be found by putting $\frac{1}{2}$ for x in the irrational part of [1], since the rational part for this value gives $y=mO$:

$$\text{hence the length of } Op \text{ is } \sqrt{-(x-2)(x+1)}=\frac{3\sqrt{2}}{2},$$

and thus knowing the lengths and directions of a pair of semiconjugates (a', b'), we may find the lengths and directions of the principal diameters as follows: Calling the given angle $[a', b']$, α, and putting θ, θ', for the angles which a', b', make with the major semidiameter a, we have (366)

$$a'^2+b'^2=a^2+b^2, \text{ and } 2a'b'\sin\alpha=2ab, \ \therefore\ a'^2\pm 2a'b'\sin\alpha+b'^2=(a\pm b)^2,$$

which determines $a+b$, and $a-b$, and consequently a, and b. And since

$$\theta'=\theta+\alpha, \text{ we have (364) } \tan\theta\tan(\theta+\alpha)=-\frac{b^2}{a^2}, \text{ which determines } \theta.$$

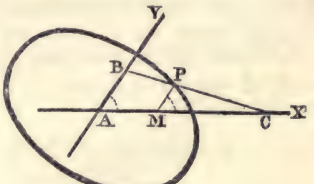
(2) Required the locus of the equation $y^2-2xy-3x^2-2y+7x-1=0$. Here, since $B^2-4AC>0$, the curve is an hyperbola: to construct it, the easiest way will be first to find the asymptotes, and then a single point on the curve: we may then determine as many more points in it as we please, and thus readily trace the locus. Solving the equation for y, we have

$$y=x+1\pm\sqrt{(4x^2-5x+2)}=x+1\pm\left(2x-\frac{5}{4}+\frac{k}{x}+\frac{k'}{x^2}+\dots\right).$$

Now the greater x is, the closer does this curve approach to the straight lines $y=x+1\pm\left(2x-\frac{5}{4}\right)$; and when $x=\infty$, they coalesce: these straight lines are therefore the asymptotes, and may easily be constructed in the usual way. And a point in the curve being determined for any assumed value of x, as, for instance, for $x=0$, as many points in the locus as may be necessary to trace out the curve may be afterwards found (386).

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402. **Problems on Loci.**—(1). The extremities B, C, of a straight line of given length, move along the sides AY, AX, of a given angle A: required the curve described by a given point P in that line, or in its prolongation.



Let $PB=a$, $PC=b$, and $\cos A=c$; then (x, y) being the point P in any position, we have (228)

$$b^2=MC^2-2MC.\ cy+y^2;$$

$$\text{but } a:b::x:MC=\frac{b}{a}x,$$

$$\therefore\ b^2=\frac{b^2}{a^2}x^2-\frac{2bc}{a}xy+y^2,$$

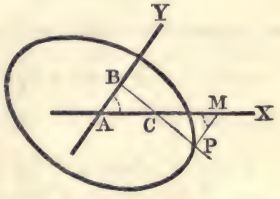
$$\therefore\ b^2x^2-2abcxy+a^2y^2-a^2b^2=0,$$

the equa. of the locus. And since

$$B^2-4AC<0,$$

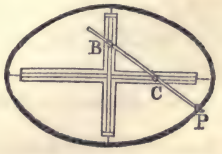
the curve is an *ellipse*, the centre of which is at the origin A, because the first powers of the variables are absent (399). If $A=90°$, then $c=0$, and the equation is

$$a^2y^2+b^2x^2=a^2b^2:$$



hence is suggested an easy method of tracing an ellipse by continued motion, when the centre and the principal diameters of it are given. The instrument employed for this purpose is called a *Trammel*.

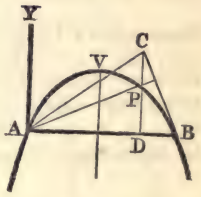
In the annexed figure, representing the trammel, the two grooved bars, usually of wood, are fixed at right angles, their intersection being over the point intended for the centre of the ellipse, and the bars themselves lying in the directions of the principal diameters. On the cross bar BCP, are moveable pins, or sliding guides, B, C; and at the point P in this bar is fixed a pencil, so that the distance BP may be equal to the semitransverse diameter, and CP equal to the semiconjugate. If the guides B, C, are allowed to move freely along the grooves, the pencil will trace the required ellipse.



(2) Given the base and altitude of a plane triangle, to determine the locus of the intersection of perpendiculars from the vertices to the opposite sides.

Putting a for the altitude CD, and c for the base AB, we know that the locus of C will be a straight line parallel to AB. Hence, taking AB, AY, for rectangular axes, and putting (x_1, y_1), for any point in the locus of C, the equa. of that locus is $y_1=a$; and the equa. of BC, passing through the two points $(c, 0)$, (x_1, y_1), is

$$y=\frac{y_1}{x_1-c}(x-c),$$



also the equa. of a perp. to this from A is

$$y=\frac{c-x_1}{y_1}x\ldots\ldots[1].$$

At the point P, where this perp. intersects CD, we must have $x=x_1$; hence, substituting in [1], x for x_1, and a for y_1, we have for the locus of P the equation

$$ay=cx-x^2, \text{ or } x^2+ay-cx=0\ldots\ldots[2],$$

which is that of a *parabola*, since $B^2-4AC=0$. When $x=0$, we have $y=0$, $\therefore$ the curve passes through A, but does not again meet the axis of y, which, prolonged, is therefore a diameter of the curve; but for $y=0$ we have the two values $x=0$, and $x=c$: hence the curve passes also through B. The chord AB is thus perp. to the diameter at A; hence the principal axis of the curve bisects AB at right angles. To find the principal vertex V,

$$\text{put } x=\frac{1}{2}c \text{ in [2], and we get } y=\frac{c^2}{4a};$$

$$\text{hence the point } \left(\frac{1}{2}c, \frac{c^2}{4a}\right) \text{ is the vertex } V.$$

If we remove the origin to this point, that is,

$$\text{if we put } x+\frac{1}{2}c \text{ for } x, \text{ and } y+\frac{c^2}{4a} \text{ for } y, \text{ in [2]},$$

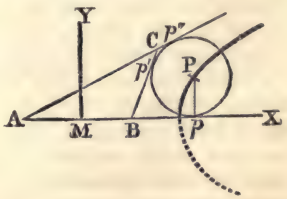
the equation will be $x^2=-ay$; therefore the length (a) of the parameter is CD.

(3) Given the base and the difference of the sides of a triangle, to find the locus of the centre of the circle touching one side (BC), and the two prolongations of the other sides (AB, AC).

Let P be the centre of any one of the circles, and p, p', p'', the three points of contact; then since

$$Ap''=Ap=AC+Cp'=AB+Bp', \therefore Ap=\frac{1}{2}(AB+AC+BC).$$

Take the origin M at the middle of AB, and let MX, MY, be the rectangular axes; then



$$Mp=Ap-\frac{1}{2}AB=\frac{1}{2}(AC+BC)=x.$$

Now $AC \sim CB$ being constant $(=2a)$, the locus of C (x_1, y_1), is an hyperbola, namely,

$$a^2y_1^2-b^2x_1^2=-a^2b^2\ldots\ldots[1],$$

and in this curve $\frac{1}{2}(AC+BC)=ex_1=x$, $\therefore x_1=\frac{ax}{c}$.....[2], since $e=\frac{c}{a}$.

Also P must be always on the line bisecting the angle BCp'', that is, on the normal through (x_1, y_1), to the hyperbola. If in the equa. of this normal we put for x_1 its value $\frac{ax}{c}$, it becomes

$$y=\frac{b^2y_1-acy_1+a^2y_1}{b^2}=\frac{(a-c)c}{a^2-c^2}y_1=\frac{c}{a+c}y_1, \therefore y_1=\frac{a+c}{c}y.....[3].$$

Hence, substituting the values [2], [3], for x_1, y_1, in [1], we have

$(a+c)^2y^2-b^2x^2=-b^2c^2$, the equa. of the locus sought.

The locus is therefore an *hyperbola*, of which the centre is M, the principal axes being $2c$, and $\frac{2bc}{a+c}\sqrt{-1}$, as found by putting $y=0$, and then $x=0$, in the equation, and doubling the results. The vertex of the curve, therefore, coincides with the focus of the hyperbola [1].

Examples for Exercise.

(1) Construct the asymptotes of the hyperbola $xy-2y+x-1=0$, and find the point where the curve crosses the axis of ordinates.

(2) Given the base of a plane triangle, and the sum of the tangents of the angles at the base, to determine the locus of the vertex.

(3) Given the base, and the difference of the tangents of the angles at the base, to determine the locus of the vertex.

(4) Given the base, and the difference of the angles at the base, to determine the locus of the vertex.

(5) Given the base, and sum of the other two sides, to find the locus of the point of intersection of lines from the vertices bisecting the opposite sides.

(6) Given the base, and sum of the other two sides, to determine the locus of the centre of the inscribed circle.

(7) A perpendicular from any point in the diameter of a circle is produced beyond the circumference, till the prolonged part is equal to one of the segments into which the diameter is divided; prove that the locus of the extremity of the perp. is an ellipse.

(8) Prove that the ellipse, determined as above, is equal in area to the circle from which it was derived.

(9) If a circle be described on the major diameter of an ellipse, and normals to both curves be drawn, from points having the same abscissa, prove that the locus of the intersection of these normals will be a circle.

(10) Prove that the locus of the centres of circles passing through a given point, and touching a given straight line, is a parabola.

(11) Prove that the locus of the centres of circles touching two given circles is an hyperbola.

(12) In a given straight line any point is taken, from which two lines are drawn :— one to a given point, and the other perp. to the given line, and the two are made equal in length; prove that the locus of the extremity of the perp. will be an equilateral hyperbola.

(13) If an ellipse and an hyperbola have the same foci, and a tangent to one curve be perp. to a tangent to the other, prove that the locus of their intersections will be a circle.

(14) If to any conic section two secants be drawn parallel to the sides of a given angle, prove that the rectangles of the parts, intercepted between their point of intersection and the curve, will always have a constant ratio.

403. **Scholium.**—The Conic Sections, as the curves discussed in the foregoing treatise are called, have received that name, as noticed at (359), because they would all be presented by the sections of a Cone, cut by a plane in different directions. The surface of a cone, in the most general acceptation of that term, is considered to consist of two *sheets*, the common cone of Mensuration being united at its apex to an opposite and equal *inverted* cone. If the base of each be a circle, every section of either sheet, parallel to the base, will also be a circle, while every section oblique to the base, but not parallel to a slant side of the cone, will be an ellipse. If the section *be* parallel to a slant side it will be a parabola, and if it be perp. to the base—cutting both sheets—the curves will be the two branches of an hyperbola. But since, as sufficiently seen in the preceding articles, these curves may all be generated on a plane, and their properties investigated without the idea of the cone being involved in the inquiry, it would be better to abandon the designation "Conic Sections," and, in reference to the degree of the equation, algebraically including them all, to call them "Lines of the Second Order." And under this title they are now very frequently treated.

Lines of the third order, of the fourth order, &c., are those the equations of which are of the third, fourth, &c., degree. Some of these higher curves still have a sort of historical interest, but very few indeed have interest of any other kind: these few will be noticed in the "Differential Calculus." The number of such curves is, of course, very great: at least eighty different forms are comprehended in the general equation of the third degree, and upwards of five thousand in that of the fourth.

Not only plane curves, but curve surfaces, and the curves described on these surfaces, may also be represented by equations, such equations involving three variables (x, y, z), instead of two. The discussion of them belongs to Analytical Geometry of three dimensions, a subject which we are compelled to exclude from the present work, and for the elements of which the student is referred to the author's separate treatise on "Curves and Surfaces of the Second Order."

END OF THE ANALYTICAL GEOMETRY.

VI. MECHANICS: Part I. Statics.

404. Mechanics is the science which investigates the conditions under which bodies are constrained to remain at rest, or compelled to move, by the operation of external causes, and these causes, whatever they be, are called *Forces*. That part of mechanics which considers the operating forces as producing the former result, namely, *rest*, is called Statics; when the result of the forces acting on the body, or which have previously acted upon it, is *motion*, the inquiry into the circumstances of that motion belongs to Dynamics. The influence called *force* is a moving influence; its effect on the body to which it is applied is either actual motion, or a tendency to motion, in the body. When two or more forces act on a body, and so act as that some of them neutralize all the others, though the *tendency* of each is to produce motion, yet their joint effect is rest; the body, thus acted upon, is said to be in *equilibrium*, and the

forces themselves to be *equilibrating* or balancing forces. Statics is wholly concerned with the doctrine of equilibrating forces.

405. **Conspiring and Opposing Forces.**—We shall at first consider every force acting upon a body as applied to only a single point of that body, called the point of application of the force. Whether this force tend to pull or to push the body from its place, the tendency to motion of the point will of course be in the *line of direction* of the force; that is, directly towards it, if it be a pulling force, and directly from it, if a pushing force. If several forces act at the same point, and all tend separately to move that point along the *same* line of direction, they are called *conspiring* forces: another set of forces applied to the same point, and each tending to move it in the *opposite* direction, are called *opposing* forces, in reference to the former set. If the opposing forces balance the conspiring forces, the point to which the two sets are applied, must evidently be in equilibrium.

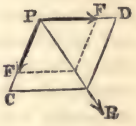
As just observed, the direction of a force is the straight line along which it acts, or along which the point acted upon tends to move: the *intensity* with which it acts may be represented by any finite portion of this line, provided we always represent the intensity of every other force, concerned in the same inquiry, by a straight line having the same ratio to the former that the latter force has to the former force. It is plain, too, that the point of application of the force, instead of being actually in the body, may be assumed as at any point in the line of direction; provided only we regard this line as incapable of extension or compression by the action of the force. We may therefore shift the point of application to any position on the line of direction, without disturbing the statical conditions. Lines representing opposite forces, are of course drawn in opposite directions; and, when expressed in symbols, are therefore marked with opposite signs; and the aggregate, or *resultant*, of such conflicting forces, is found by algebraic addition: that is, it is the algebraic sum of the several component forces.

406. It is plain that if *two* forces, and no more, act on a point, that that point can remain at rest only when the forces are opposite and equal: if the two lines of direction were not in strict opposition, but formed an angle, the point would move, in some determinate direction, just as if it were acted upon by some single force in that direction; the line (in that direction), taken of the proper length, from the vertex of the angle, to represent the intensity of this single force, is called the *resultant* force, and the two forces which furnish this resultant, the *component* forces. A line, therefore, equal and opposite to the resultant, represents the intensity and direction of the force necessary to restore equilibrium, or to keep the point acted upon by the two components at rest.

We thus see that fewer than three forces, of which no two act along the same straight line, cannot produce equilibrium; and further, that any one of the three—since it counteracts the joint influence of the other two—is equal and opposite to the resultant of those two. To determine the direction and magnitude of the resultant of two forces acting on a point—and acting other than in direct opposition—is a very important problem: the foundation, in fact, of the whole theory of Statics. To this determination the following article will be devoted, and to the several steps of the reasoning the student must give the most scrupulous attention.

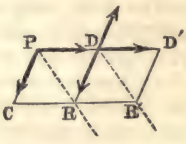
407. **The Parallelogram of Forces.**—In establishing a fundamental principle, the materials ready to our hand are generally of the scantiest kind: *some* admitted truth, however, or assemblage of truths, like the axioms of Euclid, there must always be, since all reasoning must commence its operations upon what is assented to without reasoning. The axiomatic principle in Statics is this, namely:—

Axiom. If two equal forces act in any directions upon a point, the *direction* of the resultant must bisect the angle made by the directions of the components.

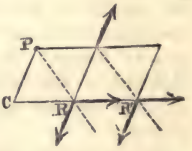


Thus, if any two equal forces F, F, represented by the black lines in the annexed diagram, act upon the point P, the direction PR, of the resultant of those forces, must bisect the angle at P. This is self-evident; for there is nothing to cause PR to be more or less inclined to PC, than to PD. Hence PR is always in the direction of the *diagonal* of the equilateral parallelogram CD, or FF.

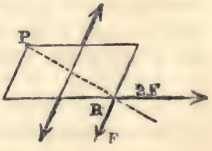
1. *Direction of the Resultant.*—If two forces F and mF, of which one is any multiple of the other, act at any angle on a point P, and if from P lines having the same directions as these forces, and proportional to them in magnitude, be drawn, thus forming two sides of a parallelogram, the diagonal of this parallelogram, from P, will be the direction of the resultant of F and mF.



First let $m=2$, and let the point P be acted upon by the force F in the direction PC, and by two forces each equal to F, in the direction PD', P itself being the point of application of one, and D, any point in the same line of direction, the point of application of the other: and construct the two equilateral and equal adjacent parallelograms PR, DR'. Moreover, let two opposing forces, each equal also to F, be introduced at D, as in the figure: these, as they destroy each other, cannot interfere with the effect of the other three. The point P will thus be acted upon by the force F in the direction PC, and by the force $F+F=2F$ in the direction PD', which may be regarded as a rigid straight line: the object is to show that the resultant of these forces, or that resultant prolonged, will pass through R':—their magnitudes being represented by PC, $PD+DD'$.

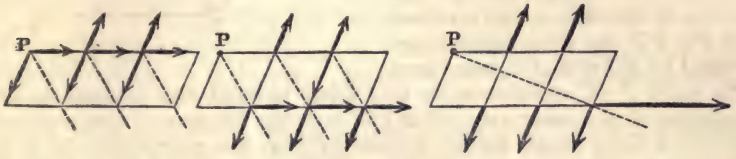


Now, by the axiom, the direction of the resultant of PC, PD, passes through R, and since a force may be considered as applied to any point in the line of direction along which it acts, the resultant of PC, PD, may be considered to act at R, in the direction of PR prolonged, or, which is the same thing, the components F, F, themselves may be transferred to R, PR being a rigid line. Similarly with respect to the two forces DR, DD', applied to D: these may be transferred to R'. Hence the forces, acting as in the first figure, must produce the same effect, when acting as in the second figure. But in this latter the two forces F, F, acting along the same line CR', may both be applied to the same point R' of that line: let them be so applied: then, as in the third figure, we shall have the two forces F and $2F$,



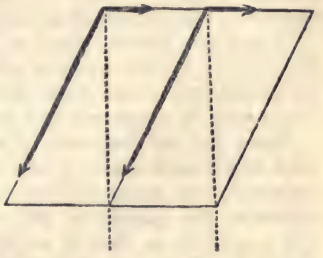
originally applied to P, producing the same effect on that point when they both act at R'. Hence the resultant of the forces now acting at R' must be in the same straight line as the resultant of the same forces originally acting at P, so that P, R', being points in this straight line, the resultant of the forces F, $2F$, applied to P, must be in the direction PR'. The two neutralizing forces, temporarily introduced, destroy each other as at first.

We may extend this reasoning to the two forces F, $3F$, originally acting as in the first of the three figures below, changing the points of application as in the second figure, and finally as in the third figure:



And proceeding in this way it follows that the proposition holds for F and mF, m being any integer.

Again: in the same way that the prop. has been shown to be true for mF, and F, may it now be shown to be true for mF, and $2F$, mF, and $3F$, and generally for mF, and nF, whatever whole numbers m and n may be. Hence the prop. holds for any two forces F, F', provided $F=mf$, and $F'=nf$, where f is some submultiple of F and of F': in other words it holds if $\frac{F}{F'}=\frac{m}{n}$, where m and n are whole numbers. Let $\frac{F}{F'}$ involve a decimal of n places; then the figures of this decimal being a_1, a_2, a_3, ..., a_n, the decimal is the sum of the fractions



$$\frac{a_1}{10}+\frac{a_2}{10^2}+\frac{a_3}{10^3}+\ldots+\frac{a_n}{10^n},$$ any finite number of which is a finite fraction $\frac{m}{n}$,

so that even though n be infinite, this fraction may be made to differ from the decimal by a value less than any that can be assigned: the prop. is therefore true generally for any two forces F, F'. That is to say—

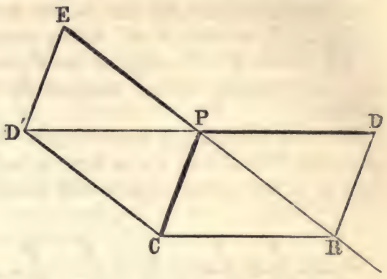
If any two forces acting, on a point, be represented in magnitude and direction by two adjacent sides of a parallelogram, the diagonal of that parallelogram, drawn from the point, will be the direction of the resultant.*

408. It remains for us to prove that this diagonal will represent the resultant, not only as to *direction*, but also as to *magnitude*.

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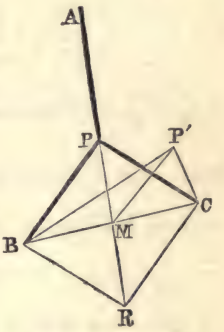
* The above is a modification of Du Chayla's celebrated proof of this proposition.

2. *Magnitude of the Resultant.*—Let PC, PD represent any two forces acting upon the point P, and complete the parallelogram of which these lines are the sides. Draw PR, which will be the direction of the resultant, and produce RP to E, making PE equal to the *magnitude* of the resultant. Complete the parallelogram $PCD'E$, and draw its diagonal PD'. Then, since PE is equal in magnitude and opposite in direction to the resultant of PC, PD, the three forces represented by PC, PD, PE, must keep the point P in equilibrium, so that either is equal and opposite to the resultant of the other two. But PD' is in the *direction* of the resultant of PC, PE, as shown above; therefore PD' is the prolongation of DP: hence the opposite sides of the figure $D'R$ are parallel; $D'R$ is therefore a parallelogram, $\therefore PR=D'C=EP$. But PE is by construction equal to the resultant of PC, PD, in *magnitude:* hence the diagonal PR is the resultant of the forces PC, PD, in magnitude and direction. We infer, therefore, the important principle that—



If any two forces, acting on a point, be represented in magnitude and direction by the two adjacent sides of a parallelogram, the diagonal of that parallelogram, drawn from the point, will represent the resultant of those forces both in magnitude and direction.

Since the two diagonals of a parallelogram always bisect each other, it follows that if PB, PC, represent two forces applied to P, the line PM, drawn to the middle of BC, will represent half the resultant, so that if we take any number of points P, P', &c., whether in the same or in different planes, and draw from each two lines to two fixed points B, C, and apply to P, P', &c., forces which these lines represent, the resultants of the several pairs of forces will all pass through the same point M, the middle of BC; for each pair of components forms with BC a plane triangle.



409. **The Triangle of Forces.**—Any three forces, PA, PB, PC, in the same plane, which keep a point in equilibrium, may always be represented in magnitude and direction by *the three sides of a triangle.* And conversely, if three lines be drawn from any point, respectively equal and parallel to the three sides of a plane triangle, the forces which those lines represent will keep the point in equilibrium.

For in the parallelogram BC, $PR=PA$, and $BR=PC$, and also parallel to it: hence the three sides RP, PB, BR, are respectively equal to, and in the same directions as, the three lines PA, PB, PC. Such a triangle is called *the triangle of forces.* In tracing the directions of the equilibrating forces, along the sides of this triangle, we proceed thus:—Commencing at R, the extremity of the line which is the prolongation

through P, of one of the forces, we proceed along this line up to P; then along the adjacent side PB; and finally along BR.

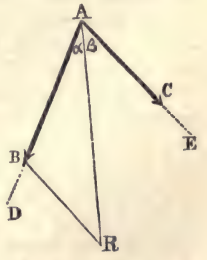
The converse is evidently true; for every plane triangle may be considered as formed of two adjacent sides, and the diagonal joining them of a parallelogram: and we know that if the two sides represent two forces, this diagonal will represent a third force, the opposite of which is in equilibrium with the former two.

If we represent the angle BPC, at which the two forces PB, PC, act, by α, and the forces themselves by P, Q, then since the angle PBR is the supplement of α, we shall have always this relation between the components, the angle at which they act, and the resultant R, namely, (228).

$$R^2=P^2+Q^2+2PQ\cos\alpha\ldots\ldots[1],$$

so that the fundamental formula of statics—the truth upon which the whole science rests—is the same as that which, in like manner, involves the entire theory of the plane triangle (p. 195).

410. **Composition of any Number of Forces acting in a Plane upon a Point.**—Hitherto the forces acting upon a point have been supposed to have their lines of direction in the same plane; but in whatever direction the forces act, it is obvious that the point solicited by them can tend to move under their combined influence only in one direction. It is impossible that a particle can start from repose in *two* directions: hence there is one and but one opposing direction along which, if a new force, of some definite magnitude, act, the point will be held in equilibrium. This new force must therefore be in magnitude equal to the resultant of all the other applied forces, and opposite to it in direction. When a set of forces is thus reduced to the single force which is the resultant of all, the original forces are said to be *compounded*, so that to compound two or more forces is to determine their resultant. On the contrary, to *resolve* a force, is to replace it by two others, of which the joint effect is the same as that of the single force they replace. The former problem is:—Given the components, to find the resultant,—a problem which, as we have just seen, admits of but one solution. The latter is:—Given the resultant, to find the components,—a problem which admits of innumerable solutions, as is evident, for we have only to draw from the extremities of the line representing the given resultant, any two lines forming with it a triangle, and a pair of suitable components will be at once exhibited, both in magnitude and direction. Thus, if the given resultant be AR, A being the point of application, we have only to draw AB, RB, at random, meeting in some point B, and then to make AC equal and parallel to BR: from what has already been proved, AR will be the resultant of AB, AC; and since the angles RAB, RAC $(=ARB)$ are quite arbitrary it follows that—*A force may be resolved into two, having any directions we please, in any plane through the direction of the given force.*

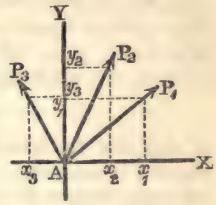


In the above diagram, AB is the resolved part of AR in the arbitrary direction AD, and AC is the resolved part of AR in the equally arbitrary direction AE: all three of the forces acting in the plane of the paper. As we may choose resolvents, or components, in any directions we please, unless other considerations direct the choice, it is best to choose them at

right angles to each other, or to resolve the force into components acting on the point along two rectangular axes, originating at that point. If $A=\alpha+\beta$, were a right angle, then P being the force along AD, and Q the force along AE, we should have

$$P=R\cos\alpha,\ Q=R\cos\beta,\ R^2=P^2+Q^2,\ \text{where}\ \cos\beta=\sin\alpha.$$

411. Let it now be required to compound into a single resultant several component forces P_1, P_2, P_3, &c., all acting in the same plane upon a point A.



AX, AY being any assumed rectangular axes, let the angles which the directions of P_1, P_2, &c., make with AX be α_1, α_2, &c., respectively, and the angles they make with AY, be β_1, β_2, &c. Then resolving each force into its two components along AX, AY, we shall have, for the sums X and Y, of these components,

$$\left.\begin{aligned} X&=P_1\cos\alpha_1+P_2\cos\alpha_2+P_3\cos\alpha_3+\ldots+P_n\cos\alpha_n \\ Y&=P_1\cos\beta_1+P_2\cos\beta_2+P_3\cos\beta_3+\ldots+P_n\cos\beta_n \end{aligned}\right\}\ \text{and}\ R^2=X^2+Y^2\ldots[1],$$

the cosines of the angles in the second equation being the sines of those above them in the first.

The group of forces is thus reduced to two, namely, the single force X, and the single force Y, acting on the point, in directions at right angles to each other; the resultant of which, and therefore the resultant of all the original forces, is $R=\sqrt{(X^2+Y^2)}$.

It is scarcely necessary to observe that as the components acting along the axes are all represented by lines in co-ordinate directions, the X-component of every force which would cause the point A to move towards the right of the axis of y is $+$, and the Y-component of every force which would urge the point A above the axis of x is also $+$: in the contrary cases the signs are $-$. If we put $[R, X]$, and $[R, Y]$, to represent the inclinations of R to the axes AX and AY, respectively, we shall evidently have

$$X=R\cos[R, X],\ Y=R\cos[R, Y],\ \therefore\tan[R, X]=\frac{Y}{X},\ \tan[R, Y]=\frac{X}{Y}\ldots[2].$$

When the forces keep the point A in equilibrium, their resultant R is of course zero: but X^2+Y^2 cannot be zero unless $X=0$, and $Y=0$; hence *when a set of forces, all acting in a plane and meeting in a point, are in equilibrium, if the components of them, along any line whatever, be taken, the algebraic sum of all will be zero.*

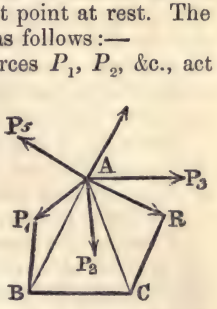
NOTE.—The components here spoken of are those which arise from multiplying the forces themselves, each by the cosine of its inclination to the line:—they are called the *projections* of the forces on that line. That the algebraic sum of these projections be zero, is *necessary* for equilibrium, but it is not *sufficient:* the sum of the projections on a line, perpendicular to the former, must *also* be zero, by [1]. And the proposition is true whether the forces be projected, as here supposed, *orthogonally* (perpendicularly) upon the line, or obliquely: for if the axes AX, AY, be oblique, then, in order that R may be zero, we must have [1] p. 365,

$$X^2+2XY\cos[X, Y]+Y^2=0,\ \text{where, since}$$
$$\cos[X, Y]<1,\ 4X^2Y^2>\{2XY\cos[X, Y]\}^2,$$

so that (68) the only *real pair* of values for X and Y, that satisfy this

necessary condition, must be $X=0$, $Y=0$, as before. Such then are the *analytical* conditions which must be satisfied in order that a set of forces, acting in the same plane, upon a point, may keep that point at rest. The corresponding *geometrical* conditions are determined as follows:—

412. **The Polygon of Forces.**—Let the forces P_1, P_2, &c., act upon the point A, as in the margin; then replacing them by their representative straight lines, we know (409) that if P_1B, equal and parallel to AP_2, be drawn, AB will be the resultant of P_1, P_2, and may therefore be substituted for these two forces. Compounding, then, this resultant with P_3, by drawing BC equal and parallel to AP_3, AC will be the resultant of P_1, P_2, P_3. Continuing thus to draw parallels, as many as there are forces—each successive parallel from the extremity of the preceding—the line AR, from A to the extremity R of the last of these, must represent the resultant of all the forces P_1, P_2, P_3, &c. If this resultant be zero, that is, if P_1, P_2, P_3, &c., keep A in equilibrium, then the last of the parallels spoken of must close the polygon $AP_1BC...A$, the point terminating this last parallel then coinciding with A, the final parallel (RA) being, in fact, equal and opposite to the last force—the force P_5 in the preceding diagram. The polygon thus formed is called *the polygon of forces:* there may, however, be as many such polygons as there are individual forces, since any one of the forces P_1, P_2, &c., may be taken for the *first* force, or for the first side of the polygon; the *last* side, however, will always be equal and opposite to the line representing the last force. And as everything here said holds, whether the original forces act in one plane or not, the crooked line $AP_1BC...A$ will always be a *closed* polygon, but a *plane* polygon only when the original forces all act in one plane.



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413. **Applications of the Preceding Theory.**—In the foregoing exposition, in order to avoid unnecessary complication in our diagrams, we have uniformly regarded the forces concerned as *pulling* forces, for a pushing force may always be replaced by a pulling force of equal intensity, acting in the opposite direction. In the following examples, a pulling force will be conceived to act on a point through the intervention of a cord or string, regarded, in itself, as without weight, and incapable of extension: the *tendency* of the cord to stretch is of course the measure of the pulling force, which is thus the force of *tension* of the string on the point. The effect of a pushing force is *pressure*, or thrust.

414. There are two principles implied in the theory which has been established above—the formal enunciation of which we have thought it prudent to reserve for this place—that demand the special attention of the student, previously to his attempting the solution of statical problems: they are the following, namely:—

1. If three forces acting at any points in a rigidly-connected system of lines, keep that system at rest, the lines along which these forces act, must, when prolonged, all meet in one point.

For (supposing first that no two are parallel) two of them must meet in a point, and these meeting lines being regarded as rigidly connected

with the system, the two forces may be transferred to that point, and acting there must have the same effect, which effect is counteracted by the third force, since the system is at rest: the direction, therefore, of this third force must pass through the point. Hence, knowing the positions of the three points on which the forces act, and also the directions of two of the forces, the direction of the third will be found by joining the point on which it acts, with the point where the other two directions meet.

Similarly, if any number of forces act on as many points, and if the directions of all but one are known to meet in a point, then the direction of that one must pass through the same point, in order that the system may be kept at rest.

If two of the three directions of the balancing forces considered above, are parallel, all three must be parallel, since if two meet, the third must pass through the same point.

NOTE.—It is also useful to remember that since three equilibrating forces are represented by the three sides of a triangle drawn in the directions in which they act, and that sides are as the sines of the opposite angles, each force may be represented by the sine of the angle between the directions of the other two: thus, if P, Q, R, equilibrate, then

$$P : Q : : \sin [Q, R] : \sin [P, R].$$

2. If any number of forces in the same plane act on as many points rigidly connected, and keep the system at rest, then the same forces, acting in the same directions on a single point, would keep it at rest.

For we may resolve each force in the same two rectangular directions, and thus replace the whole by two sets, each set consisting of parallel forces acting in opposite directions; and these opposite forces must balance, because the system is at rest. Hence the original forces, if transferred to a single point, will satisfy the conditions [1] at p. 366, and will therefore keep the point at rest.

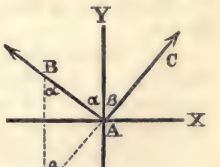
[NOTE.—These conditions prevent the system from moving bodily in any direction, but they are not sufficient to prevent a twist, or rotation: another condition is necessary for this. See (415)].

EX. 1. A weight W hanging by the string AW, tied in a knot at A, is supported by two forces acting along the strings AB, AC, which make given angles α, β, with the vertical line AY: required the tensions T, T', of these latter strings.

From any point B in AB draw a parallel BD to AW, meeting the prolongation of CA in D; then the sides AB, BD, DA, will represent the three balancing forces in magnitude and direction (409), so that we shall have

$$\frac{T}{W}=\frac{AB}{BD},\ \frac{T'}{W}=\frac{AD}{BD};\ \text{or}\ \frac{T}{W}=\frac{\sin\beta}{\sin(\alpha+\beta)},\ \frac{T'}{W}=\frac{\sin\alpha}{\sin(\alpha+\beta)},$$

$$\therefore\ T=\frac{\sin\beta}{\sin(\alpha+\beta)}W,\ T'=\frac{\sin\alpha}{\sin(\alpha+\beta)}W.$$



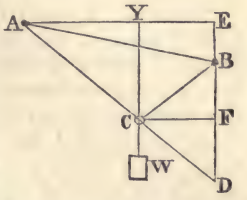
Otherwise. Resolving the three forces along the two perpendicular axes AX, AY, we have

$$T'\sin\beta-T\sin\alpha=0,\ T'\cos\beta+T\cos\alpha=W.$$

Multiplying the first by $\cos\beta$, the second by $\sin\beta$, and subtracting to eliminate T', we have

$$T(\sin\alpha\cos\beta+\cos\alpha\sin\beta)=W\sin\beta,\ \therefore\ T=\frac{\sin\beta}{\sin(\alpha+\beta)}W.\ \text{Similarly}\ T'=\frac{\sin\alpha}{\sin(\alpha+\beta)}W.$$

2. A cord ACB is fixed immovably at the ends A, B, and a weight W is at liberty to slide, by means of a ring C, freely along the cord: at what point of the string will the ring rest, and what will be the tension of the string?



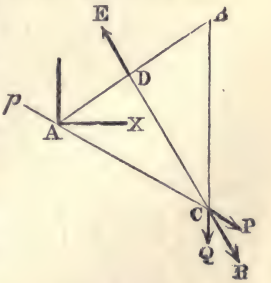
Since the string passes freely through the ring, its tension must be the same throughout. C is thus kept in equilibrium by the two equal forces acting along CA, CB, and the weight W acting vertically downwards. Let CB represent one of the equal forces or tensions; prolong AC to meet the vertical BD in D, then DC, CB, BD, represent the three forces in magnitude and direction; that is, DCB is the triangle of forces, and it must be isosceles, DC being $=CB$; so that the perp. CF bisects BD, $\therefore$ for the tension DC, or CB, we have $T=\frac{1}{2}W\div\cos D$. In order to find the angle D, we have $AE=a$, the horizontal distance between the fixed points, and also $AC+CB=AC+CD=AD=l$, the length of the string, so that $\sin D=\frac{a}{l}$. *Otherwise.* Having found D or B as above, we know its equal BCY: resolving, therefore, the two equal tensions along CY, calling the known angle $BCY=ACY$, β, we have $T\cos\beta+T\cos\beta=W$,

$$\therefore T=\frac{1}{2}W\div\cos\beta, \text{ as before.}$$

To determine the position of the point C, we have $ED=\sqrt{(l^2-a^2)}$, and EB, the distance of B from the horizontal line AE is known: hence BD is known, and $\therefore$ BF, the half of BD, and $BF\tan\beta=FC$: the point C is thus determined.

3. A weight Q is suspended from the extremity B of a rigid line (or rod) AB, which can turn freely about a hinge at its other extremity A; a string, acting perpendicularly to it, at its middle point D, sustains the system in equilibrium. What is the magnitude and direction of the pressure on the hinge at A, when the rod makes an angle of 30° with the horizontal line AX, and what is the tension of the string?

Produce ED, and BQ (if necessary), till they meet in C, and draw AC, which will be in the line of direction of the pressure on the hinge at A, or of the resistance P to that pressure acting from A towards C (see 1, p. 367). The point C, therefore, would be kept at rest if this resistance and the other two forces were applied to it. Now, since CE bisects AB at right angles, it must also bisect the angle C: hence CR, the direction of the resultant of the forces P, Q, bisects the angle between the directions of those forces, $\therefore$ $P=Q$, whatever be the angle the rod makes with the horizontal line.



Again: since AB makes an angle of 30° with the horizontal line AX, $\therefore$ $B=60°$, so that the triangle ABC is equilateral. Replacing forces by their proportional lines, since CB, CA, are equal and opposite to two of the balancing forces at C, the third force $2CD$, (p. 364) is the resultant of

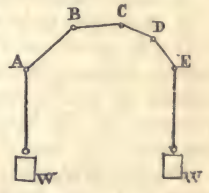
CB, CA; that is, it is $=2Q \sin B = Q\sqrt{3}$, which is $\therefore$ the tension of the string DE. Lastly, since $BAC = 60°$, $\therefore$ the direction of the pressure at A, namely, the direction CAp, is inclined at the angle CAX of 30° to the horizon, so that the inclination pAX is 150°.

Otherwise: suppose the three forces, namely, the vertical force Q, the force T perp. to AB, and the resistance P acting at A, all to act on the point D (See 2, p. 368), and let them be resolved into their components acting along the axes AB, EC. Then, calling the unknown angle which the direction of the resistance makes with AB, ϕ, we have, since the forces along AB destroy each other, $Q \cos 60° - P \cos \phi = 0$, and $Q \cos 30° + P \sin \phi = T$. Now it is plain that these conditions would remain the same, at whatever point in the line AB, the force at D acted, and yet it is equally plain that the force T could not be always the same, though applied indifferently to any point of the rod. Although therefore the equations just given are always *necessary*, as stated at the page referred to, yet when, as here, the forces really act at different connected points, they are, in general, not *sufficient*: there must be another condition before we can determine the three unknowns P, T, and ϕ. It is easy to see what this other condition is:—the components of Q and P, acting at B and A, perp. to the rod, not only balance T, but acting at equal distances from D, they must be *equal:* the third condition is $\therefore P \sin \phi = Q \cos 30°$, $\therefore T = 2Q \cos 30° = Q\sqrt{3}$, and the equations for determining P and ϕ are those annexed. Squaring these, we have $P^2 = Q^2$, $\therefore P = Q$, $\therefore \cos \phi = \cos 60°$, $\therefore \phi = 60°$: hence AC makes an angle of 60° with AB, $\therefore$ it makes an angle of 30° with the horizon.

$P \cos \phi = Q \cos 60°$
$P \sin \phi = Q \cos 30°$

4. Two equal weights W, W, balance themselves over any number of fixed points A, B, C, &c., by a string, attached to the weights, passing over them: to determine the pressure on each point.

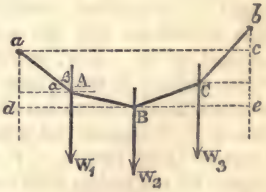
The equilibrating forces at each point are the equal tensions W, of the string on each side of it, and the resistance, equal and opposite to the pressure sustained by the point. Hence, putting α, β, γ, &c., for the angles at A, B, C, &c., the pressures on these points are respectively



$$2W \cos \frac{1}{2}\alpha,\ 2W \cos \frac{1}{2}\beta,\ 2W \cos \frac{1}{2}\gamma, \text{ &c.}$$

5. A string suspended by its extremities has weights W_1, W_2, &c., attached to it at the points A, B, &c.: required the form of the polygonal line which the string assumes.

The point A is kept at rest by three forces, namely, the tensions of the strings, Aa, AB, AW_1; and similarly of the points, B, C, &c. The vertical tensions, due to the weights, have of course no horizontal components; and the horizontal components of the other tensions are all equal, for the hor. component of Aa, must balance that of AB, which latter is the same as that of BA, which balances that of BC, and so on. Each weight is balanced by the vertical components of the tensions at the point of suspension: that is, (410) calling each of the *positive* hor. components h, and the angles which



the vertical strings prolonged make with the polygonal string, β, β_1, β_2, &c., (the angles β_1, β_2, &c., being to the *right* of A, B, &c.,) we must have

$$h(\cot\beta+\cot\beta_1)=W_1,\ h(\cot\beta_1+\cot\beta_2)=W_2,\ h(\cot\beta_2+\cot\beta_3)=W_3,\ \&c.,$$

$$\therefore \frac{\cot\beta+\cot\beta_1}{\cot\beta_1+\cot\beta_2}=\frac{W_1}{W_2},\ \frac{\cot\beta_1+\cot\beta_2}{\cot\beta_2+\cot\beta_3}=\frac{W_2}{W_3},\ \&c.\ldots\ldots[1].$$

If the angles which the several sides of the polygon make with the horizontal components be called α, α_1, α_2, &c., then since

$$h=T\cos\alpha=T_1\cos\alpha_1=T_2\cos\alpha_2,\ \&c.,\ \therefore T=h\sec\alpha,\ T_1=h\sec\alpha_1,\ T_2=h\sec\alpha_2,\ \&c.,$$

$\therefore$ the tensions of the several sides are as the secants of the angles (all measured in the same direction) which they make with the horizon. The fixed points a, b, evidently sustain the whole of the weights; the vertical pressure on these points is, therefore, the sum of those weights, while the horizontal strains balance each other. If, then, the extreme sides of the polygon be prolonged till they meet, and the sum of the weights be affixed to the point of meeting, the tensions and directions aA, bC, of the extreme sides must remain the same.

But to determine the form of the polygon, the horizontal distance ac, and the vertical distance bc of the points a, b, as also the weights and the lengths l, l_1, l_2, l_3, &c., of the several portions aA, AB, BC, &c., of the cord being given, we may proceed thus:—

Imagine a horizontal line drawn through the lowest knot (or *knots*, if there be two equally low), and terminating in the two verticals from a, b: this line de is given, it being $=ac$; also the difference $be-ad$ is given, it being $=bc$. Now the several portions of de, intercepted between the verticals from a, A, B, &c., are $l\sin\beta$, $l_1\sin\beta_1$, $l_2\sin\beta_2$, $l_3\sin\beta_3$, &c., so that

$$l\sin\beta+l_1\sin\beta_1+l_2\sin\beta_2+l_3\sin\beta_3+\ldots+l_n\sin\beta_n=de\ldots\ldots[2].$$

In like manner

$$l\cos\beta+l_1\cos\beta_1+l_2\cos\beta_2+l_3\cos\beta_3+\ldots+l_n\cos\beta_n=bc\ldots\ldots[3].$$

Now there being n weights, the equations [1] are $n-1$ in number, so that these, with the two just given, make $n+1$ simple equations, in which β, β_1, β_2, β_3,...β_n, are the only unknown quantities; these, therefore, may be found, and thence the form of the *Funicular Polygon*, as it is called, ascertained.

It is worthy of notice that it would be impossible for any forces, however great, applied at a, b, to stretch this polygon into a straight line, however small the weights may be, and however few in number. For conceive the polygon to be stretched into the straight line joining a, b, the weights, or only a single weight, hanging vertically, then the components of the forces at a, b, in a direction *perpendicular* to the cord are each *zero;* but the components of the weights, or of the single weight, in the same direction, have a certain intensity, which there is nothing to oppose, consequently, it is impossible that the line ab, to which any weight, however small, is suspended, can be a straight line, however great the stretching forces, unless, indeed, the points a, b, are one vertically under the other. And the same is true of an unloaded *material* line, since every such line has weight.

Examples for Exercise.

[Note.—In resolving forces acting at different points, in the directions of fixed axes, the student will often find it convenient to take a point

distinct from the figure; to draw from it the axes, and the forces, in their proper directions, and then to resolve the latter along the former. All perplexity, as to the signs of the components, will thus be avoided.]

(1) Two forces represented by 2 lb. and 3 lb. act on a point at an angle of 60°: required the magnitude of the resultant.

(2) Two forces expressed by 26 lb. and 127 lb. act on a point at an angle of 76°: required the resultant.

(3) Two weights, each equal to 10 lb., balance each other when hanging at the ends of a string which passes over three fixed points in a vertical plane, the lines joining these points forming a triangle, of which the base is horizontal; the vertical angle of this triangle is 90°, and the other angles 60° and 30° respectively: required the pressures on the fixed points.

(4) If three forces act perpendicularly to the sides of a triangle at their middle points, and if the forces be proportional to the sides on which they act, prove that they must be in equilibrium.

(5) In the figure at p. 369, (ex. 2), the forces at A, B, are 3 lb. and 4 lb., and the weight W, 5 lb.; the distance AB is 6 feet, and the inclination of AB to the horizontal line AE is 30°: required the lengths of the two parts AC, BC, of the string.

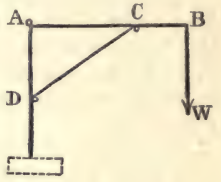
(6) A weight of 10 lb. slides freely along a cord 8 feet in length, attached at its extremities to two hooks 6 feet apart, and in a line inclined at an angle of 30° to the horizon: required the pressure upon each hook when the weight is at rest.

(7) A cord 12 feet in length is tied to two hooks 8 feet apart; the cord passes through a ring upon which a force of 50 lb. acts in a direction, making an angle of 60° with the line from one hook to the other: required the pressure upon each hook.

(8) A cord 14 feet long is fastened to two hooks 12 feet apart; to this cord another is firmly tied at a part dividing the first into two lengths of 9 feet and 5 feet; a force of 60 lb. is then applied to the second cord, and it acts in a direction making an angle of 70° with the line joining the hooks: required the pressures on the hooks.

(9) In the figure to ex. 2, p. 369, the string AC, with a weight attached, is fastened to the hook A; another string, CBD fastened to C, passes freely over the fixed point B, with a weight attached to D, the string CB makes a given angle α with the horizontal line CF: what angle θ must AC make with the vertical CY, in order that W may be kept at rest by half its weight at D?

(10) The rigid rod AB (without weight), but with the weight W attached to its extremity B, is kept from turning about the hinge A, by the supporting rod DC, also without weight, and having hinges at C, D, as in the diagram: required the strain on the hinge A when AB is horizontal



$[AB=a,\ AC=b,\ AD=c]$.

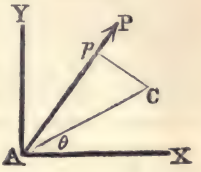
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415. **The Principle of Moments.**—In what has preceded the acting forces, all in one plane, have been regarded either as applied to a single point, or else as so applied to a system of points connected by rigid lines, that their directions all concur in a single point. In this latter case the point of concurrence may obviously be considered to be rigidly connected with the system by these concurring lines, and since the conditions of equilibrium keep this point at rest, the entire system is prevented from advancing bodily, or as a whole in any direction; or, as it is technically called, there cannot be any *motion of translation*, for no such motion could take place without every point of the system partaking of it, and yet the point of concurrence just adverted to remains fixed by the con-

ditions. But it would still remain fixed, though the rest of the system rotated round it; there is, therefore, nothing in the prescribed conditions to prevent this *motion of rotation*. Consequently, that a rigid system may be kept from all motion by forces applied to different points of it, these forces must satisfy not only the conditions (410) necessary to prevent translation, and which are sufficient for a single point, but also some other condition or conditions necessary and sufficient to prevent rotation. We had an instance of the necessity for a distinct condition in ex. 3, at p. 369; this necessity suggests the introduction of a new term—the term *moment*, which is thus defined.

The moment of a force, *in reference to a given point*, is the product of that force by the numerical value of the perpendicular from the point upon its direction.

THEOREM.—The moment of the resultant of any number of forces acting in a plane upon a point is equal to the sum of the moments of the component forces, wherever in the plane the point of reference be taken.



Let A be the point to which the forces P_1, P_2, &c., are applied, then drawing two rectangular axes from A, and *projecting* (411) the several forces upon them, we have for the sums X, Y, of the rectangular components the values [1], p. 366, or putting R for the resultant, we have $X=R\cos\alpha$, $Y=R\sin\alpha$, where α is the inclination of R to AX.

Let C be the point of reference, or *centre of moments;* call AC, c, and the angle CAX, θ; then multiplying the expressions for X, Y, at p. 366, by $c\sin\theta$, $c\cos\theta$, respectively, we have

$$Rc\cos\alpha\sin\theta=P_1c\cos\alpha_1\sin\theta+P_2c\cos\alpha_2\sin\theta+\ldots+P_nc\cos\alpha_n\sin\theta\ldots[1].$$

$$Rc\sin\alpha\cos\theta=P_1c\sin\alpha_1\cos\theta+P_2c\sin\alpha_2\cos\theta+\ldots+P_nc\sin\alpha_n\cos\theta\ldots[2].$$

Subtracting [1] from [2], and putting

$$\sin(\alpha-\theta)\text{ for }\sin\alpha\cos\theta-\cos\alpha\sin\theta,$$

$$Rc\sin(\alpha-\theta)=P_1c\sin(\alpha_1-\theta)+P_2c\sin(\alpha_2-\theta)+\ldots+P_nc\sin(\alpha_n-\theta).$$

Now P, in the diagram, being any one (as P_1) of these n forces, we have

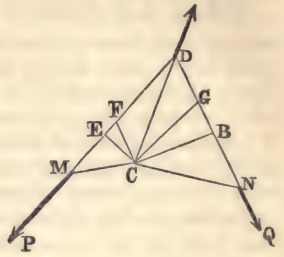
$$c\sin(\alpha_1-\theta)=c\sin CAp=\text{perp. } Cp,\text{ on the direction of that force.}$$

Hence, calling the several perpendiculars from C, p_1, p_2,...p_n, and the perp. $c\sin(\alpha-\theta)$ on the resultant R, r, we have, as the theorem affirms,

$$Rr=P_1p_1+P_2p_2+P_3p_3+\ldots+P_np_n\ldots\ldots[3].$$

Now if the lines along which the applied forces P_1, P_2, &c., act, be supposed to be rigid, as also the perpendiculars connecting them with the point C, these perpendiculars however being at liberty to obey any force which would turn them about C, as about a pivot there:—under such conditions, it is plain that the single force P in the diagram, if acting alone, would turn the system about that point; and similarly of any other one of the forces. In order, therefore, that no rotation may ensue, the efforts to produce it in one direction must just balance the efforts to produce it in the contrary direction. That the intensities of these efforts are correctly represented by what has been called the *moments* of the forces will appear from what follows.

Let two forces P, Q, act at the extremities M, N, of the rigid lines CM, CN, and let it be required to estimate the efforts of these forces to turn the lines about the point C, regarded as a fixed pivot. Prolong the directions of the forces, to meet in D, and draw CD, then a force equal to the pressure on C acting along CD will keep C at rest, though the pivot be removed. Now in order that not only C but the extremities M, N, may also remain at rest, the point D must remain at rest, and therefore MD, ND, must retain their positions, and this is the same as saying that the perpendiculars CE, CB, continue invariable. Draw the parallels CF, CG, to the fixed directions ND, MD,



then $CG : CF :: CB : CE$, since the angles at G and F are equal;

but $P : Q :: CG(=FD) : CF(=DG)$, $\therefore P : Q :: CB : CE$, $\therefore P \,.\, CE = Q \,.\, CB$,

which is the condition necessary in order that there may be no rotation about C; that is, the moments of P, Q must be equal; and conversely, if these moments are equal, the foregoing proportions have place, and $\therefore D$, as well as C, is at rest, and consequently there can be no rotation about C.

The moment of a force, therefore, in reference to a given point, fitly represents the effort of the force to cause rotation about that point. In order that there may be no rotation about the point, the sum of the moments [3], regarding the tendency to turn in one direction as $+$, and the tendency to turn in the contrary direction $-$, must be zero; and this it may be, although the point in which all the forces concur have a progressive motion, that is, although the resultant R of those forces have a finite value: for wherever upon R the point be chosen, the perp. r must be 0.

The conditions, then, that are necessary and sufficient to keep at rest a rigidly-connected system, acted upon by forces whose directions, all in one plane, meet in a point are, 1st. $R=0$: this forbids *progressive* motion; and 2nd. $P_1p_1+P_2p_2+\ldots+P_np_n=0$, which forbids *rotatory* motion. It is common to employ the symbol Σ to represent the algebraic sum of a set of like quantities, such as these last, so that writing the expressions [1], at p. 366, in this more abridged form, the following are the conditions which are necessary and sufficient to keep completely at rest a rigidly-connected system, acted upon by concurring forces in a plane:—

$$\Sigma(P\cos\alpha)=0,\ \Sigma(P\cos\beta)=0,\ \Sigma(Pp)=0\ldots\ldots[4].$$

And since, when this complete equilibrium has place, R in [3] must be zero, it follows that $\Sigma(Pp)=0$ for every point C in the plane of the equilibrating forces, or whatever be the lengths of p_1, p_2, &c.

Referring for an instant to Ex. 3, p. 369, taking D for the centre of moments, and putting $2a$ for the length of the rod, we have, for the perp. from D on AC, $p_1=a\sin\varphi$, and for the perp. from D to BC, $p_2=a\sin 60°$,

$\therefore Pa\sin\varphi - Qa\sin 60°=0$, $\therefore P\sin\varphi=Q\sin 60°$, the condition at p. 370.

Again: take A for the centre of moments; then T being the force in the direction DE, we have

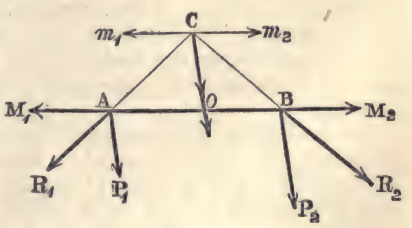
$$Ta - Q \cdot 2a \sin 60° = 0, \quad \therefore \; T = 2Q \sin 60°, \text{ as at p. 370.}$$

The last of the foregoing three equations of condition necessarily has place whether the resultant R be 0, or, the resultant being finite, if r be 0; all that we can infer $\therefore$ from that condition *alone*, is, either that R is 0, and $\therefore$ the system completely at rest, or that the system moves, without rotation, in the direction of AC, C being the centre of moments: but if the same condition equally hold for a different centre of moments C', out of the direction of AC, then there can be neither progressive nor rotatory motion, the latter being forbidden by either one of the conditions holding, and the former because there are not two directions for R. If, therefore the third condition [4] hold for *three* centres of moment C, C', C'', not in the same straight line, since one of these, at least, must be out of the direction of R, whatever this direction might be supposed to be, the system must be completely at rest. Hence, instead of the three conditions [4], if the third alone holds for three distinct points C, C', C'', not in the same straight line, there must be complete equilibrium. And if it hold for two points not in the same straight line as the point where the forces meet, it will be enough to establish complete equilibrium.

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416. **Parallel Forces acting in a Plane.** — Hitherto we have treated only of forces whose lines of direction all meet in a point; we have now to consider parallel forces, or such as applied to different points of a rigid system have their lines of direction parallel to one another.

Let P_1, P_2, be two parallel forces, and let AB be any straight line connecting the lines AP_1, BP_2, along which they act. The object is to find the resultant of P_1, P_2, that is, to determine the intensity and direction, as also the point of application in AB of a single force equivalent to the two. To the extremities of AB, conceived to be rigid, let any two equal, but opposite forces (M_1, M_2), be applied, the one acting from A towards AM_1, the prolongation of BA, and the other towards BM_2; these, as they destroy each other, can have no effect on the system. Let the force R_1 be the resultant of the two now acting at A, and R_2 the resultant of those acting at B: our two parallel forces will thus be replaced by two oblique forces R_1, R_2, having the same effect on the system. Let C be the point in which the directions of these meet, and to which, the lines being rigid, R_1, R_2, may be transferred. Resolve them, when thus transferred, into their original components m_1, m_2, equal to each other and to M_1, M_2, acting in opposition, and in a line parallel to AB, and P_1, P_2, acting along CO, parallel to AP_1, BP_2; the system of forces is thus again reduced to two: the two conspiring forces P_1, P_2, both applied in the direction CO, or at O in the line AB; the resultant $\therefore$ of the original



parallel forces is equal in magnitude to their sum, that is, $R=P_1+P_2$, and O is the point in AB to which this force must be applied to produce the same effect. Its situation is determined thus :—

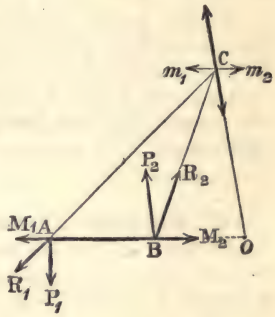
The three forces M_1, P_1, R_1, are proportional to the three sides of the triangle AOC, and the three forces at B, proportional to the sides of the triangle BOC.

$$\therefore \frac{P_1}{M_1}=\frac{CO}{OA}, \text{ and } \frac{P_2}{M_2}=\frac{CO}{OB}; \text{ but } M_1=M_2 \therefore \frac{P_1}{P_2}=\frac{OB}{OA} \ldots\ldots [1].$$

Hence the resultant of two parallel forces is itself parallel to those forces, and equal to their sum, and its direction divides the line AB between them into two parts inversely proportional to their intensities, that is,

$$P_1+P_2=R, \text{ and } P_1 : P_2 : : OB : OA \ldots\ldots [2].$$

The forces P_1, P_2, are here supposed to act both on the same side of AB, if P_1 act on one side and P_2 on the other, as in the annexed diagram, similar reasoning applies; but O will here be in the prolongation of AB, and P_1, P_2, will of course take opposite signs, so that we shall have $P_2-P_1=R$. Consequently when the two parallel forces are equal, and act in opposite directions $R=0$, and the point O, which in other cases would be the point of application of the resultant, is infinitely distant, for AR_1 then bisects the angle at A. and BR_2 bisects the equal angle at B, so that BC is parallel to AC, that is, C is infinitely remote, and $\therefore$ O, in the line bisecting the angle at C, is infinitely remote. Or thus, since [1]



$$\frac{P_2}{P_1}=\frac{OA}{OB}=\frac{OB+BA}{OB}=1+\frac{BA}{OB}, \text{ if } P_1=P_2 \text{ then } 1+\frac{BA}{OB}=1, \therefore OB=\infty.$$

Hence no single finite force, acting at a finite distance, can be equivalent to two equal and opposite parallel forces: we see, indeed, that the forces R_1, R_2, by which these have been replaced, are themselves also two equal and parallel forces. When, however, the forces are unequal, then, since the proportion [2] involves their magnitudes only, independently of their directions, we see that however the *directions* of the same pair of parallel forces may vary, the point of application O, of their resultant, is invariable.

From what has now been proved, it follows that any two *unequal* parallel forces can be counterbalanced by a single third parallel force, determinable in magnitude and in point of application: but that no *single* force can ever counterbalance two *equal* parallel forces, if they act in opposite directions. A pair of such forces is called a *statical couple*: we infer, therefore, that a rigid system cannot be kept in equilibrium by any three forces, two of which constitute a *couple*.

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417. In all the preceding articles the directions of the forces in each case have been regarded either as all uniting in a single point—which we have seen may be considered as the point to which their conjoint action is applied—or else as being parallel, and the forces themselves acting at

necessarily different points. These points, however, may be understood to be points in some rigid or solid body, or points connected together in one rigid system: this condition of rigidity is necessary, in order that the forces may not alter the form of the figure with which the points are connected. But the body, or the rigid framework, thus acted upon, must always be regarded as destitute of *weight;* for weight itself is a force, and in determining the condition of a body as to rest or motion, *all* the forces, to which that condition is due, must of course be taken into account. It is easy to see that, in the problems hitherto given, this could not have been done; for every particle of a material body has weight, or is acted upon by force, and as yet we have treated only of forces acting in one plane, and if not at a single point in that plane, yet (as in last article) at only two points, separated by an interval. In some of the problems adverted to, we have indeed employed weights, but this has been only for the purpose of transmitting the forces of those weights, through the intervention of strings, to certain points: the strings themselves we have been obliged to consider as without weight, so as to pass from one point to another without flexure, that is, in a perfectly straight direction, which no *material* string, unless it hang vertically, can do (p. 371), because it has *weight.*

We shall now consider bodies as we really find them at the surface of the earth, that is, as an aggregate of heavy particles, cohering together in one solid mass; confining our attention, at present, to that force alone which, acting upon the constituent particles, causes what we call weight.

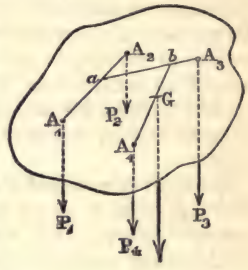
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418. **Parallel Forces acting on a Rigid Body: Centre of Gravity.**—Experience shows that all bodies, within our reach, tend towards the earth, to the surface of which, if abandoned to themselves, or left unsupported, they would fall, in a vertical direction. The reason why smoke and vapour do not, in general, thus fall, is that they are not left unsupported, being borne up by the air, in a similar way that a piece of wood is borne up by the water in a vessel. The force which thus solicits all bodies towards the earth, we call the force of terrestrial attraction—or the force of gravitation—or simply *Gravity*: it is a force that acts upon all bodies alike, so that if a piece of cork, and a lump of gold, be both let fall at the same time, from the same height, in a place exhausted of air, they will keep side by side, during the whole of the descent, and will both reach the surface of the earth at the same instant. This fact has been proved by many experiments, and justifies the conclusion that each constituent particle of every body is equally acted upon by the force under consideration. If, as in gold, the particles are more closely packed together than in a lighter substance, as cork, more force of attraction must be expended upon the gold than upon the cork, to keep them together during their descent, when simultaneously dropped from the same height, because a greater number of particles are to be brought down:—it is owing to this greater number of particles that a lump of gold would strike an object opposed to its descent with greater force than an equal bulk of cork, and that the pressure of the gold on any support is greater than that of the cork.

The line of direction of the force of gravity, acting on a particle, is the line from that particle to the centre of the earth; at least, its deviation from this direction (arising from the earth not being a perfect sphere) is

insensible. The distance, too, of this centre, from the surface is so great, and, comparatively with the bulk of the earth, the bodies which we have to consider in mechanics so small, that the lines from their several particles, thus in reality converging to the earth's centre, differ in their directions insensibly from parallelism: we have only therefore, in examining into the effects of gravity on a body, to regard the particles of that body as acted upon by so many parallel forces, all operating in the same direction.

Let P_1, P_2, P_3, &c., be parallel forces applied at the points A_1, A_2, A_3, &c., of a rigid body: it is required to determine the resultant of these forces, or its contrary, namely, the single force which, acting in the opposite direction, would prevent progressive motion.



The most obvious way of finding the resultant is this:—first find the point of application in the line A_1A_2, of the resultant of two of the forces P_1, P_2, that is, divide A_1A_2 in a, so that $P_1 : P_2 :: A_2a : A_1a$ (416): then a single force $= P_1 + P_2$, applied at a, will be equivalent to the two forces P_1, P_2, and may be substituted for them. Imagine, then, that the forces P_1, P_2, are withdrawn, and that the force $P_1 + P_2$, acting at a, in the same parallel direction, is introduced: join a, A_3, and find in like manner the point of application b, in this joining line, of the single parallel force equivalent to $P_1 + P_2$ acting at a, and P_3 acting at A_3: then the single force $P_1 + P_2 + P_3$, acting at b, may replace the three forces P_1, P_2, P_3. Proceeding in this manner, all the forces will at length be compounded into a single resultant, and the point of application (as G) of this resultant will become known: this point G is called the *centre* of the system of parallel forces: it is the point to which, if the united energies of all the individual forces were applied, the effect on the body would be the same. And it is plain, that however the directions of all the parallel forces be changed, provided only that these directions continue to be parallel, and the intensities of the forces remain unaltered, the position of the centre will continue undisturbed; for in determining this centre, the *magnitudes* and points of application, and not the directions of the forces, alone enter into consideration.

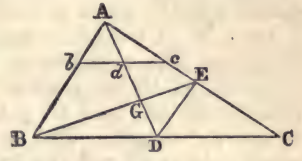
If the forces P_1, P_2, &c., be those of gravity, or in other words if they represent the weights pulling A_1, A_2, &c., downwards in a vertical direction, then G is the *centre of gravity* of A_1, A_2, &c., regarded as so many heavy particles, having the respective weights W_1, W_2, &c., replacing P_1, P_2, by these symbols for *weights;* and it is the point into which all the individual weights may be considered to be concentrated, for as just seen, it is the point upon which, if all the weights $W_1 + W_2 + \ldots$ act, the effect is the same, and it is obvious, from the way in which this point is determined, that it is the only point of which the same can be said. The figure, roughly sketched above, is intended for the outline of any solid body, every particle of which has weight; A_1, A_2, &c., are merely detached points in that body, at which points alone we have assumed weights to act: but suppose that the point A_1 is the centre of gravity of one portion of the body, A_2 the centre of gravity of a second portion, and so on, A_n being the centre of gravity of the nth or last portion; then it is plain

that G, determined as above, would be the centre of gravity of the *body*: the following problems are thus suggested.

PROB. I. To find the centre of gravity of a material straight line:—of a metal rod, for instance, so slender that its thickness may be left out of consideration. This physical line is composed of weighty particles, one of which is at its middle M; and pairs of particles, one of each pair on one side of M, and one, at an equal distance, on the other side, make up the line of particles: hence, whichever of these pairs we take, the resultant of their weights must pass through M: consequently M is the centre of gravity of the entire line, which will balance about this centre in whatever position it be placed (p. 378).

PROB. II. To find the centre of gravity of a thin triangular plate.

Let ABC be the triangle: draw AD bisecting BC, and bdc, parallel to BC: then, by similar triangles we have



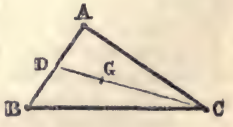
$Ad : AD :: dc : DC$, and $Ad : AD :: bd : BD$,

but $BD=DC$, hence $bd=dc$, and $\therefore$ the line bc will balance about d in all positions; and similarly will every other parallel to BC balance about the point where AD cuts it: and since the whole triangle is made up of these lines of particles, the whole must balance about the line AD, which therefore passes through the centre of gravity of the triangle. In like manner it may be shown that the triangle will also balance about BE, the line bisecting another side AC; that is, that the centre of gravity lies in BE, as well as in AD: hence G, where the two bisectors of the sides intersect, is the centre of gravity of the triangle ABC. It was shown at (p. 298) that $DG=\frac{1}{3}AD$, and $EG=\frac{1}{3}BE$, so that we have only to draw from one of the vertices of the triangle, a straight line to the middle of the opposite side, and to mark a point G on this line at one-third of the whole length of it from that middle, to get the centre required.

By dividing a polygon into its component triangles by means of diagonals, we may find the centre of gravity of each triangle, as above; and thence the centre of gravity of the quadrilateral made up of two of the triangles, thence that of the pentagon made up of three, and so on.

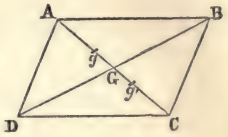
PROB. III. Three equal weights are suspended by strings at the three vertices of a triangle: whereabouts in the plane of this triangle must an opposing force be applied to keep it at rest?

The weights may be regarded as heavy particles placed at A, B, C. Bisect AB at D; then will D be the centre of gravity of A, B, so that the effect, as far as these two weights are concerned, is the same as if both were removed and placed at D. Join DC: then it remains only to find the centre of gravity of $2A$ and C, that is, to divide CD, so that $CG : GD :: 2 : 1$; hence $CG=2DG$, or $DG=\frac{1}{3}CD$, so that the centre of gravity of three equal weights at the vertices of a triangle is the same point as the centre of gravity of the triangle itself: this centre



$\therefore$ remains unchanged, whether the vertices be equally loaded, or not loaded at all.

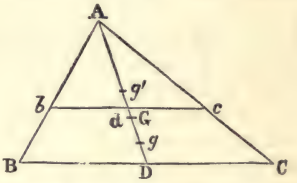
PROB. IV. To find the centre of gravity of a thin plate in the form of a parallelogram.



Draw the diagonals AC, BD, intersecting in G: then the centre of gravity of the triangle ABD is at g, where $Gg=\frac{1}{3}GA$. In like manner the centre of gravity of BCD is at g', where $Gg'=\frac{1}{3}GC$; and $GC=GA$: hence the equal weights of the two equal triangles may be considered to be concentrated in the points g, g', equidistant from G, which is $\therefore$ the centre of gravity of the parallelogram.

PROB. V. A triangle is divided into two equal parts by a line bc parallel to its base BC: required the centre of gravity of the trapezoid Bc.

Let G, g', be the centres of gravity of the whole triangle, and of the triangle cut off; and let g be the centre of gravity sought. The weight of the whole acts at G, and the weights of the two equal parts act at g', g: hence the weight at G is the resultant of two equal weights at g', g, $\therefore$ g', g, are equidistant from G, and g is in the same straight line with $g'G$,

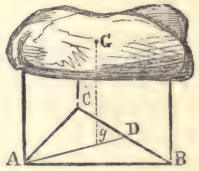


$$\therefore Ag-AG=AG-Ag',\ \therefore Ag=2AG-Ag'=\frac{2}{3}(2AD-Ad).$$

Now the similar and equal triangles ABD, and Abd, being as the squares of their like sides, we have

$$AD^2 : Ad^2 :: 2 : 1,\ \therefore Ad=\frac{AD}{\sqrt{2}},\ \therefore Ag=\frac{2}{3}\left(2-\frac{1}{\sqrt{2}}\right)AD=\frac{2}{3}\left(2-\frac{1}{2}\sqrt{2}\right)AD.$$

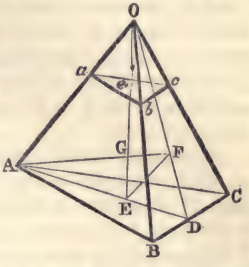
PROB. VI. A body of known weight, and of which the centre of gravity is known, rests upon three points: required the pressure upon these points.



Let A, B, C, be the points of support, and G the centre of gravity of the body: then the weight W will act in the vertical Gg, g being the point where this vertical meets the plane of the triangle ABC. Now P_1, P_2, P_3, being the pressures (or weights) at A, B, C, we have $P_1+P_2+P_3=W$: and the two pressures at C, B, are equivalent to a single pressure at some point D, such that $P_2 . BD=P_3 . CD$. In like manner, the single pressure P_2+P_3 now supposed at D, and that at A, are equivalent to the single pressure $P_1+P_2+P_3$, or W, acting at some point in the line AD: but the resultant of all the pressures is in the vertical through g: hence AgD is a straight line, and $P_1 . Ag=(P_2+P_3)Dg$. Consequently, the sides of the triangle ABC being given, as also the point g, $\therefore$ CD, BD, Ag, and Dg, are known or determinable, and we have the three simple equations just given to find the values of the three unknowns P_1, P_2, P_3.

PROB. VII. To find the centre of gravity of a triangular pyramid.

Let ABC be the base and O the vertex of the pyramid. Bisect a side BC of the base in D; draw AD, OD, and make $DE=\frac{1}{3}DA$, and $DF=\frac{1}{3}DO$; then E is the centre of gravity of the triangle ABC, and (by similar triangles) the centre of gravity e, of any section abc parallel to ABC, is in the line OE. Similarly: the centre of gravity of every section parallel to the triangle OBC is in the line AF: hence, conceiving the pyramid to be made up of thin triangles parallel to ABC, the centre of gravity of the pyramid must be in OE. Similarly: conceiving it to be made up of thin triangles parallel to OBC, the centre of gravity must be in AF: hence the intersection G of these lines must be the centre of gravity sought. The distance OG may be found thus:—



Draw EF, then $\frac{DE}{EA}$, $\frac{DF}{FO}$ are each $=\frac{1}{2}$, $\therefore DE:EA::DF:FO$,

$\therefore EF$ is parallel to AO (Euc. 2. VI.), $\therefore OGA$, EGF, are similar triangles,

$$\therefore \frac{EG}{OG}=\frac{EF}{OA}=\frac{DE}{DA}=\frac{1}{3}, \therefore \frac{OG+EG}{OG}=1+\frac{1}{3}=\frac{4}{3}, \text{ that is, } \frac{OG}{OE}=\frac{3}{4}, \therefore OG=\frac{3}{4}OE.$$

Hence, having found the centre of gravity of one of the triangular faces of the pyramid, let a line be drawn to this centre from the opposite vertex of the solid: the point G in this line, which is three-fourths of its length from the vertex, or one-fourth of its length from the opposite face, will be the centre of gravity of the pyramid. It is obvious that if a plane parallel to the base and distant from it $=\frac{1}{4}$ the height of the pyramid, cut the solid, it will pass through the centre of gravity. If the base be any polygon, we may by drawing diagonals divide it into triangles, and the polygonal pyramid may be regarded as made up of the pyramids on these triangular bases: a plane parallel to the base, and at the distance from it of $\frac{1}{4}$ the height, will pass through the centre of gravity of each triangular pyramid, and therefore through the centre of gravity of the combination. Now this centre of gravity of the polygonal pyramid must lie in the line from the vertex, which passes through the centre of gravity of the polygonal base; for the pyramid, as in the former case, may be regarded as made up of thin slices parallel to the base: hence the centre of gravity of the pyramid is that point in the line joining the vertex with the centre of gravity g, of the base, which is at one-fourth the length of that line from g; so that the height of G above the base is $\frac{1}{4}$ the height of the pyramid, whatever rectilinear figure the base may be, and however numerous the sides of it: hence what is here established for the pyramid, equally applies to a cone, of whatever base.

Examples for Exercise.

(1) Two weights of 5 lb. and 7 lb. are attached to the ends of a rod without weight, 6 feet long: required their centre of gravity; and also how far towards the smaller weight the centre will shift, if an additional 1 lb. be added to that weight.

(2) If the three sides of a triangle ABC, be bisected in the points D, E, F, prove that the centre of gravity of the perimeter of the triangle will be the centre of the circle inscribed in the triangle DEF.

(3) Three weights of 1 lb., 2 lb., and 3 lb., are applied to the vertices of a triangle, each side of which is a: find the distances of the centre of gravity of these weights from each vertex.

(4) What must be the altitude of a triangle which, when cut out of a square one side of which (a) is the base of the triangle, may be such that the vertex of the triangle shall be the centre of gravity of what remains of the square?

(5) From a circle whose radius is r, the smaller circle described upon r as a diameter is taken away: at what distance is the centre of gravity of the remaining part from the centre of the original circle?

(6) If weights proportional to the lengths of the opposite sides are placed at the vertices of a triangle: prove that their centre of gravity will be the centre of the inscribed circle.

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419. From the nature of the centre of gravity—the point in which the entire weight of the body may be regarded as concentrated—any point beneath the body, in the vertical line from this centre, may be taken as a fixed point of support on which, the line being rigid, the body will rest; but it will not rest on any single point out of this line of direction: in like manner, any point of the vertical through the centre of gravity, above the body, may be taken as a point of suspension, from which the body will hang at rest. If in the former case the body be disturbed ever so slightly, so as by displacing the position of the centre of gravity to bring it into a new vertical, then, since in this new vertical there is no point of support, the body must overset: if, however, instead of being supported on a point, the body stand on an area or base, it will not upset so long as the vertical from the centre of gravity does not fall beyond the limits of the base; for at whatever point within these limits the vertical falls, that point is a point of support in the line of direction. If instead of the body resting on an area, it be supported only on points, which being connected by straight lines would inclose an area, it is just the same; for in falling, the centre of gravity would proceed down the vertical, and motion in this direction is as much prevented by the isolated points, or legs, as if the area were filled up. If we incline the body over so as to move the vertical from its original position, till at length it just passes through the edge or point on which the body is then supported in the inclined position, it will rest in that position; but the slightest disturbance will either upset it, or cause it to return to its original position: this is said to be a state of *unstable equilibrium;* but when the state is such that a slight disturbance cannot upset the body, which recovers its first position when the disturbing cause is removed, it is in a position of *stable equilibrium.* But if when disturbed, the body rests in its new position as firmly as in its original position, that original position was one of *neutral equilibrium:* a cylinder, on its side, or a rolling stone, furnishes an instance of this kind of equilibrium.

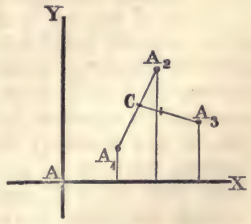
420. In all the preceding problems, except in that one concerning the pyramid, the bodies have been regarded as merely thin plates, discs, or laminæ of matter, of insensible thickness: they have been so regarded in order that the point in each, which is the centre of gravity, might be exhibited. But when, as is usually the case, the centre of gravity of the body is within the solid, that point is practically inaccessible; and it is but seldom that more than the vertical which passes through it is required to be found. In the case of the plates adverted to, it is evident that this line of direction, when the surface is horizontal, will be the same whether the plate be thin or thick; as a thick plate may be regarded as made up of equal horizontal layers, in each of which the centre of gravity is in the same vertical: the horizontal plane, which divides the thick plate into equal plates of half the thickness, will evidently cut the vertical in the point which is its centre of gravity. It is plain, too, that the centre of a circle is equally the centre of gravity of the circle itself, and of a rim of that circle: that the centre of a sphere is also its centre of gravity, as likewise the centre of gravity of a spherical shell of uniform thickness. In fact, whatever be the shape of the body, if it only have a *centre of figure*, that is, a point which equally divides *every* line passing through it that is limited in length by the exterior boundary of the figure, that point must be its centre of gravity; because, being the common centre of gravity of *each* of these diametral lines of particles, it must be the centre of gravity of *all*.

We shall now investigate the general equations of equilibrium for any system of parallel forces.

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421. **General Equations of Equilibrium of Parallel Forces.**—For convenience, and as including all the most useful cases, we shall suppose the forces to act vertically, like gravity, upon so many distinct points or particles. The directions of these forces pierce the horizontal plane in points at which the forces themselves may be regarded as applied, the lines of direction, connecting these *projections* of the points with the points themselves, being assumed to be rigid.

Let AX, AY, be two rectangular axes in the horizontal plane—the plane of the paper, suppose—and let A_1, A_2, A_3, &c., be the points in this plane to which the vertical forces P_1, P_2, P_3, &c., may be regarded as applied: the co-ordinates of the points A_1, A_2, &c., will be represented by (x_1, y_1), (x_2, y_2), &c. Now if $C\,(X_1, Y_1)$ be the centre of gravity of the forces P_1, P_2, acting at A_1, A_2, we know (p. 376) that



$$A_2C : A_1C :: P_1 : P_2;$$

$$\text{but } A_2C : A_1C :: x_2 - X_1 : X_1 - x_1,$$

$$\therefore x_2 - X_1 : X_1 - x_1 :: P_1 : P_2, \ \therefore (P_1 + P_2)X_1 = P_1x_1 + P_2x_2, \ \therefore X_1 = \frac{P_1x_1 + P_2x_2}{P_1 + P_2}.$$

Proceeding in a similar way with the force $(P_1 + P_2)$, now supposed to act at C, and the force P_3, and calling the centre of gravity of these (X_2, Y_2), we have, in like manner,

$$(P_1+P_2+P_3)X_2=(P_1+P_2)X_1+P_3x_3, \quad =P_1x_1+P_2x_2+P_3x_3, \text{ by substitution,}$$

$$\therefore\ X_2=\frac{P_1x_1+P_2x_2+P_3x_3}{P_1+P_2+P_3}.$$

And continuing this process till we have included the last force P_n, and putting (X, Y) for the centre of gravity thus determined of the entire system, we have the following general expressions for the co-ordinates of the point where the resultant of that system meets the horizontal plane, the second expression being got by merely substituting the axis of y for that of x.

$$X=\frac{P_1x_1+P_2x_2+P_3x_3+\ldots+P_nx_n}{P_1+P_2+P_3+\ldots+P_n},\quad Y=\frac{P_1y_1+P_2y_2+P_3y_3+\ldots+P_ny_n}{P_1+P_2+P_3+\ldots+P_n}\ldots[1],$$

the denominator of each fraction being R, the resultant of the system. If the forces equilibrate $R=0$, $\therefore$ $RX=0$, and $RY=0$, $\therefore$ using the notation at p. 374 the equations of equilibrium are

$$\Sigma(P)=0,\ \Sigma(Px)=0,\ \Sigma(Py)=0\ldots\ldots[2],$$

where $\Sigma(Px)$ expresses the sum of the moments of the forces with respect to the axis AY, and $\Sigma(Py)$ the sum of the moments with respect to AX. Or, if we do not include the resultant R, acting in the opposite direction under $\Sigma(P)$, these equations are then

$$\Sigma(P)=R,\ \Sigma(Px)=RX,\ \Sigma(Py)=RY\ldots\ldots[3],$$

where X, Y, as above, are the co-ordinates of the point where the resultant, or line of direction of the centre of gravity, pierces the horizontal plane.

It is obvious that although in the above reasoning we have regarded the forces P_1, P_2, &c., as those of gravity, the lines of action being vertical, yet everything would remain the same if the lines of action of the parallel forces were other than vertical, the plane of projection being, as above, perpendicular to the lines of action, though not then horizontal. Those of the forces which tend to move the body in opposite directions, take, of course, opposite signs, in the foregoing general equations.

If the points of application, instead of being projected upon a horizontal plane, had been projected upon a vertical plane, a plane to which the directions of the forces are all parallel, as, for instance, on the vertical plane passing through AY and perpendicular to AX, what above is the x of either of the points A_1, A_2, &c., would evidently be the perp. distance of the particle itself from this vertical plane: this perp. distance, multiplied by the force, is the moment of that force in reference to the plane, so that the sum of these moments would be equal to the moment of the resultant in reference to the same plane. When a force equal and opposite to this resultant is introduced, that is, when the body is in equilibrium, the sum of the moments—taking the moment of the opposing force with opposite sign—is therefore zero. Should the plane pass through the body, so that particles are on each side of it, the moments on one side must of course be taken with opposite signs to those on the other. If the sum on one side be equal to the sum on the other, the moment of the resultant must be zero, that is, the direction of the resultant must be in the vertical plane. An experimental way of determining the centre of gravity is thus suggested. If the body be placed on a horizontal table, and be pushed more and more over the edge till it

just balances itself, the centre of gravity will be in the vertical plane through the edge of the table: by help of this edge a line may be marked round the body; then turning the body into another position, we may, in like manner, mark on its surface a line through which a second plane will also pass through the centre of gravity: this centre will therefore be in the straight line joining the two points where the lines marked round the body intersect. If it be suspended by a string at either of these points, and then left to itself, it will hang at rest without any oscillation. If it be suspended at any other point, it will turn round till the centre of gravity comes into the vertical line under the point of suspension.

422. Returning now to the equations [1], [2], it is plain that if the points P_1, P_2, &c., all lie in a straight line, this line may be taken for the axis of x, and we shall then have $Y=0$, the first of [1] giving the x of the centre of gravity, and the conditions [2] of equilibrium being $\Sigma(P)=0$, $\Sigma(Px)=0$. As in the former investigations, the parallel forces are supposed to act upon particles of the body separated by an interval; but as there is no limit to the minuteness of this interval, the particles may be regarded as actually in contact, and forming in their aggregate a solid body: the number of terms in numerator and denominator of each of the fractions [1] is then infinite, as gravity acts upon each particle; but the *sum* of these terms is in each case finite; the denominator being the finite body itself, and the numerator being also finite, inasmuch as X and Y are finite. But to determine the finite expression for the numerator, consisting as it does of an infinite number of elements (like a decreasing geometrical series), requires the aid of the INTEGRAL CALCULUS, under which head the subject will be resumed. We shall here, however, give one or two applications of the preceding theory to cases in which the numerators above referred to, consisting of a finite number of terms, common algebra suffices for their discussion.

(1) Required the centre of gravity of the perimeter of any polygon, the sides of which are heavy, uniform, slender bars.

Let the weights of the n sides of the polygon be W_1, W_2, ...W_n, respectively, and let the vertices be the points (x_1, y_1), (x_2, y_2), ...(x_n, y_n): then the co-ordinates of the middle points will be

$$\frac{1}{2}(x_1+x_2),\ \frac{1}{2}(y_1+y_2)\ ;\ \frac{1}{2}(x_2+x_3),\ \frac{1}{2}(y_2+y_3)\ ;\ \&c.,$$

which points are the centres of gravity of the several sides, and $\therefore$ the weights of the bars may be regarded as concentrated there, each weight in the centre of the bar having that weight: hence [1],

$$X=\frac{W_1(x_1+x_2)+W_2(x_2+x_3)+\ldots+W_n(x_n+x_1)}{2(W_1+W_2+\ldots+W_n)},$$

$$Y=\frac{W_1(y_1+y_2)+W_2(y_2+y_3)+\ldots+W_n(y_n+y_1)}{2(W_1+W_2+\ldots+W_n)}.$$

It is plain that it is quite optional whether, as here, we represent the uniform bars by their weights, or by their lengths; since, being uniform, their lengths are proportional to their weights:—the ratio $W_1 \div (W_1+W_2+\ldots+W_n)$ is the same as the ratio $l_1 \div (l_1+l_2+\ldots+l_n)$, where l is put for length: for if $W_1+W_2+\ldots+W_n$ is m times W_1, then of course the length of the n bars, united in one, must be m times the length l_1 of the bar weighing W_1. Hence the lengths or weights of the

sides being known, and the situations of the vertices, the position (X, Y), of the centre of gravity, is given by the above expressions, putting l_1, l_2, &c., for W_1, W_2, &c. If the polygon be regular, it is obvious that the centre of the figure, that is, the centre of the inscribed circle, will be the centre of gravity, and W_1, W_2, ... W_n will all be equal, so that then

$$Y=\frac{2(y_1+y_2+\ldots+y_n)}{2n}, \quad \therefore \; nY=y_1+y_2+\ldots+y_n,$$

which expresses this geometrical property, namely:—If from the n vertices of a regular polygon perpendiculars to any straight line in its plane be drawn, the sum of them all will be equal to n times the perp. from the centre of the polygon.

Instead of perpendiculars, it is clear that the lines from the vertices may be all inclined at any constant angle to the given straight line; for each of these parallels, multiplied by the sine of the constant angle, will give the corresponding perpendicular.

Suppose the polygon to consist of but three bars l_1, l_2, l_3, forming a triangle, and let the base l_3 be taken for the axis of x, the axis of y originating at its extremity: then $x_1=0$, $y_1=0$; also for the other extremity of the base, $x_3=l_3$, $y_3=0$, the vertex being (x_2, y_2). Then we have

$$X=\frac{l_1x_2+l_2(x_2+l_3)+l_3l_3}{2(l_1+l_2+l_3)}, \quad Y=\frac{l_1y_2+l_2y_2}{2(l_1+l_2+l_3)}.$$

If the three bars be equal in length $(=l_3)$, these become $X=\frac{1}{2}l_3$, $Y=\frac{1}{3}y_2$, because then x_2 is half the base, that is, $x_2=\frac{1}{2}l_3$, so that the centre of gravity of the perimeter of an equilateral triangle is the same point as the centre of gravity of the triangle itself, that centre being the centre of figure, as noticed above.

(2) A body of given weight W, and whose centre of gravity is also given (X, Y), rests on three given points (x_1, y_1), (x_2, y_2), (x_3, y_3), on a horizontal plane: required the pressures on those points. (See Prob. VI., p. 380.)

Calling the pressures P_1, P_2, P_3, the conditions [3] for determining them are

$$P_1+P_2+P_3=W, \quad P_1x_1+P_2x_2+P_3x_3=WX, \quad P_1y_1+P_2y_2+P_3y_3=WY;$$

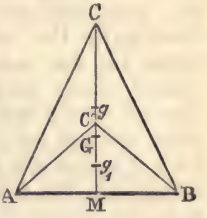
and when the given numerical values are put for all the quantities except P_1, P_2, P_3, these may be easily found from the three simple equations here represented. But if the question had been to find the pressures upon *four*, or a greater number of points, it could not have been answered, without additional equations, since *three* equations are insufficient for the determination of *four* or more unknown quantities.

[In reference to the above fact, writers sometimes say that the pressures upon the four legs of a rectangular table—the centre of gravity of which is in the middle—are indeterminate; but this is a mistake: the pressures on the legs are *equal*, each being one-fourth of the entire weight. If any *fourth* condition be introduced in reference to the pressures upon four points, the problem is always determinate.]

(3) Two isosceles triangles ABC, ABC', are on the same side of the same base: required the centre of gravity of the portion left of the larger, when the smaller is removed.

Draw $CC'M$, which will be a straight line bisecting AB at right angles.

Let $y_1=Mg_1$, be the distance from M of the centre of gravity of the smaller triangle, $Y=MG$, the distance of the centre of gravity of the larger, and $y=Mg$, the distance of the centre of gravity sought: then W_1, W_2, W_2-W_1, being the weights of these respectively, we have



$$W_1y_1+(W_2-W_1)y=W_2Y, \quad \therefore\ y=\frac{W_2Y-W_1y_1}{W_2-W_1}.$$

Now, putting H, h, for the heights of the two triangles, we know (p. 379) that $Y=\frac{1}{3}H$, and $y_1=\frac{1}{3}h$: also the weights of the triangles are as their areas, and these again are as the altitudes H, h,

$$\therefore\ y=\frac{H.\frac{1}{3}H-h.\frac{1}{3}h}{H-h}=\frac{1}{3}\frac{H^2-h^2}{H-h}=\frac{1}{3}(H+h).$$

Hence the distance of g, the centre of gravity sought, is $\frac{1}{3}$ the sum of the heights of the two triangles.

Having now established the general equations of equilibrium for a system of parallel forces, acting upon a body at different points, the way is effectually prepared for the consideration of forces acting at any points, and in any directions whatever, as will be seen in the following articles.

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423. **Analytical Conditions of Equilibrium in General.**—The line of action of every force must always be situated in some *vertical plane;* for if any vertical plane whatever be conceived to cut that line in a point, the plane, still continuing vertical, by being turned round will at length pass through the entire line, as is obvious. Imagine such a vertical plane passing through the line of action of any oblique force P_1, the point A_1 being the point of application of it. This force P_1, may be resolved into two directions in that plane, one horizontal and the other vertical; and the same in reference to any other force, P_2, P_3, &c. The *vertical* components are all necessarily *parallel* forces: the others, though all horizontal, may take various directions. Now, taking any one of these horizontal forces, we may resolve it into two horizontal directions at right angles to each other—*any* two rectangular directions we please, the two AX, AY, for instance, in the diagram at p. 383. It thus appears that, whatever be the lines of action of a set of forces applied to different points of a body, those forces can always be replaced by *three sets of parallel forces*, namely, two sets acting horizontally, one of the two in the direction AX (positively or negatively, as to any individual force), the other in the direction AY, at right angles to AX, and a third set acting vertically, or in the direction AZ, as we may call it, perpendicular to the horizon, or to the plane of the paper in the diagram at p. 383. Each of these sets of parallel forces must equilibrate separately if the body itself be in equilibrium, because, if they had a resultant, it would be impossible that it could be counteracted by any forces, *at right angles* to its line of action.

In thus inferring that each group of forces would alone keep the body at rest, even if the others were withdrawn, it must be observed that the

inference is justified only as regards *progressive* motion, the resultant of the forces being zero. In order that there may be no tendency to rotation about any given axis, the sum of the moments of the forces, in reference to that axis, must be zero, as at p. 384. Now in the case before us, the moments with respect to AY are due not only to the vertical forces as at p. 384, but also to the horizontal forces whose lines of action are parallel to AX. In like manner, the moments about AX are supplied both by the vertical forces, and by those whose lines of action are parallel to AY. The forces with which we are now dealing are the components $P_1 \cos \alpha_1$, $P_2 \cos \alpha_2$, &c., in the direction AX, the components $P_1 \cos \beta_1$, $P_2 \cos \beta_2$, &c., in the direction AY, and the components $P_1 \cos \gamma_1$, $P_2 \cos \gamma_2$, &c., in the vertical direction AZ; for the rectangular component of a force is the product of that force by the cosine of its inclination to the line along which that component is taken: hence the sum of the moments round AY, as furnished by the vertical group, and the sum furnished by that horizontal group which is parallel to AX, are $\Sigma(P \cos \gamma \,.\, x)$, and $\Sigma(P \cos \alpha \,.\, Z)$, where z is the distance from AY of the point where the direction of the force $P \cos \alpha$ pierces the vertical plane, just as x is the distance from AY where the force $P \cos \gamma$ pierces the horizontal plane. Regarding $P \cos \alpha$, acting downwards upon the horizontal plane, as taking any sign, and $P \cos \gamma$ acting on the vertical plane from right to left, as taking the same sign, the above sets of moments produce opposite effects, as to causing rotation about AY; so that the rotation resulting from their joint influence is the difference of the two; that is, it is $\Sigma P(x \cos \gamma - z \cos \alpha)$, which, therefore, in the case of equilibrium, must be zero. Hence that all tendency to motion in the body, whether progressively in either of the three rectangular directions, or rotatory round either, may be destroyed, the six following conditions must have place, namely,

$$\Sigma(P \cos \alpha)=0,\ \Sigma(P \cos \beta)=0,\ \Sigma(P \cos \gamma)=0 \ldots\ldots [1],$$

$$\Sigma P(y \cos \alpha - x \cos \beta)=0,\ \Sigma P(x \cos \gamma - z \cos \alpha)=0,\ \Sigma P(y \cos \gamma - z \cos \beta)=0 \ldots\ldots [2].$$

If the conditions [1] hold, so that progressive motion may be forbidden, and one of the conditions [2] also, so that rotation about a particular axis may likewise be forbidden, then neither can there be motion about *any* axis parallel to this. For let a parallel to AZ be drawn, meeting the horizontal plane in any point (a, b); and what, in reference to the original vertical, AZ, was the point (x, y), let now be (x', y'), in reference to the new vertical; that is, let $x = x' + a$, $y = y' + b$: then we shall have

$$\Sigma(P \cos \alpha)=0,\ \Sigma(P \cos \beta)=0,\ \Sigma P[(y'+b) \cos \alpha - (x'+a) \cos \beta]=0.$$

Now from the first two of these, $b\Sigma(P \cos \alpha)=0$, and $a\Sigma(P \cos \beta)=0$; hence the third gives

$$\Sigma P(y' \cos \alpha - x' \cos \beta)=0, \text{ which forbids rotation about the new vertical.}$$

It follows, therefore, when the foregoing six conditions hold, that there can be no tendency to rotation about either of the axes which have the directions of AX, AY, AZ, wherever these may be situated; in other words, no tendency to rotation is impressed upon the body by *either* of the three groups of parallel forces into which the original forces have been resolved, and $\therefore$ the body must be completely at rest, since these forces alone act on it. When the body to which the forces are applied is not free, but is only at liberty to move round a *fixed* point, then, taking this point for the origin, the equilibrium will be established if the equations [2] alone be satisfied, since these forbid rotation; and progression

is prevented by the fixed point. The pressure on this point must be equal in intensity to the resultant of all the applied forces, since the resistance of the point balances them all, and therefore must be equal in intensity and opposite in direction to the resultant.* It is obvious, too, from this, that when the equations [2] are satisfied, that is, when rotation cannot take place, the forces, however applied, must have but a single resultant; and this resultant would be just the same if all the forces were to be transferred parallel to themselves to any point in its direction and to act there: this is plain from equations [1], which imply no restriction as to the points of application of the forces P_1, P_2, but take account only of the *directions* in which they act. Representing the sum of the pressures upon the fixed point, in the directions AX, AY, AZ, by the symbols X, Y, Z, we have [1],

$$X=-\Sigma(P\cos\alpha),\quad Y=-\Sigma(P\cos\beta),\quad Z=-\Sigma(P\cos\gamma).$$

When instead of the body being at liberty to move round a fixed *point*, it is restricted to motion round a fixed *axis*, or axle, then taking this axis for AZ, all tendency to motion round it will be prevented if the single condition $\Sigma P(y\cos\alpha-x\cos\beta)=0$ be satisfied; and the three equations just given will express the pressures sustained by the axle in the three rectangular directions.

424. From the foregoing investigation it appears that though forces, which all applied to a single point, will keep that point at rest, provided they satisfy the conditions [1], yet that when they act at different points in a rigidly-connected system, the additional equations [2] must be also satisfied to prevent the rotation which otherwise would be communicated to the system. When, therefore, any of the conditions [2] are unfulfilled, and rotation ensues in consequence, it is impossible that any one of the applied forces, taken in opposite direction, can be the *only* resultant of all the others, as would be the case if there were no rotation. Besides this resultant there must be a pair of forces forming a *couple* (p. 376), the introduction or suppression of which could not affect the conditions [1], since the pair of forces are equal and opposite: but their existence or non-existence, in the system, determines the question as to whether or not there shall be rotation. If [2] are fulfilled, we may be sure that there is no resultant couple furnished by the applied forces, and that then any one of these forces, taken in the opposite direction, is the resultant and the only resultant of all the others.

We shall merely observe, in conclusion, that if R be the resultant (of progression) of the forces P_1, P_2, &c., and if its inclination to the three axes AX, AY, AZ, be a, b, c, then, since the opposing forces $R\cos a$, $R\cos b$, $R\cos c$, being combined with the forces $\Sigma(P\cos\alpha)$, $\Sigma(P\cos\beta)$, $\Sigma(P\cos\gamma)$, would produce equilibrium, by representing these latter by X, Y, Z, respectively, we must have

$$R\cos a=X,\ R\cos b=Y,\ R\cos c=Z,\ \text{also}\ R^2=X^2+Y^2+Z^2\ldots\ldots[3],$$

because the square of the diagonal of a right parallelopiped—and R *is* such a diagonal—is the sum of the squares of the three edges about that

* It would be well if, in books on Mechanics, what is thus always called *the resultant*, were, for the sake of precision, called *the resultant of progression*, to distinguish it from *the resultant of rotation*, which is a *couple*. A body not kept in equilibrium may have either a resultant of progression only, a resultant couple only, compelling rotation, or both a resultant of progression, and a resultant couple.

diagonal. The magnitude of the resultant is thus determinable from [3], and its direction, fixed by the angles it makes with the three axes, is found from the equations

$$\cos a=\frac{X}{R},\quad \cos b=\frac{Y}{R},\quad \cos c=\frac{Z}{R}.$$

If the original forces all act in one plane, the plane of the axes AX, AY, for instance, then the force Z is zero, and we have

$$R^2=X^2+Y^2,\ \cos a=\frac{X}{R},\ \cos b=\sin a=\frac{Y}{R},\ \therefore \tan a=\frac{Y}{X}\text{.....[4].}$$

The moments of all the forces about the axis AZ, or rather, as they are in one plane perp. to that axis, about the origin A, are

$$\Sigma P(y\cos\alpha-x\cos\beta)=R(y'\cos a-x'\cos b),$$

(x', y') being the point of application of the resultant R. Now, p. 309,

$y\cos\alpha-x\cos\beta$=the length of the perp. from the origin upon the direction of P, and

$y'\cos a-x'\cos b=$,, ,, ,, ,, R.

Calling the several perpendiculars upon the directions of the forces P_1, P_2, &c., p_1, p_2, &c., and that upon the direction of R, r, we have $\Sigma(Pp)=Rr$, that is, the sum of the moments of the components is equal to the moment of the resultant, as was shown for concurring forces at p. 373. And the equations of equilibrium, R being zero, may be written

$$X=\Sigma(P\cos\alpha)=0,\ Y=\Sigma(P\cos\beta)=0,\ \Sigma(Pp)=0\text{.....[5].}$$

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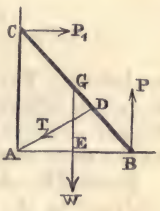
425. **Problems on Equilibrium.**—If a perfectly-smooth surface be pressed upon by the extremity of a rigid line or slender rod, in a direction inclined to that surface, the force acting along the rod will cause the end of the rod to slide along the surface: for this force may be decomposed into two forces, one acting perpendicularly to the surface, and the other in the direction of that surface. If there be no friction, that is, if the surface be perfectly smooth, there will be nothing to resist the latter force, and it must therefore take full effect, and the end of the rod will *slide:* this resolvent, therefore, of the force acting along the rod, meeting with no resistance, it is plain that the resistance of the surface, or the pressure upon the surface, is wholly in a direction perpendicular to that surface. In the following problems the surfaces brought into contact, and pressing against each other, will always be regarded as smooth surfaces; so that when the forces acting upon their points of contact are in equilibrium, those resolvents of which the directions are parallel to, or tangential to, the surfaces, necessarily destroy each other: all pressures and their opposing resistances, in the case of perfect smoothness, are perpendicular to the surfaces pressed. Friction will be taken into account hereafter.

In the problems about to be given, the forces concerned in each case will all act in one plane, and gravity will always be one of them. These forces will be replaced by their components acting in directions at right angles to each other—usually in the horizontal and vertical directions. The student will remember that although horizontal and vertical axes may be exhibited in the diagram, the sole intention of them is to show the *directions* in which the component forces, applied at different points

in the plane of those axes, act: they do not act *along* the axes, but only parallel to them. Yet the conditions of equilibrium are the very same as they would be if these component forces had their lines of action actually coincident with the axes, and were applied at the point which is their origin. The conditions here spoken of forbid all progressive motion of the body or system: there must be another condition to preclude all motion of rotation—the equation of moments. The centre of moments may be chosen at pleasure in the plane of the forces; for if the moments balance about any one point, they will balance about every other in that plane: but if there be a fixed point in the system, it is usually preferable to take that for the centre of moments. When unknown forces or pressures are to be determined, three equations will, in general, be necessary; the two equations furnished by the components, and which forbid translation, and the equation of moments furnished by the forces themselves: the components, however, may also be employed for this purpose; when the *position* merely of the system is required, one only of the former two equations—the equation between the horizontal components—will usually suffice. [See the NOTE at p. 371.]

PROB. I. A heavy homogeneous rod rests, with its lower end on a horizontal plane AB, against a smooth vertical wall AC, and is sustained in its position by a string AD, fastened to it at a given point D: required the tension of the string.

Let $2a$ be the length of the uniform bar, and W its weight, acting vertically at its centre of gravity G, the middle of the rod BC. Let α be the inclination ABC of the rod to the horizon, and θ the inclination DAB of the string: then, calling the required tension T, and the resistances (opposite to the pressures) at B, C, P, P_1, the vertical forces are W and $T\sin\theta$ acting downwards, and P acting in the opposite direction. The horizontal forces are P_1, and $T\cos\theta$ acting in opposite directions. The moments of the forces W, P, P_1, to turn the system about A, are $W.AE=W.BE=W.a\cos\alpha$, and $P_1.AC=P_1.2a\sin\alpha$ in one direction, and $P.AB=P.2a\cos\alpha$ in the opposite direction: hence the equations of equilibrium are



$$P=W+T\sin\theta,\quad P_1=T\cos\theta\ldots[1],\quad \text{and}\quad Wa\cos\alpha+2P_1a\sin\alpha=2Pa\cos\alpha\ldots[2].$$

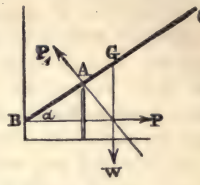
Suppressing the common factor a in the last equation, and substituting the values of P, P_1, from the preceding equations, we have

$$W\cos\alpha+2T\sin\alpha\cos\theta=2W\cos\alpha+2T\cos\alpha\sin\theta,\quad \therefore\ T=W\frac{\cos\alpha}{2\sin(\alpha-\theta)},$$

the tension required: the pressures P, P_1, are given by the first two equations. If the point D to which the string is fastened be so high up that $\theta=\alpha$, no force acting in the direction of the string will be sufficient to keep the bar from sliding; for T then becomes ∞. If D be still higher up, so that $\theta>\alpha$, the force along the string must act in the opposite direction for the equilibrium to be maintained, for T is then negative.

PROB. II. A heavy uniform rod BC presses with one extremity against an upright wall, and is supported on a fixed point A, at a given perpendicular distance c from the wall: required the pressures, and the inclination of the rod to the horizon.

Let $2a$ be the length, W the weight of the rod, acting at its middle point G, and α its inclination to the horizon; and let the pressures at B, A, be P, P_1. Resolving the forces into the rectangular directions AC, AP_1, we have



$$P\cos\alpha = W\sin\alpha,\quad P_1 = P\sin\alpha + W\cos\alpha \ldots\ldots [1].$$

Also for the moments about A, we have

$$P\,.\,AB\sin\alpha = W(BG - BA)\cos\alpha, \text{ that is,}$$
$$P\,.\,AB\sin\alpha = W(a - AB)\cos\alpha;$$

$$\text{but } AB = \frac{c}{\cos\alpha},\ \therefore\ Pc\sin\alpha = W(a\cos\alpha - c)\cos\alpha \ldots\ldots [2].$$

$$\text{From the first of [1], } P = W\frac{\sin\alpha}{\cos\alpha},\ \therefore [2],\ Wc\frac{\sin^2\alpha}{\cos\alpha} = W(a\cos^2\alpha - c\cos\alpha),$$

$$\therefore\ c\sin^2\alpha + c\cos^2\alpha = a\cos^3\alpha,\ \therefore\ \cos^3\alpha = \frac{(\sin^2\alpha+\cos^2\alpha)c}{a} = \frac{c}{a},\ \therefore\ \cos\alpha = \left(\frac{c}{a}\right)^{\frac{1}{3}},$$

$$\therefore\ P = W\tan\alpha = W\left[\left(\frac{a}{c}\right)^{\frac{2}{3}} - 1\right]^{\frac{1}{2}},\ \text{and from [1], } P_1 = \frac{W}{\cos\alpha} = W\left(\frac{a}{c}\right)^{\frac{1}{3}}.$$

The value of $\cos\alpha$ shows that the equilibrium cannot exist if $c > a$.

Otherwise. Taking the horizontal and vertical components, as also the moments about B, we have

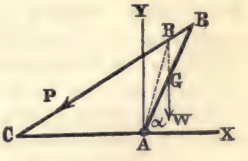
$$P_1\sin\alpha = P,\ P_1\cos\alpha = W \ldots [1],\ \text{and } Wa\cos\alpha = P_1 c\sec\alpha = P_1 c\frac{1}{\cos\alpha} \ldots [2],$$

$$\therefore\ P_1 = \frac{W}{\cos\alpha},\ \therefore\ Wa\cos\alpha = Wc\frac{1}{\cos^2\alpha},\ \therefore\ \cos^3\alpha = \frac{c}{a},\ \therefore\ \cos\alpha = \left(\frac{c}{a}\right)^{\frac{1}{3}},$$

as before. The centre of moments may, of course, be taken at pleasure: thus, we might, if we pleased, have combined the equations [1] of the former solution with the equation [2] in this, and we should have arrived at the same result.

PROB. III. A heavy bar AB, moveable in a vertical plane about A, is inclined to the horizon at a given angle α: a rope fastened at B, and pulled by a force at C, AC being $= AB$, keeps the bar in that position: required the intensity of the force at C, and the pressure on the hinge A, the weight of the bar being W, and its length $2a$.

Let P represent the force in the direction BC, and let the rectangular components of the unknown resistance R of the hinge A be represented by X, Y; then resolving the other force P acting at C in these directions, and observing that W acting at G, the middle of AB, opposes the vertical components of the other two forces, we have



$$X = P\cos\frac{1}{2}\alpha,\ Y = W + P\sin\frac{1}{2}\alpha,$$

$$\text{and } Wa\cos\alpha = P\,.\,2a\sin\frac{1}{2}\alpha.$$

From the last of these we have

$$P = W\frac{\cos\alpha}{2\sin\frac{1}{2}\alpha},\ \text{the force at C.}$$

$$\therefore\ X = \frac{1}{2}W\cos\alpha\cot\frac{1}{2}\alpha,\ Y = W + \frac{1}{2}W\cos\alpha,$$

$$\therefore X^2+Y^2=R^2=\frac{1}{4}\left(W\cos\alpha\cot\frac{1}{2}\alpha\right)^2+W^2\left(1+\frac{1}{2}\cos\alpha\right)^2.$$

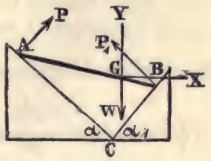
The square root of this is the intensity of the pressure on the hinge, and $\tan\theta=\frac{Y}{X}$ is the tangent of the angle at which its direction is inclined to AX. If the vertical through G be prolonged to meet the line of action of the force at C in R, AR will be the direction of the force of resistance at A (413), and RAX is the angle here represented by θ. If from any point in RW, G for instance, a parallel terminating in RA be drawn to BC, the triangle thus formed will be the triangle of forces, the sides being proportional to the equilibrating forces acting in the directions of those sides.

Prob. IV. A heavy rod of given length, not uniform in thickness, but whose centre of gravity is known, is placed with its ends on two smooth inclined planes: required the position in which it will rest, and its pressures upon the two planes.

Let AB be the rod resting on the two smooth surfaces CA, CB, inclined at angles α, α_1, to the horizon; let G be the centre of gravity of the rod, and put $AG=a$, $GB=b$. Then taking G for the origin of the horizontal and vertical axes, let the unknown pressures P, P_1, or rather the opposing resistances at A, B, be resolved in these directions, then we have, since the pressures are perp. to the planes,

$$P\sin\alpha=P_1\sin\alpha_1,\ P\cos\alpha+P_1\cos\alpha_1=W\ldots\ldots[1].$$

Also if θ be the unknown angle at which the rod is inclined to the horizon, we have for the moments about G,



$$Pa\cos(\alpha-\theta)=P_1b\cos(\alpha_1+\theta),\ \therefore[1],$$

$$\frac{P}{P_1}=\frac{\sin\alpha_1}{\sin\alpha}=\frac{b\cos(\alpha_1+\theta)}{a\cos(\alpha-\theta)},$$

$$\therefore\frac{a\sin\alpha_1}{b\sin\alpha}=\frac{\cos\alpha_1\cos\theta-\sin\alpha_1\sin\theta}{\cos\alpha\cos\theta+\sin\alpha\sin\theta}=\frac{\cos\alpha_1-\sin\alpha_1\tan\theta}{\cos\alpha+\sin\alpha\tan\theta},$$

from which we get $\tan\theta=\frac{b\cot\alpha_1-a\cot\alpha}{a+b}$, the position of the rod.

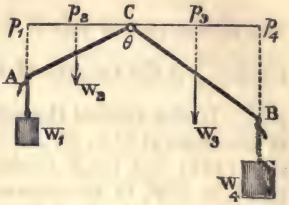
As to the two pressures P, P_1, we have from [1], $P_1=P\frac{\sin\alpha}{\sin\alpha_1}$,

$$\therefore P(\sin\alpha_1\cos\alpha+\cos\alpha_1\sin\alpha)=W\sin\alpha_1,\ \therefore P=W\frac{\sin\alpha_1}{\sin(\alpha+\alpha_1)},\ \therefore P_1=W\frac{\sin\alpha}{\sin(\alpha+\alpha_1)}.$$

If the centre of gravity be at the middle of the rod, then $a=b$, and the tangent of inclination is $\tan\theta=\frac{1}{2}(\cot\alpha_1-\cot\alpha)$; but the pressures on the planes are the same wherever this centre may be, or whatever be the length of the rod. These pressures, too, would be the same if instead of a rod any other heavy body—a sphere for instance—were to rest between the planes, and have a single point of contact with each.

Prob. V. A uniform bent bar ACB is suspended at C, about which point it is free to turn in a vertical plane: weights W_1, W_4, are attached to its ends: to find the position in which it will rest.

Taking the fixed point C for the centre of moments, it will be sufficient for the equilibrium that the moments about C are equal. The applied forces are the weight W_1 acting at A, the weight W_2, of the bar $CA=2a$, acting at its middle point, and the weight W_3, of the bar $CB=2a'$, acting at its middle point, and lastly the weight W_4 acting at B. Put α_1 for the unknown angle ACp_1, θ for the known angle ACB, then $\pi-(\alpha_1+\theta)$ will be the angle α_2 or BCp_4.



The condition of equilibrium is $W_1 . Cp_1+W_2 . Cp_2=W_3 . Cp_3+W_4 . Cp_4$,

where $Cp_1=2a\cos\alpha_1$, $Cp_2=a\cos\alpha_1$, $Cp_3=a'\cos\alpha_2=-a'\cos(\alpha_1+\theta)$,

$Cp_4=2a'\cos\alpha_2=-2a'\cos(\alpha_1+\theta)$: hence the equation of equilibrium is

$$(2W_1+W_2)a\cos\alpha_1=-(W_3+2W_4)a'\cos(\alpha_1+\theta)\ldots\ldots[1].$$

Now since $-\dfrac{\cos(\alpha_1+\theta)}{\cos\alpha_1}=\dfrac{\sin\alpha_1\sin\theta-\cos\alpha_1\cos\theta}{\cos\alpha_1}=\tan\alpha_1\sin\theta-\cos\theta$,

[1] is the same as $(2W_1+W_2)a=(W_3+2W_4)a'(\tan\alpha_1\sin\theta-\cos\theta)$

$$\therefore\ \tan\alpha_1=\frac{(2W_1+W_2)a+(W_3+2W_4)a'\cos\theta}{(W_3+2W_4)a'\sin\theta}.$$

If there are no weights at their ends, then $\tan\alpha_1=\dfrac{W_2a+W_3a'\cos\theta}{W_3a'\sin\theta}$.

And these results remain the same whether the two arms CA, CB, are of equal thickness or not: but if they are equally thick, their weights must be as their lengths, so that $\dfrac{W_2}{W_3}=\dfrac{a}{a'}$, and the last expression may then be written $\tan\alpha_1=\dfrac{a^2}{a'^2}\operatorname{cosec}\theta+\cot\theta$.

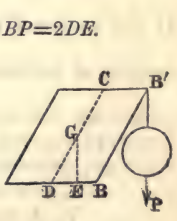
PROB. VI. An oblique cylinder stands on a horizontal plane, to which its axis CD is inclined at angle of 60°; the perp. height of the cylinder is 4 feet, and the diameter of its base 3 feet: required the diameter of the greatest sphere of the same material, that will hang suspended from the upper edge B', without overturning the cylinder.

The centre of gravity G of the cylinder is at the middle of CD, and the forces tending to turn it about B are the weight of the cylinder in the vertical GE, and the weight of the sphere in the vertical $B'P$: hence, equating the moments, we have

$BE\times$Cylinder$=BP\times$Sphere, where $BE=BD-DE$, and $BP=2DE$.

Now $DE=GE\tan G=2\tan 30°=\dfrac{2}{3}\sqrt{3}$, $\therefore\ BP=\dfrac{4}{3}\sqrt{3}$,

and $BE=\dfrac{3}{2}-\dfrac{2}{3}\sqrt{3}$:



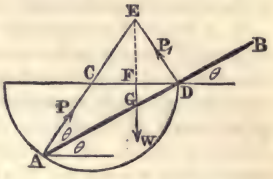
hence, substituting the volumes of the homogeneous bodies for their weights, we have

$$\left(\frac{3}{2}-\frac{2}{3}\sqrt{3}\right)\times3^2\times\cdot7854\times4=\frac{4}{3}\sqrt{3}\times\frac{2}{3}\times\cdot7854\times(\text{diameter})^3,$$

$$\therefore \text{(diam.)}^3=\frac{(9-4\sqrt{3})3^4}{12\sqrt{3}}=\frac{3^3(3\sqrt{3}-4)}{4}=8{\cdot}074, \therefore \text{diam.}=2{\cdot}006 \text{ ft.}$$

PROB. VII. A heavy uniform rod AB has one end in a smooth hemispherical bowl of radius r, the other end projecting over the rim, which is in a horizontal plane: required the position of equilibrium of the rod.

Let θ be the inclination of the rod to the horizon, the rod being kept at that inclination—first, by the resistance P at A acting perpendicularly to the curve surface at A, and $\therefore$ in a direction passing through C, the centre; secondly, by the resistance P_1 acting perp. to the rod at D; and lastly, by the weight of the rod acting vertically downwards at G, the middle of AB. Equating the horizontal components of P, P_1, we have



$$P\cos 2\theta=P_1\sin\theta, \therefore \frac{P}{P_1}=\frac{\sin\theta}{\cos 2\theta}.$$

And the moments about G are $P \,.\, a\sin\theta=P_1 \,.\, GD$, $\therefore \frac{P}{P_1}=\frac{GD}{a\sin\theta}$,

$$\therefore \frac{GD}{a\sin\theta}=\frac{\sin\theta}{\cos 2\theta}. \quad \text{Now } AD=2r\cos\theta, \therefore GD=AD-a=2r\cos\theta-a,$$

$$\therefore \frac{2r\cos\theta-a}{a\sin\theta}=\frac{\sin\theta}{\cos 2\theta}=\frac{\sin\theta}{2\cos^2\theta-1}, \therefore 4r\cos^2\theta-a\cos\theta-2r=0,$$

a quadratic from which cos θ may be easily found.

If the pressures P, P_1, are required, we may readily obtain them from the two equations furnished by the horizontal and vertical components, viz.

$$P\cos 2\theta-P_1\sin\theta=0, \text{ and } P\sin 2\theta+P_1\cos\theta=W, \text{ the wt. of the rod.}$$

Thus, to eliminate P_1, multiply the first by cos θ, the second by sin θ, and add; we then get

$$P(\sin 2\theta\sin\theta+\cos 2\theta\cos\theta)=W\sin\theta, \therefore P=\frac{W\sin\theta}{\cos(2\theta-\theta)}=\frac{W\sin\theta}{\cos\theta}=W\tan\theta,$$

$$\therefore P_1=\frac{P\cos 2\theta}{\sin\theta}=W\frac{\cos 2\theta}{\cos\theta}.$$

If the pressures are equal then the inclination θ must be 30°, since we must have $\cos 2\theta=\sin\theta$. It may be further noticed that if the length EF represent the weight of the bar, then will FD represent its pressure on A, since $P=W\tan\theta$.

EXAMPLES FOR EXERCISE.

(1) Two rafters of a roof form an isosceles triangle, each of the base angles being θ, the weight of each rafter is W: what is the horizontal thrust on the side walls?

(2) Six forces P_1, P_2, P_3, P_4, P_5, P_6, act at the centre of a regular hexagon, and are directed towards the six vertices: if the magnitudes of the forces are 1, 2, 3, 4, 5, respectively, what will be the magnitude and direction of the resultant?

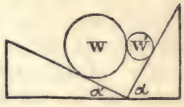
(3) A rope of given length is to be fixed to an upright pillar: what angle θ must it make with the horizon in order that a given force P, applied at the other end, may be the most effective in overturning the pillar?

(4) A ladder of uniform thickness, of which the weight is 50 lb., is inclined against an upright wall at an angle 60° with the horizon: the foot is secured from slipping by a wedge: a person weighing 12 stone ascends with a burden of 100 lbs: required the pressure against the wall and the thrust at the bottom when the middle of the ladder is arrived at.

(5) Two ladders of lengths $2a$, $2b$, respectively, and of weights W, W', are raised against two opposite vertical walls with their lower ends or feet acting against each other: required the distance between the walls when the ladders are at right angles to each other.

(6) A door of weight W hangs upon two pivots or hinges in a vertical line: the distance of the centre of gravity of the door from this vertical is a, and the distance between the pivots is b: required the horizontal pressure or strain upon the pivots.

(7) Two spheres of weights W, W', rest on two inclined planes, and press against each other: the inclinations of the planes to the horizon are α, α': it is required to find the inclination θ of the line joining the centres of the spheres to the horizon.



(8) A weight W is suspended from the edge of a hemisphere weighing W, resting on its convex surface on a smooth horizontal plane: required the position of equilibrium, that is, the angle θ, by which the axis of the hemisphere deviates from the vertical, the radius being r, and the distance of the centre of gravity from the centre of the sphere being c.

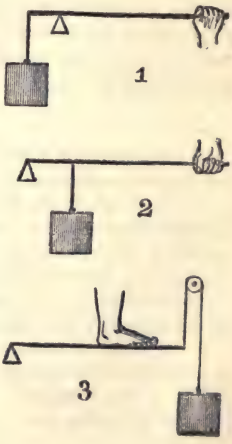
(9) Two weights W, W', connected together by a string of given length, rest on the upper surface of a sphere: if α be put for the angle at the centre, subtended by the arc to which the string is applied: required the position of equilibrium, or the angle θ at the centre formed by the vertical and the radius drawn to one of the weights W.

(10) A thin circular plate is supported on a slender vertical rod at its centre: it is required to find at what distances round its circumference three given weights W_1, W_2, W_3, must be placed so that the plate may remain horizontal.

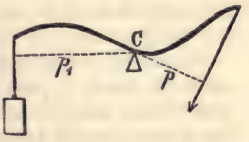
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426. **The Mechanical Powers: Machines in Equilibrium.**—The mechanical powers, as they are called, are the constituent parts or elements of which every machine is composed. They are six in number:—1. the Lever; 2. the Wheel and Axle; 3. the Pulley; 4. the Inclined Plane; 5. the Wedge; and 6. the Screw.

427. **The Lever.**—This is a rigid bar or rod:—in its simplest form it is a *straight* bar, moveable freely, in a plane, about some fixed point (or axis) of support, which fixed point is called the *fulcrum*. It is divided into three kinds, according to the position of the fulcrum in reference to the force applied and the resistance to be overcome, or balanced: the applied force is usually called the power. The three kinds of lever are represented in the annexed diagrams. Applied as in fig. 1, the lever is of the first kind, the fulcrum being between the power and the resistance. Applied as in fig. 2, the lever is of the second kind, the resistance being between the power and the fulcrum. And applied as in fig. 3, the lever is of the third kind, the power being between the fulcrum and the resistance. In the diagrams, the bar is straight, but, whatever be its shape, if acting as here represented, it is a lever: a perpendicular from the fixed point or fulcrum, to the direction in which the power acts is the *arm* of the power: the perpendicular from the fulcrum to the direction in which the resistance acts is the arm of the resistance: the portion of the lever between the fulcrum and either end is called an arm of the *lever*. There will obviously be equilibrium when the power multiplied by its arm is equal to the resistance multiplied by *its* arm, that is, when the



moments about the fulcrum are equal: the weight of the lever itself being disregarded. Thus, calling the power P, and the weight or resistance to be balanced W, the perpendiculars p, p_1, upon the directions in which the forces P, W, act, will be the arms of the power and weight, and the condition of equilibrium will be $Wp_1=Pp$, $\therefore \frac{W}{P}=\frac{p}{p_1}$; that is:



The power and resistance are to each other inversely as their arms, or the perpendiculars on their lines of direction.

If the weight w of the lever itself be taken into consideration, it must be regarded as an additional force acting at the centre of gravity of the bar: calling the perp. distance of the direction of w from the fulcrum g, we must then have

$$Wp_1 \pm wg = Pp \ldots\ldots [1]$$

the upper or lower sign being used according as w favours or opposes W: when $g=0$, that is, when the centre of gravity is at the point taken for fulcrum, the condition of equilibrium is the same as if the bar had no weight.

It will be observed that in the first kind of lever, mechanical advantage is gained or not according as the power is farther from or nearer to the fulcrum: in the second kind, mechanical advantage is always gained: in the third, it is never gained. The human arm is a lever of this last kind:—the elbow-joint is the fulcrum, the muscle is the power acting on the lever (the fore arm) near the fulcrum, and the weight supported by the hand is the resistance: the mechanical disadvantage is more than compensated by the extent of range of which the hand is thus made capable.

Besides the rough purposes to which the straight lever of the first kind is applied to ease manual labour, it is of important use in certain delicate contrivances:—especially in those for weighing.

428. **The Common Balance.**—The common balance, or a "pair of scales," consists of a lever of the first kind, the fulcrum or axis on which it turns being at the middle of the lever,—called, in this case, the *scale-beam*. When the extremities of the beam are equally loaded, it should rest on the fulcrum in a horizontal position: the loads act by means of *scale-pans*, which are suspended from the extremities of the beam, and into which the weights, and the commodity to be weighed, are respectively put. A good balance must satisfy these three conditions.

1. When loaded with equal weights the extremities of the beam should be in a line perfectly horizontal.

2. When there is any—the slightest—difference between the weights, this straight line should cease to be horizontal.

3. When disturbed, or moved out of its horizontal position, it should rapidly return to its horizontal state of equilibrium upon the disturbance ceasing.

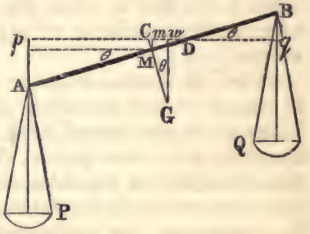
The first condition is that of horizontality; the second, that of sensibility:—the third, that of stability.

Horizontality.—To secure this the line joining the extremities of the beam must be perpendicular to that which joins the fulcrum and centre of gravity when the beam is at rest; for this latter line is always vertical. The fulcrum must not be *at* the centre of gravity, because then the beam

would remain at rest in every position (418). That the horizontal position may be retained when the extremities are loaded with equal weights, these extremities must be equidistant from the fulcrum, because the moments of the weights must be equal when there is equilibrium.

Sensibility.—To secure the greatest amount of sensibility, that is, to cause the beam to pass from a horizontal to an oblique position when there is the least possible inequality between the weights, it is necessary that the friction at the fulcrum or point of support should be reduced to the very smallest amount. With a view to this, in balances for philosophical purposes, the beam rests upon what is called a *knife-edge* (▽), supported upon horizontal plates of polished agate, and is itself usually made of brass, in order that it may not acquire any magnetic properties that might disturb perfect equilibrium and freedom of motion. It is, moreover, necessary that it be as light as is compatible with strength and rigidity, since the greater the pressure on the fulcrum the greater will be the friction.

Let C be the point of the knife-edge round which AB, joining the points to which the scale-pans are suspended, turns. If G be the centre of gravity of the beam and pans, of weight W, when unloaded, the line CG will be vertical, and will retain that position when the pans are equally loaded. But if P be a greater weight than Q, CG, perp. to AB, will be thrust out of the vertical, and the centre of gravity of the whole, when AB takes the inclined position in the diagram, will be some point vertically under C.



Let $AM=MB=a$, $CG=h$, $CM=k$, and the angle which AB makes with the horizontal line pq, θ. Then the moment of P to turn the system about C, in one direction is $P\times Cp$, and the moments of Q and W, to turn the system in the opposite direction, are $Q\times Cq$, and $W\times Cw$ respectively: the former moment, therefore, must be equal to the sum of these two moments. Now M being the middle of AB, the vertical mM bisects every line drawn through it, and terminated by the verticals from p, q; hence m is the middle of pq: moreover, the angle CMD being a right angle, the angle CMm = the angle $G=\theta$.

$$\therefore Cp=mp-mC=a\cos\theta-k\sin\theta,\ Cq=mq+Cm=mp+Cm=a\cos\theta+k\sin\theta,$$

and $Cw=h\sin\theta$: hence the equation of moments is

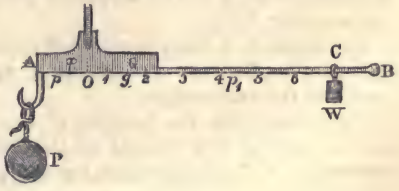
$$P(a\cos\theta-k\sin\theta)=Q(a\cos\theta+k\sin\theta)+Wh\sin\theta,\ \therefore\frac{\tan\theta}{P-Q}=\frac{a}{(P+Q)k+Wh},$$

an equation which determines θ, the angle of deviation from the horizontal line. The greater this angle is, for the same excess of weight, $P-Q$, the more is the sensibility of the balance increased; or the less be the difference $P-Q$ necessary to produce a given deviation θ, the greater is the sensibility of the balance: hence $\frac{\tan\theta}{P-Q}$ ought to be the greatest possible: it may be regarded as the measure of the sensibility: the condition, therefore, of greatest sensibility is satisfied when $\frac{a}{(P+Q)k+Wh}$ is as great as

it can be; the construction, therefore, should be such as to secure as large a value for this expression as is consistent with convenience and with the other necessary conditions; $2a$, for instance, the distance between the points of suspension should be as long as other considerations will allow; W the weight of the beam and pans should be as much as possible reduced, and the smaller the distance k of the knife-edge from the line joining the points of suspension, the more is the insensibility increased, as also by reducing the distance h between C and G. There must, however, be a limit to these reductions, for though it is desirable that a large deviation should accompany a small excess of weight, yet it is necessary that the horizontality be restored when that excess is withdrawn, which, however, it would not be if C and G coincided, and the nearer they are to coincidence the more slowly would the beam return to its horizontal position: stability requires that this return be rapid.

Stability.—In speaking of sensibility we have regarded only the effect of a small inequality of weights in producing a sensible deviation of the beam from its horizontal position, but it is necessary, when perfect equality is restored, that the beam returns to horizontality: this requires that the centre of gravity G be below the fulcrum, as we have represented it to be in the diagram. The greater the distance between these two points the longer will be the line Cw for any given disturbance, and therefore the greater the moment to bring down the arm MB when the disturbing cause is removed: in other words, the greater is the stability; this, therefore, is opposed to sensibility, since for *this* the smallness of CG is an advantage. But in the balance of commerce a quick return of the beam to horizontality is of importance, inasmuch as in business transactions *time* is. In the balance for purely philosophical purposes time is of less consequence, and sensibility is the main object aimed at; the best of these balances are so constructed as to detect a difference of weight of a thousandth part of a grain. The only way of increasing sensibility without disturbing stability is to lengthen the equal arms of the balance.

429. **The Roman Steelyard.**—This is another application of the lever to the purposes of a balance for weighing commodities. It is simply an iron or a steel bar, moveable about a knife-edge fulcrum O; the body P to be weighed is hung at the extremity of the shorter arm A, and a given weight W is moved along the other arm till it balances P, the weight of which is marked by the figure on the arm OB, at which the weight rests: the graduation of this arm is effected thus:—



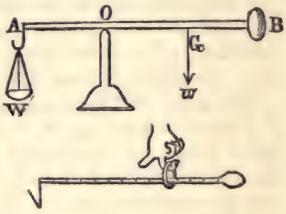
Let G be the centre of gravity of the bar and hook, and Gg the vertical from it: let $OA=p$, $OC=p_1$, $Og=g$, and put w for the weight of the bar and hook. Then by the last of equations [5], p. 390,

$$Wp_1+wg=Pp, \therefore p_1+\frac{wg}{W}=\frac{Pp}{W}. \quad \text{Make } Ox=\frac{wg}{W}, \therefore xC=\frac{Pp}{W}.$$

Let now the distance p, or OA, be measured from x to 1, from 1 to 2, from 2 to 3, and so on, and let each of these intervals be subdivided into

equal parts; then the number marked at C will be $\frac{xC}{p}$, which call n, and from the above equation, $nW=P$, so that the weight of P is found by multiplying the known weight W by the number at which it must be placed to produce equilibrium; if W be 1 lb., that number will show the number of lbs. that P weighs. When the steelyard is so constructed that the centre of gravity G is in the same vertical as O, then $g=0$, $\therefore Ox=0$, so that each division $O1$, &c., is equal to OA.

430. **The Danish Balance.**—This differs from the steelyard, in having the counterpoise fixed at one end and the fulcrum O moveable. To graduate the beam 1, 2, 3, &c., ounces are placed successively in the scale-pan, and the corresponding points 1, 2, 3, &c., at which the fulcrum must be placed to keep the beam horizontal are marked on it; intermediate weights are then put in, and the places of the fulcrum marked in like manner. Or thus: let w be the weight of the lever and scale-pan, and G their centre of gravity, and let O be the place for the fulcrum when W keeps the bar horizontal, then



$$W \,.\, AO = w \,.\, GO = w(AG - AO), \;\therefore\; AO = \frac{w}{W+w} AG,$$

so that G, the point at which the instrument would balance when unloaded, and the weight of it w, being known, the point O, at which it would balance when loaded with W, becomes known. If w, W, each express so many ounces, the several distances of the fulcrum from A, for 1 oz., 2 oz., &c., will be $\frac{w}{1+w}$, $\frac{w}{2+w}$, &c. The reciprocals of these numbers are $1+\frac{1}{w}$, $1+\frac{2}{w}$, &c., which are in arith. prog. Hence the distances from A are in harmonic prog. (86). This balance is often used for weighing letters.

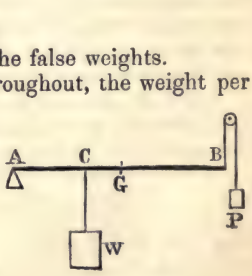
Problem I. A body is weighed successively in the two scales of a false balance: in the one it balances a weight P, in the other a weight Q: required the true weight.

Put a and b for the lengths of the two unequal arms of the balance, and x for the true weight of the body; then by the problem, $ax=bP$, $bx=aQ$. Multiplying these together, we have

$$abx^2=abPQ, \;\therefore\; x=\sqrt{(PQ)};$$

hence the true weight is a geom. mean between the false weights.

Prob. II. A heavy lever AB, equally thick throughout, the weight per inch being given, has a given weight W suspended at a given distance CA from the fulcrum A: what length must the lever have so that the weight P, to keep it in a horizontal position, may be the least possible?



The lever being equally thick, any portion of its length may be taken to represent the weight of that portion: hence, putting x for

the required length AB, its weight will be also denoted by x, acting at G, its middle point: hence, putting p_1 for CA, we have [1], p. 397,

$$Wp_1+x.\frac{1}{2}x=Px,\ \therefore\ P=\frac{Wp_1}{x}+\frac{1}{2}x=u,\ \text{a } \textit{minimum},$$

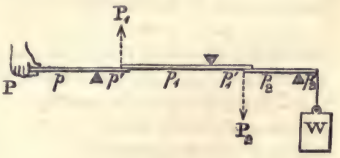
$$\therefore \text{ (p. 148), } x^2-2ux+2Wp_1=0,\ \therefore\ x=u,\ \therefore\ \frac{Wp_1}{x}+\frac{1}{2}x=x,$$

$$\therefore\ Wp_1=\frac{1}{2}x^2,\ \therefore\ x=\sqrt{(2Wp_1)},$$

where for W the *length* of lever, weighing W, is to be put. The same expression, namely, $P=x=\sqrt{(2Wp_1)}$ gives the weight of P, which is therefore that of the lever.

In each kind of lever the perpendicular pressure on the fulcrum is always equal to the algebraic sum of the other perpendicular forces which keep the lever at rest, because it is the resultant of these other forces, inasmuch as the resistance of the fulcrum, which is equal and opposite to the pressure, balances the other forces. In the case above, the pressure on the fulcrum is $W+x-x=W$, that is, it is equal to the suspended weight; P therefore merely supports the lever.

431. **Combination of Levers.**—The mechanical advantage of the single lever will be considerably increased by a combination of several: thus in the system of levers in the annexed diagram, if we call the arms which are on the same side of the power P, p, p_1, &c., and the other arms p', p_1', &c., the powers acting at the extremities of these latter being P_1, P_2, &c., we shall have, in the case of equilibrium,



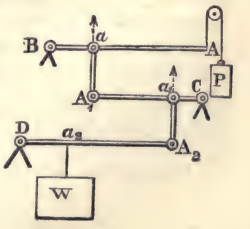
$$Pp=P_1p',\ P_1p_1=P_2p_1',\ P_2p_2=P_3p_2',\ \text{&c.},\ \therefore\ Ppp_1p_2\ldots=P_np'p_1'p_2'\ldots;$$

or, the last power P_n, being the weight W,

$$Ppp_1p_2\ldots p_n=Wp'p_1'p_2'\ldots p'_n\ldots\ldots[1].$$

Hence the applied power, multiplied by the product of all the arms nearest to it, will be equal to the resistance balanced, multiplied by the product of all the other arms. If the forces act obliquely, the several arms will be the perpendiculars on their directions from the fulcra.

In the preceding diagram the levers are all of the first kind, but the reasoning, and the conclusion [1], remain the same whatever levers are combined, and whether they are in actual contact, or are connected together by *links*, as in the diagram annexed, where B, C, D, are the three fulcra on which the levers of the second kind AB, A_1C, A_2D, rest. Putting, as before, $AB=p$, $A_1C=p_1$, $A_2D=p_2$, and $aB=p'$, $a_1C=p'_1$, $a_2D=p_2'$, we have the same expression [1] for the relation between the power and the weight, that is, $\dfrac{W}{P}=\dfrac{pp_1p_2}{p'p_1'p_2'}$.



Examples for Exercise.

(1) A uniform lever is 10 feet long, and weighs 6 lb. : its longer arm is 7 feet, and at the extremity of the shorter a weight of 21 lb. is placed : what weight must be placed at the end of the longer arm to balance the lever?

(2) A body weighs 10 lb. 9 oz. in one scale of a false balance, and 12¼ lb. in the other : what is the true weight?

(3) Two persons, A, B, of the same height, sustain upon their shoulders a weight of 150 lb., suspended on a pole 8½ feet long : the weight hangs at 3½ feet from A : what is the weight borne by each?

(4) The arms of a bent lever are equal, and $P : W :: 1 : \sqrt{2}$, the arm from whose extremity P is suspended, rests parallel to the horizon : required the angle between the two arms.

(5) At one extremity A of a uniform straight lever whose length is a, and weight w, a weight W is suspended : where must the fulcrum C be so that the lever, unloaded in the other part except by its own weight, may rest parallel to the horizon?

(6) The longer arm of a steelyard is 8 feet, and the shorter 6 inches : it is so contrived that, with its hooks, &c., a weight of 5 lb. on the longer arm, at 1 foot from the fulcrum, balances 12 lb. at the end of the shorter arm : the moveable weight is 5 lb., and it cannot be placed nearer to the fulcrum than 6 inches : what must be the graduation, so that by shifting the moveable weight from one mark to the next an additional half-pound may be weighed?—and what will be the least and greatest weights that can be ascertained by the machine?

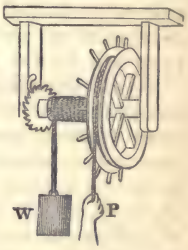
(7) In a lever of the third kind the power makes an angle of 48° with the lever, and acts with a force of 217 lb. in that direction, and the vertical resistance is 100 lb. : required the oblique strain on the hinge which forms the fulcrum.

(8) Three levers act in combination, as in the figure at p. 401 ; the arms nearest to the power P, which is 3 lb., are 9, 11, and 12 ; and the other arms, 1, 2, and 4 : what weight W will be sustained?

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432. **The Wheel and Axle.**—This machine, which is only a modification of the lever, consists of a cylinder called the *axle*, and a wheel, firmly connected with it; the whole being moveable round a common axis perpendicular to the face of the wheel.

The more immediate use of this machine is to support or raise a weight W, suspended to a rope, which is wound round the axle by means of the power P applied at the circumference of the wheel. Sometimes P acts through a cord wrapped round the wheel; in other cases, the actual *wheel* is dispensed with, and spokes only or radii inserted in the axle are used instead, as in the capstan and windlass.

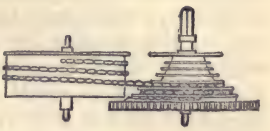


Prob. To find the relation between the power and the weight, when the wheel and axle is in equilibrium. Let the radius of the wheel be R, and that of the axle r: then, since the moment of P about the fixed axis to turn the system in one direction must be equal to the moment of W to turn it in the opposite direction, we have

$$P \,.\, R = W \,.\, r, \quad \therefore \frac{P}{W} = \frac{r}{R}, \text{ or } \frac{\text{Power}}{\text{Weight}} = \frac{\text{rad of axle}}{\text{rad of wheel}} \ldots\ldots [1],$$

the power and weight being inversely as their distances from the common fixed axis, as in the lever.

The greater R is, r remaining the same, the less will the power be which is requisite for the support of a given weight: when it is necessary that a continually-diminishing power should have a uniform effect upon a constant weight, it must act upon a series of wheels continually increasing in radius, the foregoing proportion being always kept up: thus P, P', being any two values of the powers, we must have for the corresponding values R, R', when r and W are constant, the relations

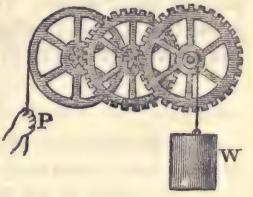


$$\frac{P}{W}=\frac{r}{R}\ \frac{P'}{W}=\frac{r}{R'}\ \therefore \frac{P}{P'}=\frac{R'}{R}.$$

In this way the varying power, exerted by the mainspring of a watch while uncoiling, and thus diminishing in elastic force, is made to produce a uniform effect, the continually-decreasing power acting with continually-increasing leverage, by means of the continually-enlarging wheels cut on the surface of the fusee, as in the diagram.

NOTE.—When the thickness of the rope, round either axle or wheel, is considerable, the radius of the rope should be added to the radius of the circumference round which it is wrapped, the power acting along the axis of the rope.

433. **Toothed Wheels.**—Wheels of this kind are employed alike in the most delicate as well as in the most ponderous pieces of machinery. The teeth or projections round the edges of the wheels acting on one another, as in the annexed figure, are all placed at equal distances, so that when the train is put in motion the teeth of one may enter the spaces between those of the other. When the axle is toothed, or rather when a small-toothed wheel surrounds the axis, the power being applied to the larger circumference surrounding it, the small wheel is called a *pinion*.



Let the radii of the pinions be r_1, r_2, &c., and those of the wheels R_1, R_2, &c.: then the power P acting on the first wheel equilibrates a weight or power P at its pinion, expressed by $P_1=\frac{R_1}{r_1}P$; this, therefore, is the power applied to the second wheel, which power equilibrates a weight or power P_2 at the second pinion expressed by

$$P_2=\frac{R_2}{r_2}P_1=\frac{R_1R_2}{r_1r_2}P.$$

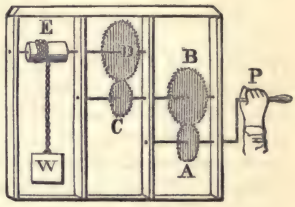
This increased weight or power P_2 is now applied to the third wheel, and thence acts with further increased effect at the third pinion, and so on: hence the power P_n at the nth pinion is

$$P_n=W=\frac{R_1R_2R_3\ldots R_n}{r_1r_2r_3\ldots r_n}P\ldots\ldots[2],$$

so that P_n is the weight that the power P applied to the first wheel will balance on the nth pinion. And it is plain that in this expression *circum-*

ferences may be put for *radii;* and then again for these, the number of teeth on each. If, as in the diagram, the first wheel and the last pinion have no teeth, the radii or circumferences of these must be inserted in the expression, or else the number of teeth they are competent to carry. The expression [2] evidently applies, whatever be the magnitudes of the wheels and pinions, or whether the pinions are enlarged to wheels. If the first wheel were toothed, and acted by these teeth on the second, and in like manner the second on the third, and so on, no mechanical advantage would be gained, since then $R_1=r_1$, $R_2=r_2$, &c., so that the mechanical effect would be the same as if the first wheel acted immediately on the last without the intervention of the other wheels.

The same expression equally applies, however far the wheels are asunder, each consecutive pair, when in motion, turning on the same axis. Thus, in the annexed diagram, the wheel P drives the wheel A, called its *follower;* the next driving wheel B, in like manner, turns its follower C, which acts upon the driver D, which finally acts on the axle E. And it may be noticed that when the wheels, in any such train, are put in motion, the symbols R, r, in the fraction which multiplies P, in the expression [2], may stand for *number of revolutions*, instead of for *radii:* it will then denote the number of revolutions of the last wheel, or of the axle E.



The foregoing expression [2], being derived from [1], assumes that the power communicated from the tooth of one wheel to that of another on which it acts is in the direction of a tangent to the circle on which the latter tooth is raised: in other words, that the forces act towards points on the circumference of each wheel in directions at right angles to the radii at those points. This requires that the faces of the teeth which come in contact should be of a peculiar curved form: the curve is what is called the *involute of the circle.*

Examples for Exercise.

(1) In the figure at p. 403, the radius of each pinion is 1 inch, and the radius of each wheel 10 inches: how much weight will 1 lb. balance?

(2) What must be the radius of the wheel (fig. at p. 402) in order that 12 lb. acting on it may support 120 lb. suspended by a rope wound round an axle of 6 inches radius?

(3) A weight of 500 lb. is sustained by a rope of 1 in. diameter, going round an axle of 8 in. radius: the radius of the wheel is 4 feet, and the power acts close to its circumference: required the weight acting on the axle.

(4) A train of four toothed wheels act as in the figure above, and can be put in motion by turning a winch of 12 in. radius: the radii of A, C, are each 4 in., and the radii of B, D, each 15 in.; also the radius of the axle E is $2\frac{3}{4}$ in.: what power must be applied to the handle P to sustain 600 lb. from a rope, of $\frac{1}{4}$ inch radius, wound round the axle?

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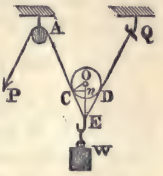
434. **The Pulley.**—This consists of a grooved wheel revolving freely about an axis passing through its centre, the ends of which axis are usually fixed in a frame called the *block* of the pulley, the grooved circle itself, which turns freely between the sides of this block, being called the

sheaf of the pulley: the sheaf is grooved to prevent the rope passing round it from slipping off when pulled by opposing forces.

Single fixed Pulley.—When the pulley is fixed, the forces P, W, which balance, when acting at the ends of a cord passing over the smooth sheaf, as in the diagram, must be equal, whatever be the directions in which they pull the flexible cord; and conversely. For the force at P being communicated without impediment to W, can be balanced only by an equal and opposite force, acting on that point of the cord. And conversely, equal forces acting at P and W must keep every point of the cord at rest, since they act upon each point of it in opposite directions. There is $\therefore$ no mechanical advantage in the single fixed pulley, but it enables us to give any direction we please to the force P, without interfering with its effect upon the weight W.



435. *Single moveable Pulley.*—In the moveable pulley, the weight is fastened to the block, the cord is fastened at one end Q, and the power P is transmitted, through the cord passing over a fixed pulley, to the moveable one: the weight W is thus sustained by the two *equal* forces at P, Q: that these *are* equal is plain, because the cord is equally tense throughout its whole course $PACDQ$, and we have to ascertain what relation these equal forces bear to the weight W which they balance.



If CA, DQ are parallel, the forces along CA, DQ, balancing W, must be each equal to half W (p. 376). If they are not parallel let their directions meet in E, at which point the forces P, Q, may be supposed to act, and as they balance W, E must be the line of action of W (p. 363). This force being equal and opposite to the resultant of the equal forces P, Q, the angle E is bisected by its line of action. If $\therefore$ α be half the angle of inclination of the cords, we must have

$$W=2P\cos\alpha,\ \therefore\ \frac{P}{W}=\frac{1}{2\cos\alpha}\text{.....[1]},$$

which is true whether the sheaf be circular or not. If it be circular, and if radius $OC=r$, then the perp. Cn will be the sine of O to that radius, or the cosine of α to that radius, so that the trig. cosine above is

$$\frac{Cn}{r},\ \text{and}\ \therefore\ 2\cos\alpha=\frac{\text{chord } CD}{r},\ \therefore\ \frac{P}{W}=\frac{r}{\text{chord } CD}\text{.....[2]},$$

that is, the power is to the weight as the radius of the pulley to the chord of the arc which is in contact with the rope.

It appears from [1] that the mechanical advantage is greatest when $\cos\alpha$ is the greatest, that is, when $\alpha=0$, or when the directions of the three forces are all parallel; and that when $\cos\alpha=\frac{1}{2}$, that, is when $\alpha=120°$, there is no mechanical advantage, for then $P=W$.

436. *System of moveable Pulleys.*—The advantage gained by a single moveable pulley may be increased to any extent by employing a system of pulleys as in the annexed figures: thus putting t_1, t_2, &c., for the tensions of the several ropes, the equation [1] gives the series of equations

$$W=2t_1\cos\alpha_1,\ t_1=2t_2\cos\alpha_2,\ t_2=2t_3\cos\alpha_3,\ \ldots,$$
$$t_{n-1}=2t_n\cos\alpha_n=2P\cos\alpha_n\ldots\ldots[3];$$

and taking the product of these, expunging common factors, we have

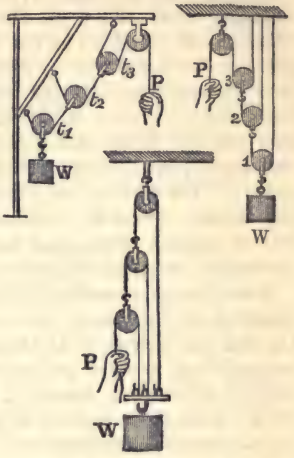
$$W=2^nP\cos\alpha_1\cos\alpha_2\ldots\cos\alpha_n\ldots[4],$$

where n is the number of moveable pulleys. If the angles are all equal, then

$$W=2^nP\cos^n\alpha\ldots[5];$$

and if the cords round the moveable pulleys are all parallel (second figure),

$$W=2^nP,\text{ or }P=\frac{W}{2^n}\ldots\ldots[6].$$



Suppose the weights w_1, w_2, &c., of the pulleys 1, 2, &c., are taken into account, then P will have to sustain these several weights suspended at 1, 2, &c., in addition to the weight W. By the formula just given, the additional power at P to sustain 1, will be $\frac{w_1}{2^n}$, since w is merely a new W: the additional power to sustain 2, will be $\frac{w_2}{2^{n-1}}$; to sustain 3, $\frac{w_3}{2^{n-2}}$, and so on, the additional power requisite to sustain the last moveable pulley w_n alone, being $\frac{w_n}{2}$: hence the power P requisite for balancing W and all the moveable pulleys, when the cords are parallel, is

$$P=\frac{1}{2^n}\{W+w_1+2w_2+2^2w_3+\ldots+2^{n-1}w_n\}\ldots\ldots[7],$$

weight of pulleys included; or if the weights of the pulleys be equal, it is $P=\frac{1}{2^n}\{W+(2^n-1)w\}$, since the sum of the geom. series

$$1+2+2^2+\ldots+2^{n-1}\text{ is }2^n-1.$$

If instead of fastening the several ropes to immoveable points as in the cases hitherto considered, each is attached to the weight to be supported, as in the third figure, then, the ropes being parallel, the pressure on the first pulley is $2P$, that on the second $4P$, and so on, that on the nth being 2^nP. Subtracting, therefore, P from the whole pressure on the hook, which is the pressure $W+P$, we have

$$W=(2^n-1)P,\ \therefore\ P=\frac{W}{2^n-1}\ldots\ldots[8],\text{ weight of pulleys neglected.}$$

But if we take into account the weights of the pulleys w_1, w_2, &c., we must observe that, contrary to the last case, these weights relieve P instead of making greater demand on it; the heavier the moveable pulleys are, the

less is the force at P that will sustain W: in the former arrangement the moveable pulleys acted on the side of the weight, here they act on the side of the power. Let w', w'', w''', &c., be the portions of W balanced by the pulleys themselves, and W' the remaining portion supported by P; then from the formula [8],

$$W'=(2^n-1)P,\ w'=(2^{n-1}-1)w_1,\ w''=(2^{n-2}-1)w_2,\ \&c.$$

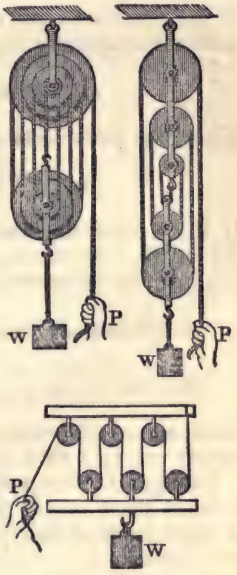
$$\therefore W=W'+w'+w''+\ldots=(2^n-1)P+(2^{n-1}-1)w_1+(2^{n-2}-1)w_2+\ldots+(2-1)w_{n-1}.$$

If the weights of the pulleys are equal, then, since

$$2^{n-1}+2^{n-2}+\ldots+2=2^n-2,\ W=(2^n-1)P+(2^n-n-1)w_1.$$

If the weight of each moveable pulley were $w_1=\frac{W}{2^n-n-1}$, the n pulleys alone would support W, since then $P=0$.

Besides the preceding arrangements there are several other systems of pulleys, in which each block embraces only a single sheaf: but a very useful mechanical power of this kind is that in which the system consists of only two blocks, each block containing several sheaves, as in the annexed figures, where only one rope is employed; and as this is supposed to pass freely over the sheaves without any impediment, it must be equally tense throughout, this tension being expressed by P. The weight W is supported by the equal tensions of the several portions of rope ascending from the moveable block, so that if this block have n sheaves, the tensions amount to $2n$, each $=P$, $\therefore W=2nP$, the power multiplied by twice the number of sheaves, where W includes of course the weight of the moveable block of pulleys. And the relation between W and P is the same when the moveable sheaves, instead of being inclosed in a single block, are placed as in the last figure, a single rope only being employed.



Examples for Exercise.

(1) A weight of 640 lb. is sustained by a power of 5 lb. acting through the system of pulleys in fig. 2, p. 406: required the number of moveable pulleys.

(2) Required the power requisite to sustain 1,020 lb. by aid of the system of pulleys in fig. 3, p. 406.

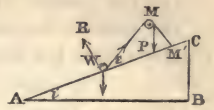
(3) In a system of pulleys six are moveable, and a single rope goes round all the pulleys: what power will be necessary to sustain 1 cwt.? (See last fig.)

(4) In a moveable block are five sheaves: 1,000 lb. is attached to one end of a rope passing round them: what weight at the other end is necessary for equilibrium?

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437. **The Inclined Plane.**—This is simply a plane surface inclined to the horizon. Let i be the inclination of the plane, ε the angle which the direction of the power P makes with the plane, and W the weight supported by P on the plane. There are three forces acting at

W, namely, the weight W in the vertical direction, the resistance of the plane in the direction WR perp. to its surface, and the force P in the direction WM, all which directions are given. Taking WC, and WR for axes, and resolving the two forces P, W, along WC, we have



$$P\cos\varepsilon - W\sin i = 0, \quad \therefore P = W\frac{\sin i}{\cos\varepsilon} \ldots\ldots [1],$$

which expresses the relation between the power and the weight it sustains. To find the pressure on the plane, or its opposite, the resistance R, we have, by resolving the same forces along WR,

$$R + P\sin\varepsilon - W\cos i = 0, \quad \therefore [1],\ R = W\frac{\cos i\cos\varepsilon - \sin i\sin\varepsilon}{\cos\varepsilon} = W\frac{\cos(i+\varepsilon)}{\cos\varepsilon} \ldots [2].$$

If the power act along the plane, as it usually does, seeing that it is then most advantageously applied, we have $\varepsilon = 0$, $\therefore \cos\varepsilon = 1$,

$$\therefore P = W\sin i, \text{ and } R = W\cos i \ \therefore \frac{P}{R} = \tan i \ldots\ldots [3].$$

When the power acts horizontally, $\varepsilon = -i$, and

$$\therefore P = W\tan i, \ R = W\sec i, \ \therefore \frac{P}{R} = \sin i \ldots\ldots [4].$$

If it act vertically, $\cos\varepsilon = \sin i$, $\therefore P = W$, and $R = 0$. If l be put for the length, h for the height, and b for the base of the plane, the relations [3], [4], may be conveniently expressed thus:—

1. When the power acts parallel to the plane, $\frac{P}{W} = \frac{h}{l}$, $\frac{R}{W} = \frac{b}{l}$, $\frac{P}{R} = \frac{h}{b}$ ...[5],
2. „ „ „ to the horizon, $\frac{P}{W} = \frac{h}{b}$, $\frac{R}{W} = \frac{l}{b}$, $\frac{P}{R} = \frac{h}{l}$ ...[6],

where we see that either set of expressions is got from the other set, by merely writing b for l, and l for b.

Problem I. The position of the pulley M, in reference to the given inclined plane AC, is given, as also the weights W and P, to determine at what point of the plane W must be placed so that there may be equilibrium.

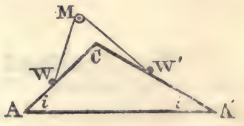
To AC draw the perp. MM', which is given because M and AC are given. Put a for this perp.; then $a = WM\sin\varepsilon$, and from [1] above,

$$\cos\varepsilon = \frac{W}{P}\sin i, \ \therefore \sin\varepsilon = \frac{\sqrt{(P^2 - W^2\sin^2 i)}}{P}, \ \therefore WM = \frac{a}{\sin\varepsilon} = \frac{Pa}{\sin i\sqrt{(P^2 - W^2\sin^2 i)}},$$

an equation which determines the distance of the required point from M.

Prob. II. Two weights, W, W', attached to the ends of a string, which passes over a fixed pulley, mutually support each other on two inclined planes: to determine the relations between W, W', the tension of the string, and the pressures on the planes.

As the string passes freely over the pulley its tension will be the same throughout, and it will exercise the force on the weights which hitherto has been called P. Marking the angles and the resistances concerned, as at p. 407, we have



$\frac{P}{W}=\frac{\sin i}{\cos \varepsilon}$, $\frac{R}{W}=\frac{\cos(i+\varepsilon)}{\cos \varepsilon}$, and $\frac{P}{W'}=\frac{\sin i'}{\cos \varepsilon'}$, $\frac{R'}{W'}=\frac{\cos(i'+\varepsilon')}{\cos \varepsilon'}$, $\therefore \frac{W}{W'}=\frac{\sin i' \cos \varepsilon}{\sin i \cos \varepsilon'}$,

which equations express the relations required. If the pulley be at the intersection of the planes, so that the string acts along each plane, then

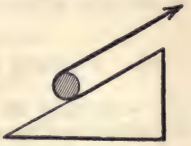
$$\varepsilon=0,\ \varepsilon'=0, \text{ and we have } \frac{W}{W'}=\frac{\sin i'}{\sin i}=\frac{CA}{CA'}=\frac{l}{l'},$$

that is, the weights must be as the lengths of the planes on which they rest. And the same is evidently true when $\varepsilon=\varepsilon'$, that is, when the string makes the same angle with one plane that it does with the other.

Examples for Exercise.

(1) The inclination of a plane to the horizon is 60°: prove that a force P will sustain four times as much weight on the plane when it acts along it, as when it acts horizontally.

(2) A person is just able to sustain by his strength a weight of 200 lb. : what weight could he sustain on a plane of 50° inclination, by means of a rope going round it, and fixed to the top of the plane, as in the annexed figure?

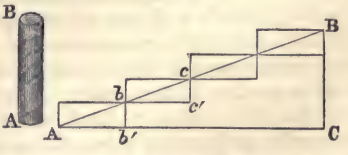


(3) Upon a plane of 30° inclination, whose height is 20 feet, a weight of 3 lb. is sustained by a power of 2 lb. acting over a pulley fixed 10 feet above the top of the plane, and in the vertical line through it: at what distance from the top of the plane does the weight rest?

(4) A heavy body is supported between two inclined planes, the angles of inclination being 60° and 30°: prove that the pressure on the former plane is to that on the latter as $1 : \sqrt{3}$.

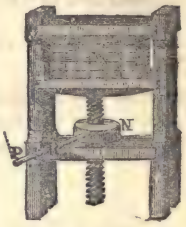
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438. **The Screw.**—This consists of a cylinder with a projection called the thread winding spirally round it, and inclined throughout at the same angle to the axis of the cylinder. When the screw is vertical, the projecting thread is nothing but an inclined plane winding round a vertical column: if we could unwind the thread, commencing at the bottom, and keeping the top end of the thread where it is, we should have the ordinary straight inclined plane, the height of which would be that of the screw, the length that of the thread, and the inclination the same as it was before the thread was unwound. Or the course of the thread might be traced on the cylinder thus: Let the base Ab' of the rectangle Ab be equal to the circumference of the cylinder, as also the base bc', &c.: then each of these rectangles being wrapped round the cylinder one above another, the united diagonals AB will trace out the spiral thread, Ab forming one turn of it, bc another, and so on: and the same power will be required to sustain a weight on the inclined thread in either of the two positions of it. If the power act horizontally, and be applied immediately to the weight, we shall have, as at p. 408,



$$\frac{P}{W}=\frac{bb'}{Ab'}=\frac{h}{2\pi r}, \text{ or } P=\frac{Wh}{2\pi r},$$

where h is the interval between the threads, and r the radius of the cylinder. The power acts horizontally, as in the annexed figure, which represents the bookbinders' press, N being the *nut*, or hollow screw, grooved within, so as to receive the projecting threads of the solid screw. A force applied at the end of the lever P, causes the nut to revolve, and thus to press the solid screw, bearing the press-board, upwards. The books to be pressed are placed between the press-board and the fixed beam at top. The whole pressure on the screw is thrown upon the thread within the nut, so that if R be the distance of P from the axis of the cylinder of the screw, and r, as above, the rad. of the cylinder itself, the efficacy of the power P will be expressed by $P\frac{R}{r}$; in other words, P acting at the extremity of R, is the same as $P\frac{R}{r}$ acting at the extremity of r: hence, substituting this for P in the preceding equation, we have



$$P\frac{R}{r}=\frac{Wh}{2\pi r}, \therefore \frac{P}{W}=\frac{h}{2\pi R}=\frac{\text{distance between the threads}}{\text{circum. described by } P}.$$

The ratio of the power to the weight is ∴ independent of the radius of the cylinder: the power is increased by diminishing h, the distance between the threads, or by increasing the leverage R.

Examples for Exercise.

(1) The distance between the threads of a screw is 1 inch; a power of 30 lb. acts at the end of a lever, at the distance of 2 feet from the axis: required the pressure produced.

(2) The distance between the threads is ¼ in., the leverage 48 in., and applied power 16 lb.: required the pressure produced.

(3) A power of 6 lb., with a leverage of 3 feet, produces a pressure of 1 ton on the press-board of a screw: required the distance between the threads.

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439. **The Wedge.**—This is a triangular prism made of some hard material: when used for the common purpose of cleavage or of splitting solid bodies in two, its edge is placed over the place of intended fracture, and the surface of the prism opposite to this edge, called the *head* of the wedge, is struck by a mallet, and the edge is thus forced forward, splitting or fracturing the body to which it is applied. Annexed is a section of the wedge perpendicular to the edge: when the wedge is used for cleavage, the section is usually an isosceles triangle. The mathematical theory of the equilibrium of this machine, employed in the above manner, is of no practical value, for it proceeds on the assumption that the resisting surfaces, and the faces of the wedge against which the resistances act, are perfectly smooth; in other words, that the friction



is nothing, whereas, in practice, the friction is everything:—it maintains the equilibrium when the applied power is withdrawn: in the absence of friction the wedge would be expelled by the pressure on its sides.

If, however, we consider a smooth wedge to be kept at rest by a power P applied perpendicularly to its head, and represented in the diagram by the line DE, and by the pressures P_1, P_2, perpendicular to its faces, that is, acting in the direction MD, ND, then completing the parallelogram IK, of which DE is the diagonal; DE, EI, ID, will then represent the three equilibrating forces; or, since the triangle DIE is similar to the triangle BAC, the three forces P, P_1, P_2, will be proportional to AB, AC, BC, that is,

$$P : P_1 : P_2 :: AB : AC : BC, \text{ or } P : P_1 : P_2 :: AB \cdot l : AC \cdot l : BC \cdot l,$$

where l is the length of the edge C, so that the three forces are proportional to the areas of the head and faces on which they respectively act. If, as is usual, the wedge be isosceles, and 2θ be the angle C of the wedge, then

$$AB : AC, \text{ or } Am : \frac{1}{2}AC :: \sin\theta : \frac{1}{2}, \therefore P = 2P_1 \sin\theta.$$

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440. In the preceding articles the simple machines treated of have all been regarded as motionless under the action of the forces applied to them, the object having been exclusively to ascertain what relations have place among forces which balance through the intervention of one or other of the so-called mechanical powers. We shall now briefly consider the working effect of these elementary machines when they are put into motion; first, however, recommending to the attention of the student the following judicious remarks from Venturoli.

The false opinions which persons unskilled in the nature and the power of machines are apt to conceive often encourages empty errors and mischievous deceptions. One of the most common of these conceits is, that of considering machines as available to increase and multiply the force of agents, which is not always true.

To form a just notion of the aid which may be expected from machines, looking to the uses to which they are most commonly put, we shall divide them into two classes:—those intended simply to sustain a weight, and those intended to draw it, or to raise it equably.

In machines of the first class, both the effect of the machine, and the immediate effect of the power, can only be estimated by the weight sustained. This being understood, it is evident that the machine increases the effect of the power, so that, for example, a force of 10 lb. will sustain, by means of a lever, 100 lb., provided that the arm of the force be 10 times as long as that of the weight. If it be asked how the force can ever produce an effect so much greater than itself, we shall perceive, if we consider well, that the force 10 does not really sustain the whole weight 100, but only the tenth part of it. Let the lever be supposed to be of the second kind; the force 100 may be resolved into two, the one equal to 90, which acts upon the fulcrum, and the other equal to 10, which acts at the point of application of the power. The first is entirely sustained by the prop; and the power sustains the second alone. Archimedes required only a fixed point to hold the terraqueous globe in equilibrium.

If he had found it, says Carnot, it would not in reality have been Archimedes, but the fixed point which would have sustained the earth.

In machines of the second class, neither the effect of the machine, nor that of the power can be estimated simply by the weight raised; otherwise the measure of the effect would be altogether vague and indeterminate. In fact, any force, however small, may carry a weight of any assignable magnitude, however great, if it only be granted that the weight admits of being divided and of being carried, one piece at a time. Wherefore it is necessary to take into account the time also in which the power can carry the weight through a given space, or the velocity with which the weight is carried; and on this account it is that the effect is measured by the product of the weight by the velocity.

Now upon this principle the machine does not increase the effect of the force. If a man with a force equivalent to 10, raise, by means of a machine, a weight of 100, he moves with a velocity 10 times as great as that of the weight, and does as much as if, operating without any machine, he carried those 100 at 10 journeys, loading himself with 10 at a time. In a word, what is gained in the quantity of the weight moved, is lost in the velocity; and the effect remains the same.

Between the two classes of machines, above described, there is then this characteristic difference, that the first add to the effect of the power, the second do not add to it.

There is another difference, not less remarkable, respecting the resistances of friction, and of ropes, and other resistances. In machines of the first class, these resistances are all of them advantageous to the power, and themselves also sustain their portion of the weight; whence there remains so much less for the power to support. On the contrary, in machines of the second class, the resistances are all of them detrimental to the power, and form part of the weight to be overcome: whence, on this account, a force is required greater than that which would be required, in the immediate application of the power.

These things being understood, we may now easily bring in review the true scope, and the real utility of machines.

Machines of the first class seem to increase the effect of the power; and they do this by conveniently distributing the weight between the power and the prop.

Machines of the second class serve—not to augment the effect of the power in quantity, but to modify its quality, as is wanted; and they do this in the following manner. The effect of the machine being the product of the weight by its velocity, we can increase, at pleasure, one of the two factors, provided that the other is proportionally diminished. Thus, by means of a machine, we can move a weight enormously great, provided that we are content to move it slowly; or, *vice versâ*, we can move a weight with very great velocity, provided that it is a small weight; whereas, by the immediate application of the force, we can hardly go beyond certain limits, either of velocity or of weight (*Venturoli's Mechanics*, by Creswell, Part II., p. 164).

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441. **The Mechanical Powers in Motion.**—The principal object of the elementary machines described in the foregoing articles, is not to balance weights or resistances at one part, by the application of

force to another part, but rather, by aid of this force, to raise or overcome them. With the applied force *motion* must be combined: both are transmitted through, and modified by, the machine; and the result is work executed. Whatever be the machine, there is always an equality between the work applied to it, and the work performed by it, understanding by *work* the power or weight multiplied by the space through which it moves, or by the *velocity*, that is, by the rate per minute, or per second, at which it moves. The effect actually aimed at, or what is called the *useful* effect, does not thus always equal the expenditure necessary to produce it, yet none of this expenditure is lost:—what does not contribute to the useful effect, may be said to be spent in wearing out the machine, by the unavoidable attrition, &c., of its parts. Only a fractional part, therefore, of the force applied is profitably accounted for by the actual result:—this fractional part is called (by Prof. Moseley) the *Modulus* of the machine:—it is the ratio of the work performed to the work applied. In what follows the fractional part of the work consumed or absorbed, as it were, by the machine, will be disregarded, so that all hindrance to the perfect discharge of its functions being hypothetically removed, the work delivered by it must be exactly equal to the work received by it, understanding by WORK the product of the force or power exerted, by the length of path or *space* through which the exertion is uniformly kept up: there is the same amount of work in carrying 100 lb. ten yards, as in carrying 10 lb. one hundred yards *in the same time*, and the great advantage of a machine is that it exchanges for us the one work for the other, never, however, giving more than a just equivalent. It is as if we delivered over copper money to the machine—investing it penny by penny—and should receive back at once, so soon as the last instalment had been made, the exact equivalent in the more concentrated form of gold.

442. **Lever in Motion.**—If a straight lever of either kind be turned uniformly round its fulcrum, or fixed centre of motion, the arc described by the point of application of the power, will be to that described by the point of application of the weight or resistance, as the arm of the power is to the arm of the resistance; because arcs measuring equal angles are as their radii: also since both arcs are described in the same time, their lengths must be as the rates or velocities per minute at which the points describing them move, for if one point move over m times the length of path of another in the same time—each moving uniformly—it must necessarily move m times as fast as that other: hence, equating work applied and work produced,

$$P \times \text{arc of } P = W \times \text{arc of } W, \ \therefore P \times \text{vel. of } P = W \times \text{vel. of } W,$$

$$\therefore \frac{\text{vel. of } P}{\text{vel. of } W} = \frac{\text{arc of } P}{\text{arc of } W} = \frac{\text{arm of } P}{\text{arm of } W}.$$

By the first of these equations the ratio of the arcs is constant, namely, $\frac{\text{arc of } P}{\text{arc of } W} = \frac{W}{P}$, however small they may be; but when they are each zero, no work is performed, or the lever is at rest: hence, when there is equilibrium, $\frac{W}{P} = \frac{\text{arm of } P}{\text{arm of } W}$.

The fraction vel. $P \div$ vel. W is called the velocity ratio, and the fraction $W \div P$ is called the advantage gained by the machine.

Required the velocity ratio of P and W in the arrangement of levers figured at p. 401 (fig. 2).

$$\text{vel. } W=\frac{p'_2}{p_2}\text{ vel. } A_2 \qquad \therefore \text{ vel. } W=\frac{p'p'_1p'_2}{pp_1p_2}\text{ vel. } P,$$

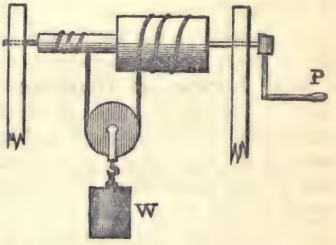
$$\text{vel. } A_2=\frac{p'_1}{p_1}\text{ vel. } A_1 \qquad \text{since vel. } P=\text{vel. } A,$$

$$\text{vel. } A_1=\frac{p'}{p}\text{ vel. } A, \qquad \therefore \frac{\text{vel. } P}{\text{vel. } W}=\frac{pp_1p_2}{p'p'_1p'_2}.$$

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443. **Wheel and Axle in Motion.**—When this machine revolves uniformly, the power moves through a space equal to one circumference of the wheel, while the weight moves through a space equal to one circumference of the axle: that is, the vel. of the power will be to that of the weight as the circ.$=C$ of the wheel to the circ.$=c$ of the axle, or as the rad.$=R$ of the former to the rad.$=r$ of the latter, so that the velocity ratio is $\frac{\text{vel. } P}{\text{vel. } W}=\frac{C}{c}=\frac{R}{r}$, or equating the work applied and performed, $Wc=PC$, $\therefore Wr=PR$: the same is obviously true when similar arcs, however small, of C, c, are moved through; but when these arcs are each zero, no work is done, and the machine is at rest: hence when there is equilibrium, we must also have $W : P :: R : r$.

The annexed figure represents the compound wheel and axle, the latter consisting of two cylinders of unequal radii, the cord coiling round one in a direction opposite to that in which it coils round the other, the weight to be raised being suspended to a moveable pulley. Every turn of the handle coils the cord once round the larger circumference, while it is at the same time uncoiled one circumference of the smaller cylinder: the part of the rope hanging down is thus shortened by the difference of these circumferences, and $\therefore$ the weight is raised through half this difference, so that by the principle of equality of work,



$$P\times C=W\times\frac{c-c'}{2},\ \therefore P\times R=W\times\frac{r-r'}{2},\text{ and }\frac{\text{vel. } P}{\text{vel. } W}=\frac{2C}{c-c'}=\frac{2R}{r-r'},$$

where c, c', are the circumferences of the compound axle, r, r' their radii, and C, R the circumference and radius of the wheel.

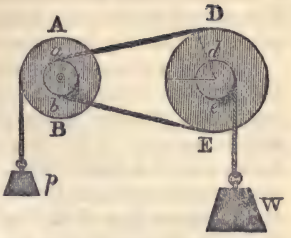
Ex. 1. Let the radius of one part of the axle be 3 in., that of the other part 2 in., and the radius of the wheel, or handle, 20 in.: required the advantage of the machine. Here

$$\frac{W}{P}=\frac{2R}{r-r'}=\frac{40}{1},\ \therefore W=40P;$$

hence the weight raised is equivalent to 40 times the power applied, but the latter must move through 40 times as much space.

2. In the annexed arrangement of a pair of wheels and axles, to determine the advantage of the machine. Let C be the circumference

of AB, c the circumference of ab, C' the circum. of DE, and c' that of de. One revolution of AB causes one revolution of its axle ab, and therefore DE is turned through an arc equal in length to c; and since de must turn through a similar arc, and that the lengths of similar arcs are as the whole circumferences, the similar arc of de is $\frac{cc'}{C'}$, which is $\therefore$ the space ascended by W: hence by the principle of the equality of work,



$$P \times C = W \times \frac{cc'}{C'}, \quad \therefore \frac{W}{P} = \frac{CC'}{cc'} = \frac{RR'}{rr'},$$

which is also the velocity ratio, or that of the spaces uniformly moved through by P and W,

$$\text{that is, } \frac{\text{vel. } P}{\text{vel. } W} = C \div \frac{cc'}{C'} = \frac{CC'}{cc'}.$$

The ratios $\frac{C}{c}$, $\frac{C'}{c'}$ remain the same, however small be the arcs turned through: but when these are zero there is equilibrium, in which case therefore we must also have $\frac{W}{P} = \frac{RR'}{rr'}$.

NOTE.—In all these cases we perceive that the ratio $\frac{W}{P}$, when the machine is at rest, is the same as the ratio $\frac{\text{vel. } P}{\text{vel. } W}$ when the machine is in uniform motion. This might have been anticipated, for since the balancing forces are inversely as their arms, and the arms (or radii) directly as the arcs described by their extremities, and these again as the velocities with which they are uniformly moved through, the inference is obvious.

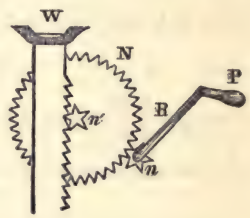
But without availing ourselves of the conditions of equilibrium established in the former articles, we may, as here shown, easily arrive at those conditions from the equation which implies equality of work when the machine is in uniform motion; for the ratio $\frac{\text{vel. } P}{\text{vel. } W}$, being invariably the same, however minute the spaces through which P and W move, this unchanging ratio must equally apply to the extreme or limiting case, in which these spaces become each zero; that is, S and s being the general symbols for the two spaces, and $\frac{S}{s} = a$, or which is the same thing, $\frac{sa}{s} = a$, the ratio must still be a, even when s, and *consequently* S, becomes 0. It is this limiting ratio of the spaces described (whether uniformly or not) by P and W, or of the spaces that would be described by P and W, if the machine were put in motion, that is called the ratio of the *Virtual Velocities* of P and W, when these are in equilibrium: it is the ratio of the *actual* velocities with which P and W would begin to move; and these initial velocities of P and W, when put in motion, are the quantities called the virtual velocities of P and W when they are at rest.

444. **Rack and Pinion.**—The circular motion of a wheel may be made to produce rectilinear motion in a bar, by supplying both with teeth, as in the annexed figure: the toothed bar is called a *rack*, which is made

to advance in its own direction by the revolution of the pinion: enclosed in a strong frame, this machine is the *Jack*, much used for the purpose of raising or moving heavy weights a small height or distance.

[By the distance between two consecutive teeth of a wheel, is meant the interval between the two radii through the middle points of those teeth, the interval being measured on the arc of the circle joining the middle points. This interval is called the *pitch* of the teeth, and the circle, whose radius extends from the centre of the wheel to either of the above-mentioned middle points, is called the *pitch-circle*. It is the radius of this circle that is the radius of the wheel, as the teeth are so shaped that their pressure is in the direction of the common tangent to the pitch-circles of the two wheels which *gear* or work together.]

Let R be the length of the handle, that is of the wheel to which the power is applied, n the number of teeth in the pinion of this wheel, N the number of teeth in the wheel driven by this pinion, n' the number of teeth in the pinion n', and let d be the distance between the teeth of the rack, or of the pinion n'. Then for one revolution of n' or N, there must be $\frac{N}{n}$ revolutions of n or P, so that the rack moves through a space $n'd$, while the power moves through $2R\pi\frac{N}{n}$,



$$\therefore\ Wn'd=\frac{2PRN\pi}{n},\ \therefore\ \frac{W}{P}=\frac{2RN\pi}{nn'd}=\frac{\text{vel. } P}{\text{vel. } W}.$$

Suppose, for ex., that $R=20$ in., the number of teeth $n=6$, $N=30$, $n'=6$, and $d=\frac{1}{2}$ in., then

$$\frac{W}{P}=\frac{40\times30\times3{\cdot}1416}{6\times6\times\frac{1}{2}}=\frac{40\times5\times3{\cdot}1416}{3}=200\times1{\cdot}0472=209{\cdot}44=\frac{\text{vel. } P}{\text{vel. } W}.$$

In all such combinations of wheels and pinions

$W : P$, or vel. P : vel. W : : prod. of circum. of wheels : prod. of circum. of pinions,

or, instead of circumference, we may use the number of teeth, and *vice versâ*. The above is only a particular exemplification of this general truth, $2R\pi$ being the circumference of the first wheel, and $n'd$ the circumference of the last pinion, so that the pressure produced by the rack is 209·44 times that applied to the handle.

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445. **Motion on an Inclined Plane.**—When a power pulls a weight up an inclined plane it must move through the length l of the plane, in order to raise the weight to the height h of the plane,

$$\therefore\ Pl=Wh,\ \therefore\ \frac{W}{P}=\frac{\text{vel. } P}{\text{vel. } W}=\frac{l}{h}.$$

Heavy bodies raised by the help of this machine would require much additional power to overcome the friction, which, in practice, however, is

reduced to a very small amount by introducing *friction rollers* between the body and the plane: by which means the mass is made to roll up instead of to slide up.

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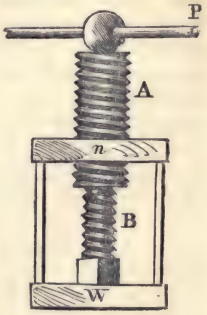
446. **The Screw in Motion.**—While the power moves through a circumference, of which the radius is its arm (fig. p. 410), the screw moves once round and causes the weight or resistance to move a distance equal to that between two threads of the screw,

$$\therefore P \times \text{circ. of } P = W \times \text{dist. of threads.}$$

The Compound Screw.—This consists of two screws A, B, the latter turning within the former, so as that when A is turned round once, one interval between its threads descends through the fixed nut n, but the same turn causes the screw B to ascend by one of its intervals through A, the latter screw being free to move in the direction of its length. Hence the weight lifted is raised through a space equal only to the difference of interval between the threads of A and B,

$$\therefore W \times \text{diff. of interval between the threads} = P \times \text{circum. of } P.$$

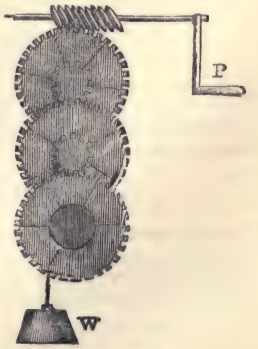
Ex. The distance between the threads of the larger screw is $\frac{3}{8}$ in., that between the threads of the smaller $\frac{1}{4}$ in., and the length of the lever is 5 feet: what is the advantage gained by the press? Here $\frac{3}{8} - \frac{1}{4} = \frac{1}{8}$, and circum. of



$$P = 60 \times 2\pi \text{ in.}, \therefore \frac{W}{P} = \frac{120 \times 3{\cdot}1416}{\frac{1}{8}} = 3016:$$

hence the weight or pressure is 3016 times the force applied, which force, however, must act through 3016 times the space that the pressure does. The screw, here described, is called, from the contriver, Hunter's Screw.

The Endless Screw.—Here the screw and one or more toothed wheels are combined. Each revolution of the handle causes the wheel acted upon by the screw to turn through the space of one tooth only, the distance between the teeth being equal to that between the threads. In the annexed figure let the radius of the handle be 20 in., and the diam. of the axle 5 in.: let the number of teeth in the three wheels be 30, 60, and 40 respectively, the number in the pinions 10 and 8. Then the screw may be regarded as a pinion, which for one revolution of its wheel (the handle) advances the wheel it drives 1 tooth: dividing, therefore, the circumferences (or no. of teeth) of the wheels by those of the pinions, we have



$$\frac{W}{P}=\frac{30\times60\times40\times40\pi}{10\times8\times1\times5\pi}=\frac{36000}{5}=7200,\ \therefore\ W=7200P.$$

Hence the velocity of P will be 7200 times that of W, and the weight lifted will be 7200 times the force or pressure applied to the handle.

447. **Systems of Pulleys.**—The principal varieties of these systems have been considered at pp. 406, 407, and since on the principle of the equality of work $\frac{W}{P}$ is always, in uniform motion, $=\frac{\text{vel. } P}{\text{vel. } W}$, we have only to substitute this latter fraction for the former in the deductions there established to get the velocity ratio in each of the arrangements referred to.

448. **Principle of Virtual Velocities.**—We have already explained at p. 415 what is meant by virtual velocity: if the forces acting at different points of a machine, or of a connected series of points keep the system at rest, and are also of such a nature that when the system is put in motion, the points of application of the forces move uniformly, then these actual velocities, the machine being in motion, are the *virtual velocities* of the several forces, the machine being at rest. But if, when put into motion, the points of application would not move uniformly, then the virtual velocities are the velocities with which the points would *begin* to move; the equation, therefore, $P\times$ virtual vel. $P=W\times$ virtual vel. W is applicable in either case to the system at rest. Since P and W necessarily tend to move the system in contrary directions, if they be marked with opposite signs, the algebraic sum of the two products here equated will be zero. This principle is general, that is, if any number of forces P_1, P_2, P_3, &c., act at any points of a compound machine, and the corresponding virtual velocities be v_1, v_2, v_3, then if the system be at rest

$$P_1v_1+P_2v_2+P_3v_3+\ldots+P_nv_n=0.$$

In the preceding articles different simple machines have been compounded into one system, and what has been called the resultant work may be regarded as terminating at, or as accomplished by any one of these parts, the power being transmitted through the preceding parts; and it is pretty evident that the principle is true at whatever piece we may regard the compound machine as terminating—as also whatever new forces be introduced to act on one or more of these pieces, provided the equilibrium be still undisturbed.*

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* Let the spaces s_1, s_2 ...s_n, be passed through with any velocities v_1, v_2, ...v_n, however varying, then by the principle of work, $P_1s_1+P_2s_2+\ldots+P_ns_n=0$, after any time t;

$$\therefore\ P_1\frac{ds_1}{dt}+P_2\frac{ds_2}{dt}+\ldots+P_n\frac{ds_n}{dt}=0.$$

The differential coefficients express the velocities that *would* have place if the forces were such as to render the actual motion uniform at the time t; but if the condition have place for *uniform* motion, it has place for virtual motion, when the system is at rest, in which case the coefficients denote the virtual velocities. (See DIFFERENTIAL CALCULUS.)

Examples for Exercise.*

(1) Required the velocity ratio of the power and weight in the system of straight levers in the fig. at p. 401, the arms nearest the power being $5\frac{1}{2}$, 6, and 7, and those nearest the weight 2, 3, and 1 respectively.

(2) In the compound wheel and axle (p. 414), the length of the handle is 2 ft., the diameter of the larger axle 10 in., and that of the smaller $9\frac{1}{2}$ in.: required the velocity ratio.

(3) What must be the difference between the diameters of the axles, in a compound wheel and axle, in order that the weight raised may be 100 times the power applied to a handle $2\frac{1}{2}$ ft. long?

(4) In the wheels and axles at p. 415, the diameter of the wheel to which the power is applied is 20 in., and that of its axle 6 in.: the diameter of the other wheel is 50 in., and that of its axle 3 in.: how high will the weight have ascended when the power has descended through 4 ft.?

(5) In the wheels and pinions at p. 403, the diam. of the axle to which the weight hangs is 3 in., and the number of teeth in its wheel 30: the number of teeth in the pinion which this wheel drives is 5, and the number in its wheel 28: the number in the last pinion is 8, and the diameter of its wheel 27 in.: through what space will the power descend if the weight ascends through 3 ft.?

(6) In the common press (p. 410), if the length of the lever is 3 ft., and the distance between the threads of the screw $\frac{1}{2}$ in., what space must the end of the lever be moved through, in order that the press-board may move through a space of 2 in.?

(7) In the compound screw (p. 417), the distance between the threads of the larger screw is 1 in., that between the threads of the smaller $\frac{7}{8}$ in., and the length of the lever 4 ft.: how many times the power applied will the weight moved be?

(8) In the mechanism represented at p. 417, let the radius of the handle be 30 in., the number of teeth in the three wheels, 40, 60, and 50 respectively, the number in the two pinions 5 and 10, and the diameter of the axle 6 in., required the velocity ratio of P and W.

(9) Wheel A, 36 in. in diameter, with 72 teeth, and making 120 revolutions per minute, drives wheel B, 12 in. diam. with 24 teeth: on the shaft of B is fixed wheel C, 10 in. diam. with 30 teeth, driving D, 6 in. diam. with 18 teeth: required the speed, or the no. of revolutions per min. of D.

(10) A driving wheel is to have 10 teeth, and the follower 40: it is required to find what must be the diameters of the *pitch circles* in order that the pitch of the teeth may be 1 in. (see p. 416).

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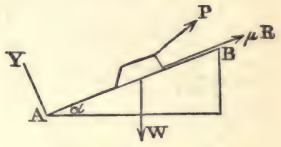
449. **Friction.**—In all that has preceded, the effects of friction have been left entirely out of consideration: the bodies acting upon one another have been assumed to be perfectly smooth, whereas they are invariably more or less rough, on which account more power must be expended on a machine to produce a given motion, than would be necessary if the resistance due to friction did not exist. Sometimes this is a very useful resistance:—useful for the purpose of checking speed or stopping motion

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* As a first book on the leading principles of mechanism, the learner is recommended to read the neat and instructive little volume of Mr. Tate, formerly of the Battersea Training College: the above few examples are selected chiefly from that work. The treatise, too, on "Practical Mechanics, and the Steam-Engine," by John Imray, M.A., C.E., in the volume on Mechanical Philosophy, in Orr's "Circle of the Sciences," will amply repay the attentive perusal of the student of Machinery—a subject which does not come within the scope of this work.

altogether; familiar instances present themselves in the ropes coiled round windlasses, and capstans, to raise heavy weights or resist great strains; in the break applied to a wheel of a heavy-laden waggon, or an omnibus, when descending a slippery declivity, as also in the friction-break of a locomotive steam-engine. But to *increase* speed, the friction must be diminished as much as possible, as it is a hindrance to motion. To estimate the amount of this hindrance, a force must be applied to the machine which will bring it into a state *bordering upon motion*, in which state it will be strictly in equilibrium.

It is found by experiment that for flat surfaces sliding one upon the other, the amount of friction is the same for the same sliding body or pressure, whatever be the dimensions of the side of the body which presses: thus, the friction of a block of material with several unequal but flat faces will be the same whichever face be downwards, if the supporting surface be the same; also that the materials being the same, the friction varies as the perpendicular pressure on the plane, or that it is equal to a constant fractional part of that pressure or of the opposing resistance. If we call this normal pressure or resistance R, the friction will be denoted by μR, the fraction μ being called the *coefficient of friction*. Practically this coefficient for given materials is found thus: let the plane AB, supporting a body with a flat surface (each plane being that of a given material), be more and more raised at one end B till the body is in a position bordering upon motion, that is, till the slightest additional elevation would cause it to slide: it is then in that state of equilibrium by the forces acting on it: these forces, resolved along the plane and perp. to it, are,



Perp. to the plane, $W \cos \alpha = R$. Along the plane, $W \sin \alpha = \mu R$, $\therefore \mu = \tan \alpha$;

so that by measuring the greatest angle α at which the plane can be inclined to the horizon, without causing the body to slide, the coef. of friction, for the particular substances experimented upon, becomes known.

For instance, the coef. of friction for polished brass and iron is $\mu = \cdot 143$, the limiting angle α being 8°; so that if a piece of such iron of 1 cwt. rested on a horizontal brass plate, it would require a lateral force of 16 lb. to cause it to slide: for the vertical (or normal) pressure R being 112 lb., we have $\mu R = \cdot 143 \times 112 = 16$ lb. If the plate were inclined at an angle of 8°, the iron would be on the point of sliding down it. Again: suppose a polished cylindrical iron shaft, weighing 1 ton, to revolve in a brass bearing: the force that must be applied to the circumference to balance the friction would be $2240 \times \cdot 143 = 320$ lb.: but as the shaft does not slide, but turns on its axis, if the radius of the shaft at the bearing be 3 in., then 3 in. is the leverage at which the friction acts to resist rotation; and since $320 \times \frac{3}{12} = 80$, it would be balanced by 80 lb. with 1 foot leverage: this force, acting in opposition to the friction, would bring the shaft into a state bordering on rotatory motion.

PROB. I. To find the force P, acting at a given inclination ϵ, to a plane which is itself inclined to the horizon at the angle i, necessary to keep a weight W in that state of equilibrium which borders on motion, the friction being taken into account.

Let R be the resistance of the plane, then μR being the force of friction acting up the plane (since it opposes downward motion), we have, by revolving the four forces perp. to the plane and along it,

$$W\cos i - P\sin\varepsilon = R, \text{ and } W\sin i - P\cos\varepsilon = \mu R, \therefore P = W\frac{\sin i - \mu\cos i}{\cos\varepsilon - \mu\sin\varepsilon}.$$

If P act along the plane, then $\varepsilon = 0$, and $P = W(\sin i - \mu\cos i)$.

The equilibrium considered above is that in which the tendency to motion is *down* the plane; since, however little P be diminished, that downward motion must ensue. But it cannot be said that, however little the force of P be increased, an *upward* motion will ensue; because the friction is equally opposed to motion either way: hence, for the other state of equilibrium:—the state bordering upon upward motion, the friction μR takes a contrary sign;

so that, changing $+\mu$ for $-\mu$, we have $P' = W\dfrac{\sin i + \mu\cos i}{\cos\varepsilon + \mu\sin\varepsilon}$,

or if, as before supposed, P' act along the plane, $P' = W(\sin i + \mu\cos i)$.

The weight W will therefore move neither up nor down the plane so long as any force, not without the extreme limits P, P', be applied to it, and act in the direction WP, making an angle ε with the plane.

For every state of the system bordering upon motion, the motion being limited to one of two directions, there is thus always what may be called a *conjugate* state, the friction in both states being the same, but acting in opposite directions: the difference between the two forces P, P', which, acting in the same line as the friction, apply to these conjugate states, must be twice the force f of that friction, and the coef. of friction might, from this consideration, be found in another way: thus, when P, P', act along the plane, as above,

$$P' - P = W\{(\sin i + \mu\cos i) - (\sin i - \mu\cos i)\} = 2W\mu\cos i,$$

$$\therefore f = W\mu\cos i, \therefore \mu = \frac{f}{W\cos i}, \text{ or } \mu = \frac{P' - P}{2W\cos i}.$$

If $i = 60°$, then μ is found from the equation $\mu = \dfrac{P' - P}{W}$.

Prob. II. To find at what angle ε the direction of the force P must be inclined to the plane, in order that P may be the least possible to prevent a given weight W from sliding down; friction being taken into account.

In order that P may be the least possible, it is plain that the denominator $\cos\varepsilon - \mu\sin\varepsilon$, in the preceding expression for it, must be the *greatest* possible.

Put $\sin\varepsilon = x$; then we are to have $\sqrt{(1-x^2)} - \mu x =$ a *maximum*.

Calling this u, as at p. 148, and rationalizing, we have $(1+\mu^2)x^2 + 2\mu ux + u^2 - 1 = 0$,

$$\therefore x = -\frac{\mu u}{1+\mu^2}; \text{ also (see p. 148), } \sqrt{U} = \sqrt{(1+\mu^2-u^2)} = 0, \therefore u^2 = 1+\mu^2,$$

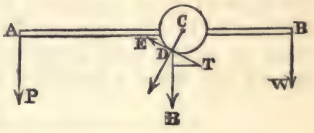
and since any greater value for u^2 renders $\sqrt{U}$ imaginary, this is the greatest value possible, and these values of x and u satisfy the proposed equation. Hence

$$x = \sin\varepsilon = -\sqrt{\frac{\mu^2}{1+\mu^2}}, \therefore \sin^2\varepsilon = \frac{\mu^2}{1+\mu^2}, \therefore \cos^2\varepsilon = \frac{1}{1+\mu^2}, \therefore \tan^2\varepsilon = \mu \quad \therefore \tan\varepsilon = -\mu.$$

Consequently, the least force P, which will restrain the body W from falling down the plane, must have its direction inclined to the plane at an angle whose tangent is equal to the coef. of friction; and this angle must lie *below* the plane. The direction of the least force which will just keep the body from moving up the plane, is got from the above by changing the sign of μ; that is, the direction ε is such that $\tan \varepsilon = \mu$, the angle lying *above* the plane.

PROB. III. To find the equation bordering on motion for a lever free to turn on a cylindrical axle working in a hollow cylinder, or bearing.

Let the body be acted on by the vertical pressures P and W: let C be the centre and r the radius of the axle, and D the point on which the axle will begin to turn in the hollow cylinder in contact with it there, on which point the pressure $R = P + W$ acts, in the vertical direction. The tangent DE is the direction in which the friction acts, and CD is the direction of the normal pressure, or that perp. to the surface pressed, and this multiplied by μ is the force of the friction. If θ be put for the angle RDT, then p. 420,



$$\mu = \tan T = \cot \theta \ldots\ldots [1].$$

Now the normal pressure is $R \sin \theta = R \frac{1}{\sqrt{(1+\cot^2 \theta)}}$, and $\therefore$ the friction f is [1],

$$f = \mu R \frac{1}{\sqrt{(1+\cot^2 \theta)}} = \frac{\mu}{\sqrt{(1+\mu^2)}} (P+W).$$

The opposing force f, acting in the direction DE, being thus known, we have, by taking the moments of the three forces P, W, and f, about C, putting r for CD, and observing that f conspires with W to produce rotation in the direction DE, we have

$$P \,.\, AC = W \,.\, BC + fr = W \,.\, BC + \frac{\mu}{\sqrt{(1+\mu^2)}} (P+W) r$$

for the equation of equilibrium bordering on motion in the direction DE. The equation of equilibrium, bordering on motion in the contrary direction, is got from this by merely changing the sign of fr, or the sign of μ.

It may be remarked, in reference to the foregoing equation for the equilibrium, when nearest to motion, that μ being always a fraction—usually a small one—the second power of it will, in general, have a value so small as to be undeserving of notice: expunging μ^2, therefore, the equation is

$$P \,.\, AC = W \,.\, BC + (P+W)\mu r, \;\; \therefore \;\; P = W \frac{BC + \mu r}{AC - \mu r}.$$

NOTE.—The friction to be overcome, in passing from a state of rest to a state of motion, is always greater than that which opposes the continuance of the motion when once commenced, although in the case of very hard bodies the excess is very trifling. The experiments of Morin have shown that the *friction of motion* is wholly independent of the velocity of the motion: the former kind of friction has been called the *friction of quiescence*.

To diminish as much as possible the resistance occasioned by friction is the great object of practical engineers in working machinery, every

part of which is therefore kept free of dirt and dust, while the rubbing surfaces are lubricated with oil, or some other unctuous matter. "When a steam-engine is employed as the prime mover of any machine, the power communicated can be readily ascertained by the *Indicator*. The engine is first worked alone, or with merely the train of wheel-work, in order that the power necessary to overcome friction may be estimated. It is then worked in connection with the machine, and the driving-power required for the machine is ascertained by subtracting the force necessary to overcome friction from the total power, including friction and the resistance of the machine. When machinery is driven by some other power, or when the indicator cannot be conveniently applied, the *dynamometer* (power-measurer) is employed."*

[Additional investigations connected with Statics will be hereafter given as applications of the DIFFERENTIAL and INTEGRAL CALCULUS.]

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450 **Scholium.**—The foregoing treatise contains as much of the general theory of equilibrium as can be reasonably looked for in an elementary work like the present, in which the higher calculus has not as yet been taught. Within the extent to which our space has compelled us to limit it, as full an amount of detail, and as great a variety of topics, have been introduced as is consistent with the general plan and moderate pretensions of this volume. The student, however, must regard what is here done as little more than a theoretical preparation for the study of those extensive practical subjects that come under the heads of Architecture, Engineering, and Mechanism: on these matters he is referred for ample information to the following works: Imray's "Practical Mechanics and the Steam-Engine," Moseley's "Architecture and Engineering," Whewell's "Mechanics of Engineering," Willis's "Treatise on Mechanism," and the neat introductory book on the same subject by Mr. Tate. On the "Strength of Materials," Barlow's is the standard English work; the practical and professional man will also find a vast amount of available results in Professor Hodgkinson's numerous "Experimental Researches," as contained in his published contributions to the "Manchester Memoirs," to the British Association, to the Royal Society, &c. The Author of this work has reason to believe that Prof. Hodgkinson is now occupied in collecting together and arranging the chief of these contributions for publication, in a connected form, in one great work on "The Strength of Materials," on "Suspension Bridges," &c.

END OF THE STATICS.

VII. MECHANICS: PART II. DYNAMICS.

451. The first part of Mechanics considers bodies at rest, or in a state of equilibrium under the operation of applied forces; the second part treats of the motion which ensues when a body is not thus kept at rest.

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* Imray's "Practical Mechanics and the Steam-Engine:" Orr's "Circle of the Sciences."

We have seen in Statics that in every body there exists one point, and one only, which is distinguished from all others by a remarkable peculiarity, which is this:—that, provided only the forces acting upon the body be all applied at that point, the tendency to progressive motion, and the direction and intensity of that tendency, will remain the same: this point is the centre of gravity of the body; it is the point into which the entire mass of the body may be conceived to be compressed and condensed. If under the operation of forces, thus conceived to be all applied at the centre of gravity, progressive motion actually takes place, the movement of that centre must therefore commence in the direction of the resultant of those forces, and being only a point its path must be *a line*. If the forces cease to act, the instant this point starts into motion, it must not only commence, but continue to move in the direction of the resultant, that is, its path must be *a straight line*, so long at least as no external obstacle or influence interferes with its onward progress, and in the absence of all such interference its onward motion must be uniform. These truths must be admitted as self-evident: they are usually enunciated in the form of an axiom in the following terms, and constitute what is called—

The first Law of Motion.—Mere matter, if at rest and unacted upon by any external force, or only by forces of which the resultant is nothing, must remain at rest. If it be in progressive motion, and unacted upon by any external force, it must continue in motion: the motion must be uniform, and the centre of gravity must describe a straight line. And this is only affirming that inanimate matter is incapable *of itself* of altering the state into which it is put by any external cause, whether that be a state of rest or a state of motion: this incapability is called the *inertia* of matter.

When in what follows we speak of the path of a moving body it will be understood that we mean the line described by its centre of gravity.

452. **Uniform Motion: Velocity.**—The motion of a body is said to be uniform when it passes over equal lengths of path in equal times: the term *uniform* so sufficiently implies this that we have not hesitated to use it previously to formal definition. What is commonly called the rate or speed of a body's motion is what is here meant by its *velocity*; in uniform motion it is estimated by the length of path described in a given interval of time, as for instance, in one second, one minute, &c. If a body moving uniformly pass over ten feet in every second of time we say that its velocity is ten feet per second. In dynamical inquiries it is the *second* that is generally chosen for the unit of time, and the *foot* for the unit of length. For the *number* of seconds in any portion of time the symbol t is employed, and in like manner the initial letters s and v stand for space (*linear* space) and velocity; but this distinction must be carefully observed, namely, that t is always an *abstract number*: it does not stand for *time*, but only for the number of seconds in that time, while, on the contrary, s and v each denote the concrete quantity of *feet*, the former symbol expressing the whole length of path described in t seconds, and the latter the portion of that length described in *one* second. It is plain that in uniform motion these three symbols are related to one another as follows, namely,

$$v=\frac{s}{t},\ s=vt,\ t=\frac{s}{v}\text{.....[1]},$$

t being measured from the commencement of the motion.

If, however, t is not reckoned from the instant that the motion commences, but only after a certain space s' has been passed over by the body, then the space described in t seconds is only $s-s'$, instead of s, so that the three equations will then be

$$v=\frac{s-s'}{t},\ s=s'+vt,\ t=\frac{s-s'}{v}\ldots\ldots[2].$$

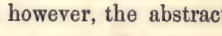
453. *Angular Velocity.*—If a body rotate about an axis or point equably, it turns through equal portions of an entire revolution in equal times, and the rate at which it rotates is the angular velocity of the body. This, like linear velocity, is also usually measured in feet, whenever the foot is taken for the linear unit: it is the length of arc, of which the radius is one foot, that is turned through in a second. However different two straight lines may be in length, if each turning about its fixed extremity as upon a pivot complete a rotation in one and the same time, the angular velocities are equal, though the linear velocities of the moving extremities are unequal. If the lengths of the lines or rods be R, r, and the linear velocities of the moving extremities V, v, the angular velocities will be $\frac{V}{R}=\frac{v}{r}$ *feet.* For the uniform linear velocities are as the circumferences described with them, and these are as the radii, so that calling the angular velocity ω, that is, the linear velocity at a foot distant from the centre of motion, we have $\frac{V}{R}=\frac{v}{r}=\frac{\omega}{1}$. Sometimes, however, the abstract number, which this ratio expresses, is regarded as the angular velocity, the linear velocity at the distance r from the centre of rotation being expressed by $v=\omega r$.

The following are some applications of the foregoing formulæ:—

(1) A railway train travels at the uniform rate of 40 miles an hour: what is its velocity per second?

$$\text{Here } v=\frac{s}{t}=\frac{5280\times 40 \text{ feet}}{60\times 60}=\frac{528}{9}=58\frac{2}{3} \text{ ft.}$$

(2) Two bodies a, b, animated by the uniform velocities v, v_1, set out at the same instant from the points A, B, and move in the direction of AB prolonged: in what time will a come up to b? Suppose that a comes up with b at the point C, then putting s for AC, and s_1 for AB, we have $s=vt$, $s-s_1=v_1t$;



$$\text{Subtracting this from the former, } s_1=(v-v_1)t,\ \therefore\ t=\frac{s_1}{v-v_1},$$

so that the number of seconds is found by dividing the space between the bodies at starting by the difference of the spaces denoting their velocities.

(3) A wheel whose radius is 3 feet makes 120 rotations in a minute: required its angular velocity per second. Here every point of the circumference moves uniformly through a space

$$s=120\times 6\pi \text{ in 60 seconds, } \therefore \text{ its linear velocity is } v=12\pi,$$

$$\text{and } \therefore \text{ the angular velocity is } \frac{v}{r}=4\pi=12{\cdot}566 \text{ ft. per second.}$$

454. **Variable Motion.**—When the moving body does not pass over equal spaces in equal times its motion is not uniform but variable;

and its velocity at any instant is the space it *would* pass over in the following second, provided it were to move uniformly during that second at the speed with which it is actually moving at the instant. When the variable motion is such that equal accessions of velocity are acquired in equal intervals of time, the motion is said to be uniformly *accelerated;* if instead of being equally increased the velocity is equally diminished in equal intervals of time, the motion is said to be uniformly *retarded.*

From the axiom that a body cannot of itself alter its state of rest or of uniform motion, whenever this motion is accelerated or retarded, the change must be due to some external cause:—we call it *force;* regarding it as a constant force, if its effects be uniformly the same, that is, if there be the same increase or decrease of velocity in equal times, and a variable force if its effects be variable.

455. **Uniformly-Accelerated Motion.**—Let the uniform increment of velocity which a body receives from second to second be represented by f; then if v_1 be its actual velocity at any instant, at the end of 1 second from that instant the velocity will be v_1+f, at the end of 2 seconds, v_1+2f, and at the end of t seconds, $v=v_1+tf$; and measuring causes by their effects, the constant increment f of the velocity is the quantity employed to represent the intensity of the *force*, or that constant influence which thus continuously and uniformly accelerates the motion of the body; the acceleration f of the velocity is hence often spoken of as the *accelerating force.* The student, however, will bear in mind that f denotes a *length* in feet; the length by which the space described in one second, with the velocity v_1+f, exceeds the space described in one second with the velocity v_1, these spaces themselves being respectively v_1+f, and v_1. The two propositions which follow contain the entire theory of the rectilinear motion of a body uniformly accelerated.

1. If a body be moved from rest, and f be the constant acceleration of its velocity (or the accelerating force continuously influencing it) during t seconds, at the end of which time the velocity is v,

$$\text{then } v=ft \ldots\ldots [1].$$

This is included in the inference above, v_1 being 0, since t is here measured from the commencement of motion.

2. If s be the space through which a body is moved from rest, f being the constant acceleration of its velocity, or the constant accelerating force, then at the end of t seconds, $s=\frac{1}{2}ft^2$. This is proved as follows:—

Let the t seconds be divided into n equal intervals, each less than a second, and call each of these smaller intervals τ, so that $n\tau=t$. Then since equal velocities are generated in equal times, the velocities at the *end* of each of these intervals will be respectively

$$f\tau,\ 2f\tau,\ 3f\tau, \ldots,\ nf\tau \ldots\ldots [A],$$

and at the *commencement* of the same intervals, the velocities will be

$$0,\ f\tau,\ 2f\tau, \ldots,\ (n-1)f\tau \ldots\ldots [B].$$

Now if the several velocities [A] were uniform during the several intervals, that is, if the velocity throughout each interval were the same as that at the end, then the whole space S, described in the time t would be

$$S=f\tau.\tau+2f\tau.\tau+3f\tau.\tau+\ldots+nf\tau.\tau$$

$$=f\tau^2(1+2+3+\ldots+n)=\frac{1}{2}f\tau^2n(n+1)=\frac{1}{2}fn^2\tau^2\left(1+\frac{1}{n}\right)=\frac{1}{2}ft^2\left(1+\frac{1}{n}\right).$$

And if the several velocities [B] were uniform, the velocity throughout each interval being the same as that at the beginning of it, the whole space S', described in the time t, would be

$$S'=0.\tau+f\tau.\tau+2f\tau.\tau+\ldots+(n-1)f\tau.\tau$$

$$=f\tau^2\{1+2+3+\ldots+(n-1)\}=\frac{1}{2}f\tau^2n(n-1)=\frac{1}{2}fn^2\tau^2\left(1-\frac{1}{n}\right)=\frac{1}{2}ft^2\left(1-\frac{1}{n}\right).$$

But neither of the spaces S, S', is that actually described by the moving body in the time t, but some space s intermediate between them, that is, s has a value always between

$$S'=\frac{1}{2}ft^2\left(1-\frac{1}{n}\right), \text{ and } S=\frac{1}{2}ft^2\left(1+\frac{1}{n}\right),$$

however great n may be, which value can be no other than $s=\frac{1}{2}ft^2$, the value which each of the extremes S, S', becomes when n is infinite, since s must be that quantity which coincides with S, S', when these become *equal*. We thus have the following relations connecting the *terminal* or last acquired velocity, and the space described, with the time and the uniform acceleration, or force f, namely,

$$v=ft,\ s=\frac{1}{2}ft^2,\ \therefore v^2=2fs,\ s=\frac{1}{2}vt\ldots\ldots[2].$$

From these relations the following interesting truths are immediately deducible, namely:—

1. The space described in any time, reckoning from the commencement of the motion, is half the space that would have been described in that time, if the body had moved from rest with a uniform velocity equal to the terminal or last acquired velocity. The last of the preceding equations expresses this truth.

2. The spaces described in equal successive intervals of time from the commencement are to one another as the odd numbers 1, 3, 5, 7, &c. For let t measure the seconds first in one interval, then two, then three, and so on; then from the equa. $s=\frac{1}{2}ft^2$, the corresponding spaces described are

$$\frac{1}{2}f.1,\ \frac{1}{2}f.4,\ \frac{1}{2}f.9,\ \frac{1}{2}f.16, \text{ &c.}$$

Consequently, subtracting each of these from that which immediately follows, the spaces described in the several individual intervals are

$$\frac{1}{2}f.1,\ \frac{1}{2}f.3,\ \frac{1}{2}f.5,\ \frac{1}{2}f.7, \text{ &c.},$$

which spaces are as the numbers 1, 3, 5, 7, &c. Suppose for ex., there be an accession of velocity of 4 feet every second; then f would be represented by 4 feet; and the spaces described during the first second, the second second, the third second, ...the tenth second, would be 2, 6, 10, ..., 38. The sum of these, or the whole space s, would be

$$s=2(1+3+5+\ldots+19)=(19+1)10=200 \text{ feet},$$

which of course is the same as would be given at once by the formula

$$s=\frac{1}{2}ft^2, \text{ when } f=4 \text{ ft., and } t=10.$$

If, after sufficient observation of the rectilinear motion of a body, we are convinced that that motion is due to a constant force acting upon it, we may readily estimate the intensity of that force by observing the space the body passes through, from rest, in a given time, from the condition $f=\frac{2s}{t^2}$: thus, if the time of passing through the space s be one second, then $f=2s$; that is, the constant accelerating force is measured by twice the space described, through the influence of it, in the first second of the body's motion. When the rectilinear motion of a body is *uniform* no force can be acting on it *during* that motion, which can have originated only from an impulse—a force which expires, as it were, in the act: while moving uniformly the body is subjected to no influence whatever.

From the equations [1], it is plain that any two of the four quantities f, t, v, s, being given, the remaining two may always be found, and that for each of the four, there are three distinct but equivalent expressions:—they are exhibited in the margin for convenience of reference.

$$v = ft = \frac{2s}{t} = \sqrt{(2fs)}$$
$$s = \frac{1}{2}ft^2 = \frac{1}{2}vt = \frac{1}{2}\frac{v^2}{f}$$
$$t = \frac{v}{f} = \frac{2s}{v} = \sqrt{\frac{2s}{f}}$$
$$f = \frac{v}{t} = \frac{2s}{t^2} = \frac{1}{2}\frac{v^2}{s}$$

456. The most important accelerating force with which we are acquainted is the force of terrestrial gravity, or the attractive influence which the earth exercises on all bodies, and which all experiments show to be a constant force at the same place, and for all distances above the surface that are within our reach. Its intensity is such as to cause a body to fall from rest, in the latitude of London, a distance of 16·1 feet in the first second of time; consequently in this latitude the expression for the force of gravity, which force is usually represented by g, from what is shown above, must be $g=32{\cdot}2$ feet, that is, g is such as to uniformly increase the velocity of a falling body by 32·2 feet at the end of every second during its descent. The *exact* force of gravity, in a few widely-differing latitudes, as determined by very careful observation, is given in the annexed table, the experiments from which the different values of g are deduced, having all been made at the level of the sea. It is found that the influence of this force diminishes the further the body acted upon is removed from the centre of the earth, but as there is no sensible diminution within those limits accessible to man, gravity may be regarded, thus far, as a force which in the same latitude is constant.

Station.	Latitude.	Gravity.	Observer.
London . .	51° 31′ 8″ N.	32·1908	Sabine.
Paris . . .	48 50 14 N.	32·1820	Borda.
Leith . . .	55 58 41 N.	32·2040	Kater.
C. G. Hope .	33 55 15 S.	32·1412	Freycinet.
New York .	40 42 43 N.	32·1600	Sabine.
Unst . . .	60 45 25 N.	32·2173	Biot.

In the foregoing group of equations, the body is supposed to have been put in motion solely by the force f, from a state of *rest;* but we may conceive the actual motion to be due to a two-fold cause:—to an impulse, given to the body (either concurring with or in opposition to the direction, in which f acts), and to the force f; as for instance, when a body is projected vertically downwards or vertically upwards, gravity also acting upon

it all the time. Suppose v_1 to be the velocity with which a body is projected in the direction in which f acts. If no force modified its motion, it would move uniformly with the velocity v_1 that it commenced to move with. Now there is evidently some length of time t_1 at the end of which, if the body had moved from *rest*, by the action of f alone, it would have acquired the velocity v_1. Let s_1 be the space due to this velocity, then $v_1=ft_1$, and $s_1=\frac{1}{2}ft_1^2$. Hence t being the time and s the corresponding space reckoned from the termination of t_1 and s_1, that is, measured from the instant when, and the place where the impulse to the body is given, we have at the end of t seconds

$$v=f(t_1+t)=v_1+ft,\ s_1+s=\frac{1}{2}f(t_1+t)^2=\frac{1}{2}ft_1^2+ft_1t+\frac{1}{2}ft^2=s_1+v_1t+\frac{1}{2}ft^2,$$

$$\text{that is, } v=v_1+ft\text{.....[1]}, \qquad s=v_1t+\frac{1}{2}ft^2\text{.....[2]}.$$

To eliminate t from these, we have, by squaring the first,

$$v^2=v_1^2+2v_1ft+f^2t^2=v_1^2+2f\left(v_1t+\frac{1}{2}ft^2\right),\ \text{that is, [2], } v^2=v_1^2+2fs\text{.....[3]}.$$

These are the formulæ to be used when the body is projected towards the force with a velocity v_1. If the projection be in the direction opposed to the force, then f retards the motion, and its sign must be changed in the preceding formulæ, which are then

$$v=v_1-ft,\ s=v_1t-\frac{1}{2}ft^2,\ v^2=v_1^2-2fs\text{.....[4]}.$$

When gravity is the accelerating force, the symbol g is put for f.

457. **Falling Bodies: Gravity.**—The following are a few examples on the vertical motion of bodies under the influence of gravity.

(1) A body is 4 seconds falling from rest to the earth: required the height from which it fell.

$$\text{Here } f=g=32{\cdot}2,\ \therefore s=\frac{1}{2}gt^2=16{\cdot}1\times4^2=257{\cdot}6 \text{ feet}.$$

(2) If a body be projected vertically upwards with a velocity of 400 feet; how high will it ascend? The height will be that from which, if the body were let fall, it would acquire a velocity of 400 feet upon reaching the earth,

$$\therefore \text{since } s=\frac{1}{2}\frac{v^2}{g},\ \text{we have } s=\frac{1}{2}\frac{400^2}{32{\cdot}2}=248{\cdot}4 \text{ feet}.$$

(3) From an elevated position a body is projected vertically upwards with a velocity of 80 feet: required its position at the end of 6 seconds.

$$\text{Here [4], } s=v_1t-\frac{1}{2}gt^2=80\times6-16{\cdot}1\times6^2=6(80-16{\cdot}1\times6)=-99{\cdot}6,$$

hence the body will be 99·6 feet *below* the point of projection.

Examples for Exercise.

(1) In what time will a heavy body descend 400 feet?

(2) A body is projected vertically upwards with a velocity of 1500 feet; how high will it ascend?

(3) With what velocity must a body be projected vertically downwards from the top of a tower 150 feet high to arrive at the bottom in two seconds?

(4) Through what height must a body fall to acquire a velocity equal to the space fallen through?

(5) A body is projected vertically upwards with a velocity of 100 feet: at what time will it be 100 feet from the earth, as well in its descent as in its ascent?

(6) At the same instant that a body begins to fall from a height of 100 feet, another body is projected upwards with a velocity that would carry it 150 feet: at what height would they meet?

(7) With what velocity must a stone be projected down a well 350 feet deep that it may arrive at the bottom in 4 seconds?

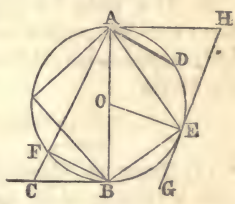
(8) When a body has fallen a feet, another, b feet below where the first then is, is let go: how many feet will this latter have fallen when the former overtakes it?

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458. **Motion of Bodies on Inclined Planes.**—When a body is placed upon a smooth plane, of inclination i to the horizon, the force which urges it down the plane is only that component of gravity which is in the direction of the plane, namely, $g \sin i$; the other component, being perp. to the plane, produces statical pressure, not motion. Consequently, we have only to substitute $g \sin i$ for f, in the formulæ at p. 428, to render those formulæ as applicable to this oblique motion of bodies as they now are to their vertical motion.

Let l represent the length of the plane, and h its height, then $\sin i = \frac{h}{l}$, so that the acceleration down the plane is $f = g\frac{h}{l}$, and the velocity acquired in descending down the whole length is (p. 428) $v = \sqrt{2fl} = \sqrt{2gh}$, the same as that acquired in falling down the height h; and we see that provided the height of the plane be constant, it matters not what be its length:—the velocity acquired in descending down it must be always the same, namely, the velocity acquired in falling through the height.

1. To determine what length down an inclined plane a body will descend while another body falls through the height of the plane. Putting s for the space descended down the plane, in the time t that the body falls vertically through the height, we have $h = \frac{1}{2}gt^2$, $s = \frac{1}{2}g \sin i \,.\, t^2 = h \sin i$. This expression for the length of plane, descended in the time that would be occupied in falling through the height, suggests an interesting property, namely,



2. If from the extremities A, B, of the vertical diameter of a circle, any number of chords be drawn, a body would fall down either of them in the same time that it would fall down the vertical diameter AB. For prolong any one of the chords from A, as AF, till it meets the horizontal line BC; then the part of AC, fallen down in the time that AB would be fallen down, is $AB \sin C = AB \cos BAC = AF$: and similarly of any other chord from A. The same is true for the chords from B: take BF for instance: then the chord AD from A, parallel to BF, is equal to BF, since the angles ABF, BAD, and $\therefore$ the arcs AF, BD, are equal: hence the lines AD, FB, being equal in length, and equally inclined to the horizon,

the times down them must be equal, ∴ the time down FB is the same as that down the vertical diameter.

3. To find the position of an inclined line through a given point, such that a body falling down it, from that point, may arrive at a given straight line in the shortest time possible. Let A be the given point, and GH the given line. Draw through A the horizontal line AH, meeting the given line in H, and make $HE=HA$: then will AE be the inclined line required. For let a perp. to AH be drawn from A, and a perp. to GH from E: these will intersect in some point O; and if OH be drawn, we shall thus have two equal right-angled triangles, so that the two perpendiculars will be equal: hence a circle from centre O, with radius OA, will touch HA, HG, in A, E, and the diameter AB will be vertical. Now AE is the only chord of the circle that reaches the given line; and the time down any other chord AD from A, and which is too short to reach the given line, is equal to that down AE, which does reach it, ∴ the line is reached down AE in less time than down any other straight line from A.

Examples for Exercise.

(1) If the length of an inclined plane be 60 feet, and its inclination 30°, what velocity would a body acquire in falling down it for 2 seconds?

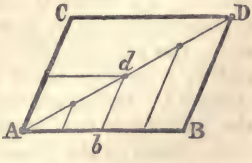
(2) How long would a body be in falling down the plane $l=100$ ft. $i=60°$?

(3) How far up the plane, $h=\frac{1}{6}l$, would a body ascend in 6 secs. with a vel. of 50 feet?

(4) Find the line of quickest descent from a given point to a circle in the same vertical plane.

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459. **The Parallelogram of Velocities.**—The velocity of a moving body at any instant is the rate or speed at which it is *then* moving, and this is measured by the space it *would* pass over in one second, if during that second its speed were to continue the same as at the instant referred to. That a body would continue to move with the same velocity that it has at any instant, it is only necessary that at that instant all influence and obstruction should be withdrawn, and the body be left to itself: its motion *then* will be the same as if it were originally communicated by a single momentary impulse. Suppose the body when at A were animated with a velocity (or received an impulse) that would alone carry it to B in a certain time, and that it were simultaneously animated with another velocity (or received another impulse), that would alone carry it, in the same time, to C: the joint effect of the two velocities, or of the two impulses, will be to carry it uniformly along the diagonal of the parallelogram to D in that time. For conceive the line AC with the moving body upon it, to be carried parallel to itself, and with the first uniform velocity, up to BD; then at the instant it arrives at BD, the body will have arrived at the extremity of the moving line, that is, it will be found at D. In consequence of its compound motion, it will have arrived there by the path AD: for in every position d of the moving body, $Ab : bd :: AB : BD$, since from the uniformity of the motions, the advance Ab of the body in



the direction AB, to its simultaneous advance bd in the direction AC, is always in this ratio: the point d is therefore always on the diagonal AD: and thus the single velocity AD is equivalent to the two component velocities AB, AC. Hence, what is proved in Statics, as to the composition and resolution of forces, may in like manner be proved in reference to the composition and resolution of velocities. If θ be the angle A between the directions of two component velocities v, v', then for the resultant velocity V, we shall have

$$V^2=v^2+v'^2+2vv'\cos\theta. \quad \text{If } \theta=90°, \text{ then } V^2=v^2+v'^2, \text{ and } \tan[V, v]=\frac{v'}{v},$$

the last expression being the tan of the angle which the resultant velocity makes with v, one of the rectangular component velocities.

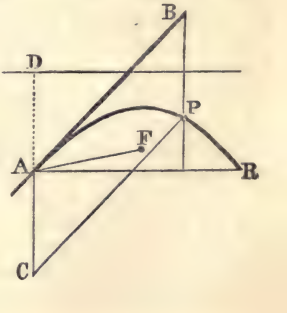
It has been assumed above that the body arrives at D from A, in consequence of its having been subjected, when at A, to two influences, of which one alone would have carried it to B, with a uniform motion, and of which the other would alone have carried it to C, also with a uniform motion, and in the same time. But the reasoning that shows that the body would arrive at D, by the joint effect of these two uniform motions, would equally apply, however irregular the motion that would carry it to B, and however irregular that, that would carry it to C, the time of the arrivals being the same. If the moving line AC just reaches BD, when the body moving along AC just reaches C, it is plain, whether the motions are uniform or not, that the body must, at the end of that time, be found at D; although its path to that point would be the diagonal of the parallelogram only when the component motions are uniform motions.

The principle upon which the foregoing proposition rests is that called *The Second Law of Motion:* namely, when a force acts upon a body in motion, the effect of this action, in magnitude and direction, is the same as it would be if it acted upon the body at rest.

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460. **The Motion of Projectiles.**—Hitherto the path of a body, under the influence of gravity, has been considered only in those cases in which it is a straight line—the two cases, namely, in which its motion is either vertical or down an inclined plane. But if it were projected obliquely in space, and all resistance and obstruction removed, its path would be a parabola, as will appear from the following investigation.

Let AB be the direction in which a body is projected from A with a velocity v_1: then, if no other motion were communicated to it, it would move uniformly along the straight line AB with that velocity, and in t seconds would arrive at B, provided $AB=v_1t$. But as gravity continues to act on the body from the instant of its motion, this force alone, in the t seconds, would draw the body down along the vertical AC to C, AC being $=\frac{1}{2}gt^2$ (p. 428). Consequently, completing the parallelogram CB, the body in the t seconds would be found at P, as shown above. Now, since



$$CP=AB=v_1t, \text{ and } AC=\frac{1}{2}gt^2, \therefore \frac{CP^2}{AC}=\frac{2v_1^2}{g}, \therefore CP^2=\frac{2v_1^2}{g}AC;$$

that is, regarding AC as the axis of x, and AB as the axis of y, $y^2=\frac{2v_1^2}{g}x$,

the equation of a parabola (396), the vertical AC being a diameter, and AB a tangent to the curve at the vertex A of that diameter.

Therefore, abstracting the resistance of the air, the path of a projectile is always a parabola, with its axis vertical. The coefficient of x in its equation is $=4FA$, F being the focus, and FA the radius vector of the origin, p. 343, or it is $=4h_1$, h_1 being the distance AD of A from the directrix, which here is horizontal. But the equation $\frac{2v_1^2}{g}=4h_1$, gives $v_1^2=2gh_1$, which expresses the square of the velocity a body would acquire in falling through the height h_1: consequently the velocity with which the body is projected is equal to that which it would acquire from gravity by falling down from the directrix of the parabola described to the point of projection. And since any point in the curve may be regarded as the point of projection, it follows that the velocity of the projectile at any point of its path is equal to that which it would acquire by falling to that point from the directrix. Hence, at equal heights, ascending and descending, the velocities must be equal, and the velocity will be least at the highest point, that is, at the vertex of the parabola, h_1 being then the least.

461. **Equation of Path in Horizontal and Vertical Co-ordinates.**—Taking the point of departure A for the origin of horizontal and vertical co-ordinates, the equation of AB, the path the projectile would pursue but for gravity, is $y=x\tan\alpha$; but in the time t seconds, gravity diminishes y by $\frac{1}{2}gt^2$, therefore the actual value of y is $y=x\tan\alpha-\frac{1}{2}gt^2$, α being the angle which the direction of the impulse makes with the horizontal axis, or the angle of elevation of the piece.

The velocity of projection being v_1, the space AB in t seconds is v_1t, and the x of B is therefore $v_1t\cos\alpha$, which is the x of P in that time; hence,

$$t=\frac{x}{v_1\cos\alpha}\text{.....[1]}, \quad \therefore y=x\tan\alpha-\frac{g}{2v_1^2\cos^2\alpha}x^2\text{.....[2]},$$

the equation of the path, which is therefore a parabola (400).

If h_1 be the height due to the velocity of projection, then $v_1^2=2gh_1$, and the equation becomes

$$y=x\tan\alpha-\frac{x^2}{4h_1\cos^2\alpha}\text{.....[3]}.$$

Let (a, b) be the vertex of the parabola, then the origin of the axes will be transferred to that point by putting $x+a$ for x, and $y+b$ for y. These substitutions give

$$y=\left(\tan\alpha-\frac{a}{2h_1\cos^2\alpha}\right)x-\frac{1}{4h_1\cos^2\alpha}x^2+\left(a\tan\alpha-\frac{a^2}{4h_1\cos^2\alpha}-b\right),$$

and that this may have the form $y=Qx^2$, we must have $\tan\alpha=\dfrac{a}{2h_1\cos^2\alpha}$,

and $a\tan\alpha-\dfrac{a^2}{4h_1\cos^2\alpha}=b=a^2\left(\dfrac{1}{2h_1\cos^2\alpha}-\dfrac{1}{4h_1\cos^2\alpha}\right)=\dfrac{a^2}{4h_1\cos^2\alpha}$.

From these we get

$$a=2h_1\sin\alpha\cos\alpha=h_1\sin 2\alpha,\ b=h_1\sin^2\alpha\ldots\ldots[4],$$

which are the co-ordinates of the vertex. And the equation of the parabola, the origin being at the vertex, is

$$y=-\frac{1}{4h_1\cos^2\alpha}x^2,\ \text{or}\ x^2=-(4h_1\cos^2\alpha)y,$$

the curve being wholly *below* the tangent at the vertex, or the axis of x: the latus rectum of the parabola is therefore $4h_1\cos^2\alpha$ (388).

But the co-ordinates of the vertex may be obtained more readily thus: Since the ordinate of the vertex is greater than that of any other point, the expression [3] for this value of y must be a maximum,

$$\therefore\ (193),\ \tan\alpha-\frac{x}{2h_1\cos^2\alpha}=0,\ \therefore x=2h_1\cos^2\alpha\tan\alpha=h_1\sin 2\alpha,$$

$$\therefore\ [3],\ y=2h_1\cos^2\alpha\tan^2\alpha-h_1\cos^2\alpha\tan^2\alpha=h_1\sin^2\alpha,$$

which values of x and y are the co-ordinates of the vertex, as before.

462. **Range of the Projectile.**—The horizontal range is the distance AR from the point of projection, on the horizontal plane through that point, at which the projectile falls: it is of course got at once by putting $y=0$, in the equation of the path [3]; we thus get for x,

$$\text{Range}=4h_1\cos^2\alpha\tan\alpha=2h_1\sin 2\alpha\ldots\ldots[5].$$

This is the greatest possible when $\sin 2\alpha$ is the greatest, which it is when $2\alpha=90°$, that is, when the elevation of the piece is 45°, the length of this range being,

$$\textit{Maximum range}=2h_1=\frac{v_1^2}{g},\ \text{equal to double the height due to the initial velocity.}$$

As the sine of $2(45°-\theta)$ is the same as the sine of $2(45°+\theta)$, it follows from [1] that the range is the same for angles above 45° as for angles as much below 45°, the charge being the same.

Calling the maximum range R, we have from [5], for any other ranges r, r', due to elevations α, α', the expressions

$$r=R\sin 2\alpha,\ r'=R\sin 2\alpha',\ \therefore\frac{r'}{r}=\frac{\sin 2\alpha'}{\sin 2\alpha},\ \therefore r'=\frac{\sin 2\alpha'}{\sin 2\alpha}r\ldots\ldots[6];$$

so that, 1. When the maximum range is known for a given charge, the range for any elevation, with the same charge, is also known; and, 2. When the range r is found for any elevation α, the range r' for any other elevation α' with the same charge becomes also known.

463. **Time of Flight.**—By the equations [1], and [5], the time of flight is

$$t=\frac{2h_1\sin 2\alpha}{v_1\cos\alpha}=\frac{4h_1\sin\alpha}{v_1}:\ \text{but}\ h_1=\frac{v_1^2}{2g},\ \therefore t=\frac{2v_1\sin\alpha}{g},\ \text{time of flight.}$$

464. **Greatest Height.**—The point of the parabolic path which is highest above the horizontal plane being the vertex of the parabola, and the ordinate of it being the value of b, above [4], we have

$$H=h_1\sin^2\alpha,\ \text{the greatest height}\ldots[7].$$

Otherwise: without referring to this expression for the greatest ordinate, we have, for the vertical component of the velocity of projection, $v_1 \sin \alpha$: the projectile will $\therefore$ continue to rise till this velocity is destroyed by gravity. Now the height or space s, due to any velocity v, is

$$s=\frac{v^2}{2g} \text{ (p. 428)}, \therefore H=\frac{v_1^2 \sin^2 \alpha}{2g}=h_1 \sin^2 \alpha.$$

And from similar considerations the time of flight may be found: for as the body reaches its highest point, or describes half its path, in the same time that the vertical component of the velocity of projection would carry it to that height, and since, whatever be the vertical velocity v, this time is

$$t=\frac{v}{g} \text{ (p. 428)}, \therefore \text{half the time of flight}=\frac{v_1 \sin \alpha}{g}, \text{ as before.}$$

Still employing the same principle—the resolution of velocities—we may, in like manner, now deduce the horizontal range: for the horizontal component of the velocity of projection is $v_1 \cos \alpha$, and since the whole time of flight is

$$\frac{2v_1 \sin \alpha}{g}, \therefore r=\frac{2v_1 \sin \alpha}{g} \times v_1 \cos \alpha=\frac{v_1^2}{g} \sin 2\alpha=2h_1 \sin 2\alpha, \text{ as before found.}$$

And thus from the principle at (459), and the common theory of the vertical descent of bodies, may all the results of the foregoing articles be arrived at, without knowing even the form of the curve.

We shall add a problem or two on the foregoing theory.

PROB. I. Given the initial direction, to determine the velocity so that the projectile may strike a given point.

Let (x', y') be the given point, then, by the equation of the path [2],

$$y'=x \tan \alpha-\frac{g}{2v_1^2 \cos^2 \alpha}x^2, \therefore v_1=\frac{x'}{\cos \alpha}\sqrt{\frac{g}{2(x' \tan \alpha-y')}}.$$

PROB. II. Given the velocity of projection (or the charge), to determine the elevation of the piece so that the projectile may strike a given point.

In the equation of the path [3], $y=x \tan \alpha-\frac{x^2}{4h_1 \cos^2 \alpha}$, putting (x', y') for the point,

and $1+\tan^2 \alpha$ for $\frac{1}{\cos^2 \alpha}$, we have

$$\tan^2 \alpha-\frac{4h_1}{x'} \tan \alpha=-\left(\frac{4h_1 y'}{x'^2}+1\right), \therefore \tan \alpha=\frac{2h_1 \pm \sqrt{(4h_1^2-4h_1 y'-x'^2)}}{x'}.$$

Hence, whenever the problem is possible, there are in general two different elevations, at each of which the mark will be hit by the same charge: but if the initial velocity, or, which is the same thing, if the value of h_1, be such that

$4h_1^2=4h_1 y'+x'^2$, or $(2h_1-y')^2=x'^2+y'^2$, there is then only one direction, namely,

$\tan \alpha=\frac{2h_1}{x'}$. And when v_1 is such that $4h_1^2<4h_1 y'+x'^2$, or

$(2h_1-y')^2<x'^2+y'^2$, the problem is impossible:

the mark cannot then be hit with the proposed charge, whatever be the elevation of the piece. For the time elapsed from the instant of discharging the piece till the object is struck, we have from the equation (461)

$$y=x \tan \alpha-\frac{1}{2}gt^2, \text{ for } t \text{ the value } t=\sqrt{\frac{2x' \tan \alpha-2y'}{g}}.$$

Examples for Exercise.

(1) A body is projected in a direction making an angle of 15° with the horizon, with a velocity of 40 feet per second: required its horizontal range, greatest altitude, and time of flight.

(2) Prove that if a body be projected at an elevation of 15° or 75°, the horizontal range is equal to the space fallen through to acquire the velocity of projection.

(3) A body is projected at an angle of 60° and descends to the horizon at 100 feet distant: what was the initial velocity, the greatest altitude, and the time of flight?

(4) With a given initial velocity, what must be the angle of elevation in order that the area of the parabola described may be the greatest possible?

(5) A number n of balls, of weights $w_1, w_2, \ldots w_n$, are all projected at the same instant from the same place with velocities $v_1, v_2, \ldots v_n$, at elevations $\alpha_1, \alpha_2, \ldots \alpha_n$: required the height of their centre of gravity at the end of t''.

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465. **Range on an Oblique Plane.**—Instead of the plane passing through the point of projection being horizontal, let it be inclined to the horizon at the angle i: then the equation of the line of oblique range will be $y = x \tan i$; and substituting this value of y in the equation [3] of the parabolic path, we shall have the abscissa of the point where the curve meets the oblique line from the resulting equation

$$x \tan i = x \tan \alpha - \frac{x^2}{4h_1 \cos^2 \alpha}, \text{ namely } x = 4h_1 \cos^2 \alpha (\tan \alpha - \tan i) = \frac{4h_1 \cos \alpha \sin (\alpha - i)}{\cos i};$$

consequently the length of the oblique line itself will be

$$\frac{x}{\cos i} = \frac{4h_1 \cos \alpha \sin (\alpha - i)}{\cos^2 i}, \text{ the oblique range.}$$

If the plane be a *descending* plane, i will be negative, so that $\alpha + i$ must be put for $\alpha - i$.

To find for what elevation the range is a maximum, since $\cos^2 i$ is constant as well as h_1, we have to find the value α that renders $\cos \alpha \sin (\alpha - i) =$ a max. Now (p. 200),

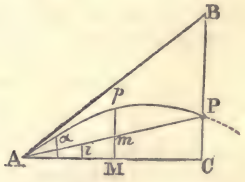
$$2 \cos \alpha \sin (\alpha - i) = \sin \{\alpha + (\alpha - i)\} - \sin \{\alpha - (\alpha - i)\} = \sin (2\alpha - i) - \sin i,$$

$$\therefore \sin (2\alpha - i) = \text{max.}, \ \therefore 2\alpha - i = 90°, \ \therefore \alpha = 45° + \frac{1}{2} i;$$

$$\text{or, when the plane is descending, } \alpha = 45° - \frac{1}{2} i.$$

Hence, the initial direction bisects the angle between the plane and the vertical.

466. **Time of Flight.**—Let t be the number of seconds occupied in describing the arc AP, AB being the initial direction; then the space $BP = \frac{1}{2} g t^2$, and $AB = v_1 t$, $\therefore t = 2 \frac{BP v_1}{AB g}$. Now



$$\frac{BP}{AB} = \frac{\sin (\alpha - i)}{\sin APB},$$

$\therefore$ substituting for $\frac{BP}{AB}$, and putting for $\sin APB$ its equal $\cos i$, we have

$$t = \frac{2v_1}{g} \cdot \frac{\sin (\alpha - i)}{\cos i} = 2 \sqrt{\frac{2h_1}{g}} \cdot \frac{\sin (\alpha - i)}{\cos i},$$

where $\alpha + i$ is to be put for $\alpha - i$ if the plane be a descending plane.

467. **Greatest Height above the Plane.**—Let mp be the greatest height above AP; then $pM-mM$ is a maximum; that is, since

$$mM=x\tan i,\ mp=x\tan\alpha-\frac{x^2}{4h_1\cos^2\alpha}-x\tan i=\text{max.},$$

$$\therefore\ (193),\ \tan\alpha-\frac{x}{2h_1\cos^2\alpha}-\tan i=0,$$

$$\therefore\ x=2h_1\cos^2\alpha\,(\tan\alpha-\tan i)=\frac{2h_1\cos\alpha\sin(\alpha-i)}{\cos i},$$

which, substituted in the expression for mp, gives the greatest height. Since

$$\tan\alpha-\tan i=\frac{\sin(\alpha-i)}{\cos\alpha\cos i},\ \therefore\ mp=\frac{x}{\cos\alpha}\left\{\frac{\sin(\alpha-i)}{\cos i}-\frac{x}{4h_1\cos\alpha}\right\}.$$

Under this change of form, the substitution of the above value of x gives for the maximum value of mp,

$$mp=\frac{2h_1\sin(\alpha-i)}{\cos i}\left\{\frac{\sin(\alpha-i)}{\cos i}-\frac{\sin(\alpha-i)}{2\cos i}\right\}=\frac{h_1\sin^2(\alpha-i)}{\cos^2 i}.$$

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468. The foregoing articles contain all necessary particulars concerning the motion of a projectile in a non-resisting medium: the several results have been arrived at quite independently of experiment, except only that by which the force of gravity is determined; *special* experiments, however, would be altogether superfluous, as the motion of the projectile is governed by influences well ascertained and strictly determinate:—the instantaneous force of propulsion, and the continuous force of gravity, and by these alone. But any disturbing cause, or the action of any additional force upon the moving body, must of course vitiate the purely mathematical conclusions here reached. In the practice of gunnery, therefore, this theory must be considerably modified, on account of the resistance of the air, a force always as present as the force of gravity itself, and which is of such an effect on projectiles of considerable velocity, that "some of those which in the air range only between 2 and 3 miles, at the most, would in vacuo range about ten times as far, or between 20 and 30 miles." This conclusion is justified from numerous experiments (by Dr. Hutton) specially undertaken for the purpose of arriving at safe practical rules of calculation for determining the range, time of flight, &c., of Military Projectiles: the principal of these practical rules are the following:—

I. *To find the velocity of any Shot or Shell.*—Rule. Divide three times the weight of the charge of powder, by the weight of the shot, taking both weights in either ounces or pounds, and multiply the square root of the product by 1600.

II. *Given the range for one elevation, to find it for another.*—Rule. As the sine of double the first elevation is to its range, so is the sine of double another elevation to its range.

III. *Given the range for one charge, to find it for another.*—Rule. One range is to its charge as any other to its charge: the elevation being the same.

It is to be observed, that of these three rules the first only is entirely the conclusion from experiment; the others are deductions from the parabolic theory: the third rule is equivalent to the statement that for

the same elevation the range is as the impetus, or height due to the initial velocity.

The following examples, chiefly from Dr. Hutton, are to be solved by combining the first rule above, founded upon actual experiment, with the approximate rules, derivable from the parabolic theory of projectiles, established in the preceding articles.

Ex. If a ball of 16 oz. weight be projected with a velocity of 1600 feet per second when fired with $5\frac{1}{2}$ oz. of powder, it is required to find with what velocity each of the several kinds of shells will be discharged by the full charges of powder, namely:—

Diameters of the Shells in inches .	13	10	8	$5\frac{1}{2}$	$4\frac{2}{5}$
Their weights in lbs.	196	90	48	16	8
Charge of Powder in lbs.	9	4	2	1	$\frac{1}{2}$
The velocities of discharge .	594	584	565	693	693

These are got from Rule I.: thus, for the first velocity we have

$$v_1 = 1600 \times \sqrt{\frac{27}{196}} = \frac{800 \times 3\sqrt{3}}{7} = \frac{800 \times 5{\cdot}1962}{7} = 594 \text{ nearly.}$$

Examples for Exercise.

(1) If a shell range 1000 yards when discharged at an elevation of 45°, how far will it range with the same charge at an elevation of 30° 16′?

(2) The range of a shell at 45° elevation being found to be 3750 feet, at what elevation must the piece be set, to strike an object at the horizontal distance of 2810 feet?

(3) With what impetus (h_1), velocity (v_1), and charge, must a 13-inch shell be fired, at an elevation of 32° 12′, to strike an object at the horizontal distance of 3250 feet?

(4) A shell being found to range 3500 feet when discharged at an elevation of 25° 12′: how far will it range at an elevation of 36° 15′, with the same charge?

(5) If with a charge of 9 lb. of powder a shell range 4000 feet: what charge will suffice to throw it 3000 feet, the elevation being 45° in both cases?

(6) How far will a shot range on an ascending plane of inclination 8° 15′, and on a descending plane of the same inclination, the impetus being 3000 feet, and the elevation of the piece 32° 30′?

(7) How much powder will throw a 13-inch shell 4244 feet up an inclined plane of 8° 15′ inclination, the elevation of the mortar being 32° 30′?

(8) Required the weight of the shot, in order that 6 lb. of powder may fire it with a velocity of 1200 feet.

(9) What must be the elevation of the mortar to throw a 13-inch shell 4000 feet up an inclined plane that ascends 10° 40′, the charge being $4\frac{3}{4}$ lb.?

Note.—It is found from experiment that the greatest horizontal range, instead of being constantly that at an elevation of 45°, as in the parabolic theory, will be at all intermediate degrees between 45° and 30°, according to the velocity and weight of the projectile; the smaller velocities and larger shells ranging farthest when projected at an elevation of almost 45°; while the greater velocities, especially with the smaller shells, range farthest with an elevation of about 30°. "But it usually happens, in the operation of natural causes, that near the point where any quantity is greatest or least, its variation is slower than elsewhere: a small difference, therefore, in the angle of elevation, is of little consequence to the extent

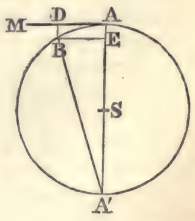
of the range, provided that it continue between the limits of 45° and 35°; and, for the same reason, the angular adjustment requires less accuracy in this position than in any other, which, besides the economy of powder, makes it in all respects the best elevation for practice, where the object is to carry a ball or shell to the greatest possible distance."* But for very ample practical information on the subject of Gunnery, the student is referred to the third volume of Dr. Hutton's Tracts.

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469. **Circular Motion, Centrifugal Force.**—If a body move uniformly in the circumference of a circle, the force which confines it to that curve must reside in, or be directed to, the centre of the circle: for resolving its actual motion at any instant into two—one in the direction of the tangent at the point where it then is, and the other in the direction perpendicular to that tangent—the latter direction will always pass through the centre of the circle: the former will be that direction in which the body tends to move at the instant, and therefore that in which it would move if left to itself. It is thus constrained to deviate from its wonted rectilinear path, solely in virtue of a force soliciting it towards the centre: this is hence called the *centripetal force:* the equal and opposite of this, that is, that opposing force which, with equal intensity, urges the body *from* the centre of the circle, is called the *centrifugal force:*—it is the force with which the body tends to increase its distance from the centre.

Of course, it is here supposed that the body is confined to its circular path, or orbit, solely by the unimpeded action of these opposing forces: if a material resistance be interposed between the circumference and the centre, so that the body be thus *prevented* from yielding to the central force, then this may exceed the centrifugal force to any amount; on the contrary, if the body's approach to the centre be unobstructed, but a material resistance is opposed to its further departure from it, the centrifugal may exceed the central force to any amount.

Let S be the centre of the circular path, and let AB be the arc described by the moving body from A in the time t seconds, where t may be a fraction of unlimited smallness. If the attractive force f at S, which when the body is at A acts perpendicularly to its wonted path AM, had continued to act perpendicularly to that line during the t seconds, then the body, since it is found at the expiration of that time at B, would have fallen perpendicularly to AM, the distance $DB=AE$: so that v being the velocity of the body in the direction AM, we should have



$$AD=vt, \text{ and (455) } AE=\frac{1}{2}ft^2, \quad \therefore \text{ eliminating } t, \ f=2v^2\frac{AE}{AD^2}, \text{ or } f=2v^2\frac{AE}{EB^2}.$$

Draw the diameter AA', and the chords BA, BA' (the former of which, though not in the diagram, may be supplied); then

$$BA^2=AA'\times AE, \ \therefore \ AE=\frac{BA^2}{AA'}, \text{ hence by substitution, } f=\frac{2v^2}{AA'}\left(\frac{BA}{EB}\right)^2.$$

Now the force acting on the body, during its progress from A to B, is

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* Illustrations of the Celestial Mechanics of Laplace (By Dr. T. Young), p. 132.

more nearly perpendicular to the tangent AD, the more the distance AB is diminished; that is, the expression for f differs less and less from the exact expression, as t becomes less and less, and there is no limit, in the above reasoning, to the smallness of t. But when t decreases down to 0, the vanishing fraction $\frac{BA}{EB}=\frac{\text{chord}}{\text{sine}}=1$. This is plain, because, calling the diam. of the circle $2r$, and AE, x, we have

$$\frac{BA^2}{EB^2}=\frac{2rx}{x(2r-x)}=\frac{2r}{2r-x}=1, \text{ when } x=0.$$

Hence the expression for the constant centrifugal force—equal and opposite to the centripetal force at S—is $f=\frac{v^2}{r}$. Or calling the whole time of describing the circle, that is, the periodic time, T, and remembering that the uniform velocity v is equal to the whole space divided by the whole time of describing it, we have

$$v=\frac{2\pi r}{T},\ f=\frac{v^2}{r}=\frac{4\pi^2 r}{T^2}\ldots[1],$$

so that the centrifugal force varies as the radius of the circle directly, and as the square of the periodic time inversely.

If the periodic times be the same in two different circles of radii r, R, and if f, F, be the respective centrifugal forces, we shall therefore have

$$\frac{f}{F}=\frac{r}{R}\ldots\ldots[2].$$

It is worthy of notice that the expression for f above, namely, $f=\frac{v^2}{r}$, coincides with the expression [2] for f at p. 427, namely,

$$f=\frac{v^2}{2s}, \text{ when } s=\frac{1}{2}r:$$

hence the uniform velocity v, of the revolving body, is the same as that which the body would acquire by falling from rest towards the centre of attraction down half the radius of the circle, observing that f denotes indifferently, as to intensity, either the central or the centrifugal force, since the two forces balance.

470. **Centrifugal Force at the Surface of the Earth.**—In the preceding article the centripetal and centrifugal forces are accurately balanced, so that neither prevails; but bodies on the surface of the earth preserve their distance from its centre during its diurnal rotation, though these forces are unbalanced—the centripetal or attracting force greatly exceeding the centrifugal force: if this were not the case, bodies at the surface of the earth would have no weight: the fact of their having weight proves that the central attraction—gravity—exceeds the opposing centrifugal force; the superior attraction being further opposed by the resistance of the solid earth, by which bodies are prevented from approaching nearer to the centre, as they otherwise would do, seeing they have weight.

But the foregoing deductions equally apply to cases of this kind:—the revolution of a body in a circle generates a centrifugal force: there is an equivalent and opposite centripetal force which, as above, just balances it, and by so much is the actual force at the centre of the circle counteracted.

In what follows we have to ascertain the amount of this counteracting force at different places on the earth's surface.

By its diurnal rotation, the earth carries round with it, with a uniform velocity, every point on its surface in 86,164 seconds, whether the point be nearer to or farther from the axis of rotation: as the *time* of rotation of all points on the surface is the same, the centrifugal force, which by [1] increases as the radius or distance from the axis of rotation increases, must be greatest at the equator, of which the radius is R=20,923,596 ft. *There* the centrifugal force is

$$F=\frac{4\pi^2R}{T^2}=\frac{4\pi^2\times 20923596}{86164^2}=\cdot 11126 \text{ feet.}$$

As this force opposes that of gravity, it follows that if the earth had no rotation on its axis, gravity at the equator, instead of being g=32·088, as experiment determines it to be, would be $G=g+\cdot 11126$ foot; so that the weight of a body there would be the $\frac{\cdot 11126}{32\cdot 088}$ part more than it actually is.

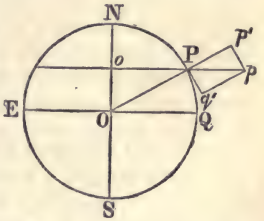
Since $G : F :: 32\cdot 199\ldots : \cdot 11126$, or as 289 : 1 nearly, $\therefore F=\frac{G}{289}$...[3];

or, since $g : F :: 32\cdot 088 : \cdot 11126$, or as 288 : 1 nearly, $\therefore F=\frac{g}{288}$...[4].

Representing the centrifugal force in the parallel to the equator, of which the latitude is l and the radius r, by f, we have [2],

$$\frac{f}{F}=\frac{r}{R},\ \therefore f=F\frac{r}{R}=\frac{G}{289}\cos l, \text{ or } \frac{g}{288}\cos l, \text{ because } \frac{r}{R}=\cos l\ldots[5].$$

The force of gravity G is not diminished by the whole of the centrifugal force except at the equator, since elsewhere this force does not act in a direction from the centre of the earth through the body, but from the centre of the rotating circle on which that body is—from the centre of the parallel of latitude. Thus, in the annexed diagram, while gravity, or the centripetal force on P, acts in the direction PO, the centrifugal force acts in the direction Pp. If, therefore, we decompose the force Pp in the perpendicular directions Pp', Pq', the former component being a force directly opposed to that of gravity, and the latter a force acting tangentially, and therefore urging the body P towards the equator EQ, we shall have



Force opposing gravity, $Pp'=Pp\cos p'Pp=f\cos l=\frac{G}{289}\cos^2 l$.

Hence that part of the centrifugal force at any point of the earth's surface which is exercised in directly opposing gravity, varies as the square of the cosine of the latitude of that point: the fraction $\frac{G}{289}$ has been already shown to be ·11126 foot.

The other component Pq' of the centrifugal force at P being in the direction of a tangent at that point, is exerted in driving the particles

from the region of the pole N, to that of the equator, and to cause the oblate figure which the earth is known to assume on account of the accumulation of matter about the equatorial regions. The expression for this tangential force is

$$\text{Tangential force, } Pq' = f \sin l = \frac{G}{289} \sin l \cos l = \frac{G}{578} \sin 2l = \cdot 05563 \sin 2l.$$

This value is the greatest possible when $\sin 2l = 1$, that is, in the latitude of 45°, where the tangential force is equal to half the centrifugal force at the equator: on each side of lat. 45° the tangential force diminishes, and it vanishes altogether at the equator and at the poles.

From the above principles we may readily determine the time in which the earth must perform its daily rotation, in order that the centrifugal force at the equator may be exactly equal to G, the force of gravity there; that is, in order that a body at the equator may lose all its weight. Let T' be the required time of rotation, the time at present being T; then, the circle of rotation being the same, the centrifugal forces are inversely as the squares of the periodic times (p. 440); hence, the centrifugal force corresponding to the time T' being G,

and that corresponding to the time T being $\frac{G}{289}$, we have

$$G : \frac{G}{289} :: T^2 : T'^2, \quad \therefore T' = \frac{T}{\sqrt{289}} = \frac{T}{17}.$$

Consequently, if the diurnal rotation of the earth were performed in the 17th part of the time that it really occupies, that is, if it were to rotate 17 times as rapidly, all bodies at the equator would be without weight, and, if placed at a small distance above the surface of the earth, would remain there unsupported. If the rotation were still more rapid, bodies would be repelled from the equatorial parts of the earth's surface.

Examples for Exercise.

(1) What is the effect of centrifugal force in diminishing that of gravity in the latitude of London, 51° 31′ N.?

(2) Express the value of that component of the centrifugal force in the latitude of London which is exerted in transporting the materials on the surface towards the equator.

(3) Calculate both forces for the latitude of Dublin, 53° 21′ N.

(4) If the length of a sling be 2 feet, with what velocity must a stone placed in it revolve in a vertical circle in order that it may be just prevented from falling out?

(5) If the central be n times the centrifugal force, prove that the velocity of revolution is the same as that which the body would acquire in falling through a distance $\frac{1}{2}nr$ towards the centre, r being the radius of the circle.

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471. **Moving Forces.**—In the foregoing articles on motion, no account has been taken of the bulk or weight of the moving body, except, indeed, in the few remarks concerning military projectiles: we have been chiefly occupied with the path described, the time of describing it, and the velocity of the motion, particulars which, in reference to an individual body, moving under the influence of gravity, or of any similar accelerating force, remain the same whether the body be great or small—

whether it be a single particle or the aggregation of any number of particles, seeing that gravity affects all alike.

But if instead of a single isolated body, left free to obey this force, two or more bodies are so connected together that the motion communicated to either modifies that which, without such connection, would be communicated to the others, then the investigation of this motion renders the consideration of the quantity of matter moved indispensable.

472. *Mass.*—This term is used instead of the words "quantity of matter." If a body be compressed into one-half of its original bulk or volume, the quantity of matter would be the same in each of its states; but a cubic inch or foot of the compressed material would contain twice the quantity of matter in a cubic inch or foot of it in its original state, and we therefore say that the mass of the former inch or foot is double that of the latter inch or foot. But, as in other cases of measurement, it is necessary that some definite unit of measure should be fixed upon by general consent, and accordingly it has been agreed upon that, taking the cubic foot as the unit of volume, the quantity of matter in that bulk of pure distilled water should be regarded as the unit of mass.

473. *Density.*—This has reference to the closeness, more or less, with which matter is packed—to the volume in which it is comprised. In the illustration above, though the quantity of matter is the same whether it occupy a cubic foot of space, or be compressed into only half a foot, yet in the latter case, since it is packed into half the space or bulk it originally filled, it is said to be twice as *dense*. Hence, if there be two bodies A, B, of equal volumes, and A be found to be m times the *weight* of B, we infer that there is m times as much matter in A as in B; and, therefore, that the matter in A has m times the density of the matter in B. But although, by thus weighing two bodies of the same volume, at the same place, we could discover the ratio of their masses, and also the ratios of their densities, yet it must be observed that weight, mass (or quantity of matter), and density, are three very different things: weight is the statical measure of the external influence exercised upon mass by gravity; an influence which varies according as we transport the mass nearer to, or farther from, the centre of the earth: the *same mass*, at the pole, has a different weight from what it has at the equator. Mass and Density are, therefore, terms irrespective of, and having no necessary reference to, weight:—if bodies had no weight, these terms would be equally significant. In like manner, as distilled water, at a certain temperature (60° Fahrenheit), supplies, by universal convention, the unit of mass, so the same fluid equally supplies the unit of density; that is, we regard the density of water as 1; so that when the density of any material is said to be D, we understand that it has D times the density of distilled water; any given volume of it will therefore weigh, at the same place, D times an equal volume of distilled water.

Mass, density, and volume, are therefore thus related, namely,

$$\text{Mass} = \text{Density} \times \text{Volume}, \text{ or, in symbols, } M = DV:$$

mass is therefore the numerical expression for units of magnitude or volume:—not of the volume V, but of the volume that V would become if the body originally of volume V were expanded or compressed till it had the density of distilled water.

Suppose, for example, the volume of a body to be 10 cubic feet, and

that in each cubic foot there is three times the quantity of matter that there is in a cubic foot of distilled water, that is, that the body has three times the density of distilled water, then, for the mass M of the body, we have the value $M=DV=3\times10=30$ cubic feet of matter equal in density to distilled water. And it is easy to see that, for the purposes of calculation, there is great convenience in thus reducing different denominations to a single one:—replacing all matter—of which the varieties are so great—by its equivalent of a specific kind—pure water.

474. *Momentum.*—The momentum of a moving body is the product of its mass by its velocity: it properly represents the force of the blow with which the moving mass would strike an obstacle. Thus: taking the mass M in the example above, suppose it to move uniformly with a velocity v of 12 feet per second, then we should have $Mv=$ the momentum of 30 cubic feet of a substance of the same density as water, moving 12 feet per second. Calling the momentum of one cubic foot of water (or of any matter of the same density as water) moving one foot per second, the unit of momentum, we have in the case before us $Mv=360$ of these units.

475. *Moving Force.*—In uniform velocity the momentum of the mass moved is constant: in uniformly accelerated velocity, additional momentum is generated every second, the increments of momentum being constant like the increments of velocity. This constant increment of the momentum is called the *moving force* of the mass. Moving force is thus related to momentum, as acceleration is to velocity; that is, putting, as heretofore, v for velocity, and f for acceleration (or accelerating force), we have

$$v : f :: Mv : Mf, \text{ the moving force, } F.$$

The statical effect of a constant force on a mass M is continued constant pressure or weight: the dynamical effect is constant moving force:—unresisted pressure. It is important that the student acquire correct notions of the terms here defined. If a body in motion be acted upon by no force, its velocity is constant: this constant velocity affects equally every particle of the mass: the sum of all the particles, that is, the whole mass, multiplied by this constant velocity, is the constant momentum Mv. If a constant force f act on the mass, it acts on every particle, and constant acceleration of the velocity takes place: this multiplied by all the particles, or the whole mass, is, in like manner, the constant increment of the momentum, and is therefore represented by Mf, and called the *moving* force. The symbol f here, as elsewhere, though called the constant *force*, stands solely for the *effect* of that force—acceleration of velocity: in the case of gravity it is $32{\cdot}2$ *feet*. When we speak of multiplying this by M, we mean the multiplying it by the *number* of cubic feet of distilled water to which M, as a mass, is equivalent. The only concrete quantity entering these expressions is therefore *feet*, but we may, if we please, eliminate even this, and employ abstract numbers only, thus:

Let the quantity of matter in a cubic foot of pure water be taken for the unit of mass, and the density of pure water for the unit of density.

Let the momentum of the unit of mass (one cubic foot of pure water, or of matter equally dense) moving with a uniform velocity of one foot per second, be taken for the unit of momentum.

Let the constant increment of momentum of the unit of mass, generated in each second, when the constant acceleration is one foot per second, be taken for the unit of moving force. Then V representing the number of cubic feet in the volume of a body B, M the number of units of mass, v

the number of feet per second of velocity, f the number of feet per second of acceleration, and s the number of feet passed through in t seconds, we shall have

$M=DV$=the number of units of mass in the body B of density D and volume V.

Mv=the number of units of momentum of B moving with velocity v.

$F=Mf$=the number of units of moving force of B, when its acceleration is f.

$s=\frac{1}{2}ft^2$=the number of units of space passed through in t seconds when the acceleration is f.

From the equation moving force $F=Mf$, we get acceleration $f=\frac{F}{M}$.....[I.],

that is, it is the moving force divided by the mass moved.

NOTE 1. In solving particular dynamical problems, the student will in general find it to add to the clearness and intelligibility of his symbolical results if one concrete quantity—the concrete quantity linear space—be preserved; that is, to regard s, v, g, f, as representing *feet*. If the problem be statical, s and v can neither of them enter the inquiry, but g and f may, not however as generating motion, but *weight*, which is the only concrete quantity in Statics (see the problem which follows).

2. It is well too for the student to notice that, as the velocity acquired in t seconds, when the constant acceleration is f, is $v=ft$, the expression Mv, for the momentum of a moving mass, is equally expressed by Mft.

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476. **Examples of Moving Forces.**—PROB. I. Two heavy bodies W, W', whose masses are M, M', are connected by a string passing over a fixed pulley: required the circumstances of the motion.

Let M be the greater of the two masses. If M were supported, the pressure on the support would be the difference of the weights: this pressure, $\therefore$ when unresisted—that is, the moving force—is $F=(M-M')g$. The mass moved is $M+M'$; for not only is M moved downwards by this force, but M' is equally moved upwards. Hence [1], $f=\frac{M-M'}{M+M'}g$. Also for the velocity of M downwards, or of M' upwards, and for the space passed through equally by each, from the commencement of motion, we have [2], page 427,



$$v=ft=\frac{M-M'}{M+M'}gt,\quad s=\frac{1}{2}\frac{M-M'}{M+M'}gt^2,$$

in which, since masses are proportional to their weights, W, W', may be substituted for M, M': we may therefore write $f=\frac{W-W'}{W+W'}g$.

Tension of the String.—If the string T were fastened to the pulley so as to prevent motion, the whole force of gravity acting then upon M units of mass, it would stretch the string with the whole weight W: but as it is, only a part of gravity acts upon M to stretch the string, for the part $f=\frac{W-W'}{W+W'}$ of gravity, is expended in producing motion. Hence, calling the tension T, we have $W:T::g:g-f$, that is,

$$W:T::1:1-\frac{W-W'}{W+W'},\ \therefore T=W-\frac{W-W'}{W+W'}W=\frac{2WW'}{W+W'}.$$

The same expression for T would of course be given by considering the force acting upon the other mass M'; this force is not only that of gravity acting downwards, but also the force f, or the $\frac{W-W'}{W+W'}$ part of gravity acting upwards, both conspiring to stretch the string: we thus have

$$W' : T :: g : g+f, \text{ that is,}$$

$$W' : T :: 1 : 1+\frac{W-W'}{W+W'}, \therefore T=W'+\frac{W-W'}{W+W'}W'=\frac{2WW'}{W+W'}.$$

477. The foregoing investigation involves considerations partly dynamical and partly statical, tension being of the nature of weight, or statical pressure. *This* effect of gravity cannot, with any propriety or consistency, be represented by g, which has always been employed hitherto to stand for 32·2 *feet*, at least in the latitude of London, nor yet by the *moving* force Mg, unless due warning be given to the reader that the symbol is to take a different meaning. Without such change of meaning the statement $W=Mg$, given in most books on the present subject, is simply absurd. It is plain that the weight of any mass must be $W=M\times$ the *weight* of the unit of mass, M being the *number* of such units in the mass. Now, by general consent, the unit of mass is 1 cubic foot of distilled water, which is found to weigh exactly 1000 ounces; *this*, therefore, is the value of g in the expression $W=Mg$, which, with this *statical* interpretation of g, correctly expresses the weight in ounces of any body when its mass is given. It is evident that it must do so, for (by def.) M is the number (whole or fractional) of cubic feet of pure water to which the body is equivalent, as to weight; but 1 cubic foot of this weighs 1000 oz.; hence the weight of the body is

$$W=M\times 1000 \text{ oz.}=Mg=DVg \text{ (p. 443).}$$

And in like manner, if any other unit were taken for that of mass, g would stand for the weight impressed by gravity on that unit.

With this interpretation of g and of f (which above is a fractional part of g), let us return to the statical part of the present problem.

The tension, or weight on the string, is $T=M(g-f)=M\left(g-\frac{M-M'}{M+M'}g\right)=\frac{2MM'}{M+M'}g.$

But $M=\frac{W}{g}$, $\therefore T=\frac{2WW'}{W+W'}$, as before. [Here g and f both represent *weight*.]

Pressure on the axis of the Pulley.—Since the tension or weight acting on the string is the same on each side of the pulley, therefore the pressure or weight on the axis is

$$\frac{4MM'}{M+M'}g=\frac{4WW'}{W+W'}.$$

Suppose the weights W, W', were 7 lb. and 5 lb., then the tension of the string would be $5\frac{5}{6}$ lb.; that is, a weight of $5\frac{5}{6}$ lb., if suspended to a fixed string, would produce the same tension as is actually produced in the moving string. And the axis of the pulley would in like manner sustain a pressure of $11\frac{2}{3}$ lb. Also the acceleration f would be

$$f=\frac{7-5}{7+5}32\cdot2 \text{ ft.}=5\cdot3\dot{6} \text{ ft. per second,}$$

being $\frac{1}{6}$ of the accelerating force of gravity. [*Here* f represents linear *space*.]

478. **Atwood's Machine.**—This was a machine based upon the principle of the preceding problem, and contrived by Atwood to render the acceleration produced by gravity, or the dynamical value of g, very easily deducible from observation. Omitting here the vertical scale to measure the space descended, the friction wheels to diminish the friction on the axis of the pulley, &c., it is plain that the weights W, W', may be so chosen as to render the fraction $\frac{W-W'}{W+W'}$ of any value we please, that is, so that the acceleration f, of the descending weight, may be any fractional part we please of the acceleration g due to gravity. Hence the motion of the descending weight may be rendered sufficiently slow for the space passed over in one second, or in any number of seconds, to be accurately observed; and since we know this space s to be

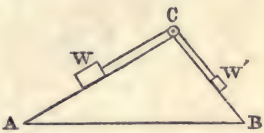
$$s=\frac{1}{2}ft^2, \text{ we thus readily find } f, \text{ and thence } g=\frac{W+W'}{W-W'}f.$$

Prob. II. Two weights connected by a string passing over a small pulley C are placed upon two smooth inclined planes CA, CB: it is required to determine the circumstances of their motion.

As before, putting M, M', for the masses of the weights W, W', the expression for the moving force of W, in the direction CA, is $Mg \sin i$; and the moving force of W', in the direction CB, is $M'g \sin i'$: these forces oppose each other, so that the moving force F, to which the motion is due, is their difference, namely,

$$F=(M \sin i-M' \sin i')\,g, \therefore [\text{I.}],\ f=\frac{F}{\text{mass moved}}=\frac{M \sin i-M' \sin i'}{M+M'}g.$$

This, therefore, is the expression for the acceleration of W down the plane CA, and consequently for the acceleration of W' up the plane BC. And since the velocity generated in t seconds from the commencement of motion is ft, and the space passed over $\frac{1}{2}ft^2$, the velocity acquired by W downwards (supposing this weight to prevail), or that acquired by W' upwards, in t seconds, is



$$v=\frac{M \sin i-M' \sin i'}{M+M'}gt, \text{ also } s=\frac{M \sin i-M' \sin i'}{2(M+M')}gt^2,$$

where $g=32{\cdot}2$ feet: the only concrete quantity in these expressions for the velocity and space.

If the two planes were vertical the problem would be identical with the former one, $\sin i$ and $\sin i'$ being each $=1$. But if one only of the planes were vertical, then the problem would be to determine the motion when one body W, hanging vertically, draws another W' up an inclined plane. In this case we have only to make $\sin i=1$ in the foregoing results. In like manner, if one of the planes, AC, were vertical, and the other horizontal, then in the same expressions,

$$\sin i=1, \text{ and } \sin i'=0, \therefore f=\frac{M}{M+M'}g.$$

And in all these expressions W, W', may be substituted for M, M', since

the masses are as their weights, and since the foregoing fractions remain the same, whatever quantities proportional to M, M', be substituted for them.

Tension of the String.—Under the original conditions, W being the descending weight, we have, for the tension of the string, the whole pressure of W in the direction of CA, diminished by so much of it as is expended in producing the motion, that is, it is

$$T=M\left(g\sin i-\frac{M\sin i-M'\sin i'}{M+M'}g\right)$$

$$=\frac{MM'}{M+M'}(\sin i+\sin i')g=\frac{WW'}{W+W'}(\sin i+\sin i'),$$

which becomes the same expression as that in last problem, when i and i' are each 90°, that is, when the weights hang vertically.

PROB. III. The height and length of the plane AC (last fig.) are given: to find the length of the plane BC, so that a given weight W may draw another given weight W' up BC in the least time possible.

Let h and l represent the height and length of the given plane, and put x for the length of the required plane; then since

$$\sin i=\frac{h}{l},\text{ and }\sin i'=\frac{h}{x},\ \therefore f=\frac{Whx-W'hl}{(W+W')lx}g,\text{ and since }s=x,$$

$$\therefore t^2=\frac{2s}{f}=\frac{2x}{f}=\frac{2(W+W')lx^2}{(Wx-W'l)hg}=\text{a }\textit{minimum},\ \therefore\ \frac{x^2}{Wx-W'l}=\text{a minimum},$$

$$\therefore\text{ (p. 148), }x^2-Wux+W'lu=0,\ \therefore\text{ (p. 148), }x=\frac{1}{2}Wu,\text{ and }4W'lu=W^2u^2,$$

$$\therefore u=\frac{4W'l}{W^2},\ \therefore x=\frac{2W'l}{W},\text{ corresponding to the minimum required.}$$

If the weights are equal, then $x=2l$; hence, that a weight W may draw an equal weight (W') up the plane BC, in the least time possible, the length of BC must be double the length of AC.

PROB. IV. A weight W is attached to the axle of a wheel, and another W' to the wheel itself: to determine the motion.

Put r, R, for the radii of the axle and wheel, then, since the weights act at these distances from the axis of motion, their efficacy in producing motion round that axis will be expressed by $W.r$ and $W'.R$, so that, putting M, M', for the masses moved, the moving forces will be $Mg.r$, and $M'g.R$, and since these oppose each other, the actual motion is due to only the moving force $(M'R-Mr)g$, a portion of which is expended in moving the mass M' downwards, and the remaining portion in moving the mass M upwards.

Let the accelerations of M, M', be f, f', respectively, then the moving forces, or moving pressures, on these masses—acting as they do at the distances r, R, from the axis of motion—are $Mf.r$, and $M'f'.R$. But the whole moving force is $(M'R-Mr)g$, hence

$$(M'R-Mr)g=M'f'.R+Mf.r\text{.....[1].}$$

Now the accelerations f', f, are to each other as the radii R, r, the velocities themselves being always in this constant ratio, since the wheel and axle both rotate in the same time,

$$\therefore f'=\frac{R}{r}f,\ \therefore (M'R-Mr)g=\left(M'\frac{R^2}{r}+Mr\right)f,\ \therefore f=\frac{M'Rr-Mr^2}{M'R^2+Mr^2}g.$$

This is the expression for the acceleration of the ascending weight W; and since

$$f'=\frac{R}{r}f,\ \therefore f'=\frac{M'R^2-MRr}{M'R^2+Mr^2}g,\ \text{is the acceleration of } W'.$$

Also, since the masses are as the weights, W, W', may be put for M, M', that is, we may write

$$f=\frac{(W'R-Wr)r}{W'R^2+Wr^2}g,\ f'=\frac{(W'R-Wr)R}{W'R^2+Wr^2}g.$$

For the velocity of the ascending weight, at the end of t seconds from the commencement of the motion, we have

$$v=ft=\frac{(W'R-Wr)r}{W'R^2+Wr^2}gt;\ \text{and for the space, } s=\frac{1}{2}ft^2=\frac{1}{2}\frac{(W'R-Wr)r}{W'R^2+Wr^2}gt^2.$$

In like manner, for the descending weight W', we have

$$v'=\frac{(W'R-Wr)R}{W'R^2+Wr^2}gt,\ s=\frac{1}{2}\frac{(W'R-Wr)R}{W'R^2+Wr^2}gt^2.$$

If $R=r$, the problem will be the same as Prob. I., and f and f' above will become each converted into

$$\frac{W'-W}{W'+W},\ W' \text{ being } \textit{here} \text{ the weight moving downwards.}$$

If we divide the linear velocity v (acquired in any time t''), of the circumference of the axle, by the radius of it, r, we shall get the angular velocity of the axle (453); or if we divide the linear velocity v', of the wheel, by its radius R, we shall get the angular velocity of the wheel: but as each performs a revolution in the same time, the angular velocities must be equal (453), and accordingly we find, whether we take the axle or the wheel, that

$$\text{angular velocity}=\frac{W'R-Wr}{W'R^2+Wr^2}gt,\ \text{and ang. accel.}=\frac{W'R-Wr}{W'R^2+Wr^2}g.$$

It is plain, from the property of the lever, that the numerator expresses the efficiency of the moving force to produce rotation:—the denominator expresses the resistance to angular motion, and is analogous to the mass moved.

479. **Principle of D'Alembert.**—When masses are perfectly free, whatever forces may be impressed upon them must have full effect; but when they are connected together, as in the foregoing problems, so that one mass is, as it were, a drag upon another, the effect of the force impressed upon each must fall short of what it would be if no such connection between the masses existed. Let the forces which would actuate the masses if they were perfectly free be called the *impressed* forces; and let those which would suffice to produce the actual effects, be called the *effective* forces: these latter applied to the several masses, simultaneously with the former—if applied in a contrary direction—would obviously prevent all motion in the system; that is, there would be equilibrium. This axiomatic truth constitutes the *Principle of D'Alembert:* it may be thus formally enunciated:—

There must always be equilibrium between the impressed and the effective forces, provided the latter be made to act opposite to their real directions.

The foregoing problems may all be solved by aid of this general principle, which we see reduces every dynamical inquiry, involving the consideration of moving forces, to a statical condition. Let us take the problem just solved:—

The impressed forces, in obedience to which the masses would move if

free, are Mg, acting with the leverage r, and $M'g$, acting with the leverage R, the motions being downwards. The effective forces, in obedience to which they actually move, are for M, upwards, $-Mf$, and for M', downwards, $M'f'$, the former acting with a leverage r, and the latter with a leverage R. Taking these with opposite signs, we have for the equation of equilibrium

$$(Mg+Mf)r=(M'g-M'f')R,$$

which is substantially the same as the equation [1] in the former solution. The student is recommended to solve Prob. II. by this principle.

Examples for Exercise.

(1) A weight of 4 lb. is attached to a string passing over a pulley, and draws up a weight of 1 lb. fixed to the other end of the string: how far will the heavier weight descend in 3 secs.?

(2) Two weights of 1 lb. each are suspended at the ends of a cord passing over a pulley: an ounce weight is added to one of them: how long will it be in descending 12 feet, and what velocity will it then have acquired?

(3) What weight must be added to one of the equal weights W in last example, in order that in 6 secs. it may acquire a velocity of 48 feet?

(4) The weight W (Prob. I., p. 445), instead of descending from rest, has a velocity of v_1 feet per second communicated to it: what velocity will it have acquired at the end of t seconds?

(5) A weight hanging vertically from a string passing over a small pulley fixed at the top of an inclined plane draws an equal weight up the plane. If the length of the plane be 12 feet and its height 10 feet, in what time will the ascending weight reach the top of the plane?

(6) Given the height h of an inclined plane, to find its length l, so that a given weight W, descending vertically, shall draw another given weight W' up the plane in the least time possible.

(7) The common height of two inclined planes is 20 feet; the length of one is 30 feet, and that of the other 40 feet: equal weights connected by a string passing over a pulley at the top (as in Prob. II., p. 447), are placed on these planes: how long will the descending weight be in falling from the top of the plane along which it moves to the bottom?

(8) A weight of 20 lb. raises a body of twice the volume and of three times the density by means of a wheel and axle: the radius of the wheel is 3 feet, and that of the axle 4 inches: through what space will the descending weight fall in 10 secs.?

(9) A given weight P raises Q by means of a wheel and axle, of which the radii are R, r: what must the weight of Q be, so that in t seconds the momentum of it may be the greatest possible? [See Note 2, page 445.]

(10) The radius r of the axle only is given: find the radius R of the wheel, so that a given weight P may raise another given weight Q through any assigned space in the least time possible.

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480. **Moment of Inertia.**—It has been seen in the solution to Prob. IV. that when masses are so connected together that the result of the forces acting upon them is *rotation*, the denominator of the fraction for the acceleration f is no longer the sum of the masses moved, but the sum of the products of each mass into the square of its distance from the fixed axis. In the problem referred to, the forces act, through the intervention of the strings, as if they were immediately applied at the extremities of the radii of the wheel and axle, so that these radii may be

regarded as the respective distances from the axis, of the masses acted upon, the masses being viewed as condensed into mere particles. If by the distance of a mass from the axis of rotation we always understand the distance, from the axis, of the point into which if that mass be condensed, the rotation will be the same, then the product of the mass by the square of its distance is that which is called the *moment of inertia* of the mass, with respect to that axis. A point having the property here alluded to may always be found for every mass of mathematical form, and in reference to any axis:—it is called the *centre of gyration* of the body; and the distance of this centre from the axis—which distance is usually represented by k—is called the *radius of gyration:* hence the moment of inertia of any mass, in reference to a given axis, is the product of that mass by the square of the radius of gyration, that is, in symbols, the moment of inertia is Mk^2.

The determination of k, for different forms of bodies, and for any given axes, requires the Integral Calculus, under which head the necessary formulæ will be investigated.

Since not only the masses themselves, but the rotating parts of the machine with which they are connected, are put in motion by the impressed forces, the moment of inertia of the latter should, in strictness, be taken into account. In the problem above, on the wheel and axle, this has been omitted; so that the denominator of the fraction f contains only a part of the whole moment of inertia. Supplying therefore the part neglected, namely, mk^2, m representing the mass, put in motion, of the machine, and k its radius of gyration, we have for the correct values of f, f',

$$f=\frac{(M'R-Mr)r}{M'R^2+Mr^2+mk^2}g,\ f'=\frac{(M'R-Mr)R}{M'R^2+Mr^2+mk^2}g.$$

If $r=R$, the problem becomes that of the weights and pulley (Prob. I.), and the correct expression for f, or f', these being then equal, is

$$f=\frac{(M'-M)R^2}{(M'+M)R^2+mk^2}g,$$

M' being here the descending mass. As mass $\propto$ weight, W, W', w, may be put for M, M', m, in these expressions, where it is to be observed that w is not the weight of the machine, but of that part of it only which is made to rotate by the action of the impressed forces: the denominator of the expression for f denotes the sum of all the resistances to rotation, that is, to angular motion.

If m' be the mass, which, placed at the extremity of R, would offer the same resistance to angular motion as the mass m of the rotating part of the machine, m' would be such that

$$m'R^2=mk^2,\ \therefore\ m'=m\left(\frac{k}{R}\right)^2:$$

the mass m, when multiplied by the proper fraction $\left(\frac{k}{R}\right)^2$, is usually so small, comparatively, that, in the mechanical powers, it may in general be disregarded.

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481. **Impact.**—When a moving body strikes another, whether at rest or in motion, the shock which takes place is called *impact* or collision. A body may so strike an immoveable obstacle as to adhere to it,

if sufficiently soft, or else to rebound and return along the line of its original motion; or in the case of a second moving body it may follow in its path, or meet it in that path, so that whatever motion after collision takes place, the common path of that motion may not be departed from. All these are cases of *direct* impact, or direct collision: these we shall consider first, previously, however, making a few remarks in reference to the elasticity of bodies of different material. It is found by experiment that when impact takes place between two bodies the shock occasions some amount of compression of the parts brought into contact, so that an instantaneous change of figure is produced. If, after the collision, the original form of the bodies is as instantaneously restored by a force of restitution, equal and opposite to the force of compression, the bodies are said to be perfectly elastic. On the contrary, if no force of restitution be brought into action, the bodies are said to be perfectly *inelastic*.

There is no reason to suppose that any such bodies actually exist in nature: whatever be the substances experimented on, it is always found either that the force of restitution is less than that of compression, or that the effect of this latter force is less instantaneous, so that if the original figure be restored, the restoration occupies more time than the compression, the bodies being driven asunder after the impact before all the force of restitution is expended, and therefore with less violence than that with which they struck. And although the material may be such that the impinging body may adhere permanently to that struck, and unite with it in one mass—as in the case of a lump of clay or putty—yet experiment finds that *some* force of restitution is always exercised, and that even clay and putty are not perfectly inelastic. The ratio which the force of restitution bears to the force of compression, as respects any substance, is called the *modulus of elasticity* of that substance. Newton's contrivance to ascertain this modulus for hard substances was this: he suspended two balls of the same material from two slender threads, so that when they hung vertically they might just rest in contact: upon being drawn asunder each would describe a circular arc. These arcs were accurately graduated, and each body being drawn from the vertical, through the same extent of arc, was let go at the same instant: the force of their rebound after impact was always found insufficient to carry them back through the same extent of arc. It was inferred from the experiments that the modulus of elasticity for ivory was $\frac{8}{9}$ of perfect elasticity, and for glass $\frac{15}{16}$. From experiments of a different kind, Prof. Hodgkinson determined the ratios to be for ivory ·89, and for glass, ·94, results that differ very little from those of Newton.

If e be put for the modulus of elasticity, V for the velocity of direct impact upon an immoveable plane, or the velocity of approach, and v for the velocity with which the body is driven back, that is, the velocity of recoil, we shall have

$$v = eV \text{.....[1].}$$

482. **Direct Impact.**—Let two bodies, of which the masses are M, M', moving with velocities V, V', strike with direct impact; and first suppose that there is no elasticity. The momentum of M is MV, that of M', $M'V'$, $\therefore$ the momentum of the compound mass, when they both unite, will be $MV + M'V'$, where the symbols V, V', involve their proper signs, these signs being like, if the motions of both bodies are in the same

direction, and unlike if the motions are in opposite directions:—motions from left to right we shall as usual regard as +.

If the common velocity after impact be V_1, then, since the two masses are united, we must have

$$(M+M')V_1=MV+M'V', \therefore V_1=\frac{MV+M'V'}{M+M'}\text{.....[2].}$$

If the bodies move in opposite directions, and the momenta before collision be equal, then V, V' taking opposite signs, $V_1=0$, that is, each body will be brought to a stand-still by the shock.

If one of the masses M' be at rest before impact,

$$\text{then since } V'=0,\ V_1=\frac{MV}{M+M'}.$$

The principle assumed above, that the momentum after impact is the algebraic sum of the momenta before impact, is implied in what is called *The Third Law of Motion*, which, as stated by Newton, is this:—"Action and reaction are equal and contrary," that is, A cannot act mechanically on B, without A itself being reacted upon equally, but in an opposite direction. In the case before us the momentum lost by one of the bodies is the momentum gained by the other. Let M be the mass which loses momentum, then

$$\left.\begin{array}{l}\text{vel. of } M \text{ before impact} = V\\ \text{,, \quad after \quad ,,} = V_1\end{array}\right\} \therefore \text{vel. lost by } M = V-V_1.$$

$$\text{But [2], } V-V_1=V-\frac{MV+M'V'}{M+M'}=\frac{(V-V')M'}{M+M'}\text{...[3].}$$

The difference $V-V'$ of the velocities of approach (always taking these with their proper signs) is the *relative* velocity of approach, so that we may put the foregoing conclusion thus:—

$$\frac{\text{Vel. lost by } M}{\text{Relative vel. before impact}}=\frac{M'}{M+M'}.$$ And, in like manner,

$$\frac{\text{Vel. gained by } M'}{\text{Relative vel. before impact}}=\frac{M}{M+M'}.$$

And from these equations it follows that

$$\text{The relative vel. before impact} = \text{vel. lost by } M + \text{vel. gained by } M'\text{...[4]},$$

for they give Vel. lost by M : Vel. gained by M' :: M' : M,

$$\therefore \frac{\text{Vel. lost by } M}{\text{Vel. lost by } M+\text{Vel. gained by } M'}=\frac{M'}{M+M'}.$$

Ex. A body of 5 lb. weight moves from left to right with a vel. of 12 ft., and comes into direct collision with another of 9 lb. weight moving also from left to right, with a vel. of 8 ft.: required the velocities lost and gained.

$$\text{The masses being as their weights, vel. lost by } M \text{ [3], } =\frac{9}{5+9}(12-8)=2\tfrac{4}{7}.$$

$$\text{And the relative vel. being 4, } \therefore \text{[4], } 4-2\tfrac{4}{7}=1\tfrac{3}{7}=\text{vel. gained by } M'$$

The common vel. after impact is [2],

$$V_1=\frac{60+72}{14}=9\tfrac{3}{7},$$

agreeing, as of course it ought to do, with the above results;

$$\text{for } 12-2\tfrac{4}{7}=9\tfrac{3}{7}, \text{ and } 8+1\tfrac{3}{7}=9\tfrac{3}{7}.$$

If the bodies moved in opposite directions, the second mass M' moving from right to left, then the sign of 8 would be *minus*, and the relative velocity would be $12+8=20$: we should then have

$$\text{vel. lost by } M=\frac{9}{5+9}(12+8)=12\tfrac{6}{7}, \text{ and vel. gained by } M'=20-12\tfrac{6}{7}=7\tfrac{1}{7}.$$

$$\text{Hence } M \text{ is driven back with a vel. } 12-12\tfrac{6}{7}=-\tfrac{6}{7},$$

$$\text{and } M' \text{ proceeds with a vel. } -8+7\tfrac{1}{7}=-\tfrac{6}{7},$$

that is, both proceed, in union, from right to left, with the common velocity $-\frac{6}{7}$.

Let now the bodies have elasticity, the modulus of elasticity being e. Let the velocity of M after impact be V_1, and that of M', V_1', then the momentum lost by M (that is, communicated to M') is $M(V-V_1)$, and that gained by M' is $M'(V_1'-V')$,

$$\therefore M(V-V_1)=M'(V_1'-V').....[5].$$

Also $V-V'$ is the velocity of approach, and $V_1'-V_1$ the velocity of recoil,

$$\therefore V_1'-V_1=e(V-V').....[6].$$

Solving [5] and [6] for V_1 and V_1', we have

$$V_1=\frac{MV+M'V'-eM'(V-V')}{M+M'},\ V_1'=\frac{MV+M'V'-eM(V'-V)}{M+M'},$$

each of which coincides with [2] when $e=0$, or the bodies are non-elastic. If $e=1$, or if the bodies be perfectly elastic, then the velocities after impact are

$$V_1=\frac{(M-M')V+2M'V'}{M+M'},\ V_1'=\frac{(M'-M)V'+2MV}{M+M'},$$

so that if in this case $M=M'$, we must have $V_1=V'$, and $V_1'=V$; that is to say, if two perfectly elastic and equal bodies strike with direct impact, each, after the stroke, has the same velocity that the other had before the stroke.

In the case of perfect elasticity,

$$[6] \text{ is } V_1'-V_1=V-V', \therefore V+V_1=V_1'+V'.$$

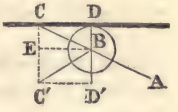
Multiplying [5] by this, we have

$$M(V^2-V_1^2)=M'(V_1'^2-V'^2), \therefore MV^2+M'V'^2=MV_1^2+M'V_1'^2.$$

Hence, when the bodies are perfectly elastic, the sum of the products of each into the square of its velocity is the same after as before impact. The product of a mass by the square of its velocity is called the *vis viva* (or *living force*), so that in the collision of perfectly elastic bodies, no loss of *vis viva* is occasioned by the impact.

483. **Oblique Impact.**—1. Let a ball whose elasticity is e strike a perfectly hard and immoveable plane: to find the motion of the ball after impact. Let AB be the path of the centre of the ball before impact, and BC' its path after impact: then if DBD' is perpendicular to the plane CD, the angle $ABD'=\theta$ is called the *angle of incidence*, and

$D'BC'=\theta'$, the *angle of reflexion*. Let V, the velocity along ABC, be represented by BC: the components of this—one perpendicular to the plane, and the other parallel to it—are BD, BE. The component BE, being parallel to DC, is unaffected by the action of the plane; but BD, by the force of restitution, is converted into BD', ∴ compounding the velocities BE, BD', the path and velocity V_1, after impact, will be represented by BC'.



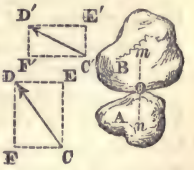
Now vel. $BD'=eBD=eV\cos\theta$
vel. $BE=V\sin\theta$
$\therefore V_1^2=(eV\cos\theta)^2+(V\sin\theta)^2=V^2(e^2\cos^2\theta+\sin^2\theta)$.

$$\text{Also } \tan\theta'=\frac{DC'}{BD'}=\frac{V\sin\theta}{eV\cos\theta}=\frac{1}{e}\tan\theta, \therefore e=\frac{\tan\theta}{\tan\theta'}.$$

Thus knowing the modulus e, and the direction and velocity of incidence, the direction and velocity of reflexion may be easily computed, or, observing the angle of reflexion consequent upon a given angle of incidence, the modulus e may be ascertained. If $e=1$, that is, if the elasticity be perfect, then $V_1=V$, and $\theta'=\theta$, showing that then the velocity is the same after impact as before, and that the angle of reflexion is equal to that of incidence. But if $e=0$, then $V_1=V\sin\theta$, and $\theta'=90°$; that is, if the body have no elasticity, and the plane be perfectly hard, the former, after oblique impact, will slide or roll along the plane with the diminished velocity $V\sin\theta$. [In the figure, BD' should be less than BD.]

It may be further noticed that in the case of imperfect elasticity the vel. after impact will always be less than that before impact, but the angle of reflexion will always be greater than that of incidence.

2. The annexed figure represents two bodies A, B, impinging upon each other at the point O: CD represents the magnitude and direction of the velocity of A, and $C'D'$ the magnitude and direction of the velocity of B. Let mn be the line perpendicular to the surfaces at O. Then, by the resolution of the velocities, A may be regarded as animated by two velocities, one CE parallel to nm, and the other CF perpendicular to nm. In like manner, B may be regarded as animated by the velocity $C'E'$, parallel to nm, and the velocity $C'F'$, perp. to nm. If A, B, at the instant of contact, were animated solely by the velocities CF, $C'F'$, they would merely slide past each other, and would experience no shock; the shock they do experience is therefore due solely to the velocities CE, $C'E'$; and even then, that any shock may take place, the former velocity must exceed the latter.



The intensity of the impact is the same, therefore, as if the two velocities CE, $C'E'$, alone existed; and the bodies will move as in the case of direct impact, only that these motions will be combined with the unchanged velocities CF, $C'F'$. Hence we shall merely have to compound the velocity of A, in the direction nm, after the direct impact spoken of, with the CF, perp. to nm, in order to obtain the magnitude and direction of A's velocity after the shock. And in a similar way are the magnitude and direction of B's velocity to be found.

Examples for Exercise.

(1) An inelastic body A weighing 10 lb., and moving with a velocity of 3 ft. per second, impinges directly upon another B weighing 7 lb., moving in the same direction with a

velocity of 2 ft.: required the common velocity after impact, the velocity lost by A, and the velocity gained by B.

(2) A weighing 2 lb. and moving with a velocity of 6 ft., comes into collision with B, weighing 3 lb., and moving in the opposite direction with a velocity of 5 feet: the coef. of elasticity is ·6: required the velocities after impact.

(3) If a ball falling from the height of 100 ft., on a perfectly hard plane, rebound to the height of 70 ft., what will be the coef. of elasticity?

(4) An ivory ball—the modulus of elasticity of which is $\frac{15}{16}$—falls from the height of 100 feet upon a perfectly hard plane: including the several rebounds, what will be the whole space described before the motion ceases?

(5) A ball is to be projected from a given point, so as that after reflexion from a given plane it may pass through another given point: it is required to find where the ball must strike the plane: the elasticity between the ball and plane being e.

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484. **Scholium.**—The foregoing short treatise on Dynamics comprehends as much of this extensive science as can be conveniently introduced without the aid of the Differential and Integral Calculus:—a department of pure mathematics which is in especial demand in the higher class of inquiries respecting motion. Exemplifications of this will be furnished in the subsequent treatise on that subject.

END OF THE DYNAMICS.

VIII. HYDROSTATICS.

485. Hydrostatics treats of the equilibrium of fluids, of which there are two distinct kinds, namely, compressible fluids and incompressible fluids:—the latter being generally called *liquids*, of which the most important is water. Of the former kind the most important is the air we breathe.

Air, the gases, and vapours, are called compressible fluids, because by the application of pressure they may be forced to contract in volume, and thus to occupy less space; and since on the removal of the pressure they resume their former bulk, they are also called *elastic fluids*.

Water, on the contrary, and liquids in general, are called incompressible fluids, because they will sustain a large amount of pressure without any sensible diminution of volume. By the application of enormous force, water has indeed been made to contract its bulk in a slight degree; but the pressure must be so great, and even then the contraction is so trifling, that no sensible error can arise from regarding it as absolutely incompressible: it is therefore said to be an *inelastic* fluid.

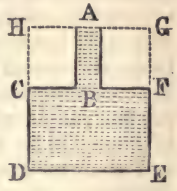
486. **Transmission of Pressure.**—The fundamental property of all fluids is this, namely, that *pressure applied to any part of the fluid is transmitted equally in all directions throughout its entire volume.* This remarkable truth is proved experimentally as follows:—Let a covered vessel, of any shape, be filled with any fluid, as water; and let any two perforations, fitted to receive pistons, whose faces A, B, have equal areas, be made anywhere in the vessel. If these pistons be introduced, a certain amount



of pressure must be applied to each to keep the fluid in its place; but if any additional pressure be applied to one, it will always be found that an equal additional pressure must be applied to the other to preserve the equilibrium:—any greater pressure on one than on the other would drive the latter out.

Suppose, for instance, one side of a closed vessel be flat, and that its superficial area be 10 square inches: then if a single piston, with a plane surface of 1 square inch, be introduced into a perforation, which it fits, at any other part of the vessel, and the vessel be filled with water, then, if besides the pressure on the piston necessary to keep the fluid in its place, an additional pressure of 10 lb. be applied to the piston, the flat side of the vessel will sustain a pressure of 100 lb., because each square inch of it sustains a pressure of 10 lb. Instead of a piston, we may conceive a tube of one-inch section inserted in the top of the vessel, and 10 lb. of water poured in: the pressure will be the same.

Let *AB* be such a tube inserted in the top of the vessel *CE*, the area of its base *B* being 1 sq. inch, and that of the side *FE*, or of the base *DE*, 10 sq. inches: then the 10 lb. of water in the tube *AB* will cause an additional pressure of 100 lb. on *EF*, and also upon *DE*, if the area of the bottom be equal to that of the side *FE*. If the tube be enlarged to 2 sq. inches section, and 20 lb. of water be poured in, these pressures will remain unaltered, because there will still be only 10 lb. of additional pressure on each sq. inch. And, *BA* being the height of the tube, if it be so enlarged as to fill up the entire space *CG*,—in other words, if the vessel *DF* be enlarged to *DG*, and then filled with the fluid, the bottom *DE*, and the portions *EF*, *DC*, of the sides, will each sustain no more pressure than when, by means of the 1-inch tube, the shape of the vessel was *ABCDEFB*.



It thus appears that the pressure on the bottom of a vessel does not at all depend on the quantity of fluid introduced into it, but only upon the height of the upper surface of the fluid: if the bottom of a vessel support the pressure of a ton of water, then, without interfering with the bottom, we may so alter the shape of the sides that a pound of water shall exercise the very same pressure. In the figure above we see that the pressure on *DE*, when the vessel of shape *DG* is filled, is no greater than the pressure on the same base when the shape of the vessel is altered to *ABEC*, and *that* filled with the liquid.

487. **The Hydrostatic Paradox.**—This is the name given to the remarkable property just mentioned, namely, that any quantity of liquid, however small, may be made to balance or overcome a pressure, however great. In the diagram, *A* is a vessel, the top and bottom of which are stout boards, connected together by loose leathern sides, like the boards of a pair of bellows:—the contrivance is, in fact, called the *Hydrostatic Bellows*. Into this vessel water is poured through the bent tube communicating with the interior: the upward pressure of the water thus introduced raises the board above *A*, and upon stopping the supply, the water will stand at the same level *bE* both in the tube and in the receptacle *A*.



Suppose now that a heavy weight *W* is placed upon the board, a weight

equal to that of the mass of water that would fill up the entire space CE above the board E, supposing this space to be closed in; and at the same time let the tube be filled to the same height c: the small additional quantity of water, thus poured into the tube, will balance the additional weight W; and if more be poured in, the weight will gradually rise. And all this will take place, however narrow the tube may be; for the downward pressure of the slender column of water cb is transmitted as an upward pressure upon every portion of the under surface of E, which is equal in area to the area of a section of the tube; so that if the area of the under surface of the board E be m times the area of a section of the tube, then the weight W, balanced by the column of water cb, will be m times the weight of that column.

Instead of pouring additional water (bc) into the tube, we may apply a pressure to the surface b, equivalent to the weight of that water, and similar effects will follow. It is upon this principle that Bramah's Hydrostatic Press is constructed: the *bellows* A is replaced by a strong hollow cylinder CF of metal, to which a piston E is fitted; the tube bc, which is very much smaller in section than the cylinder, and which communicates with it as in the figure, has also a piston fitted to it. A quantity of water being introduced under the two pistons, the rod of the small piston in bc is connected with a lever, to the arm of which a moderate force being applied, a considerable pressure is communicated downwards on the surface of the water b: this pressure acting upwards upon the piston E raises it and whatever heavy weight may be connected with it, in the manner above described. Suppose for ex. the diameter of the smaller cylinder bc to be $\frac{1}{2}$ inch, and that of the larger cylinder CF be 1 foot, and that a pressure of 10 lb. only be applied to the smaller piston: then since the areas of circles are as the squares of their diameters, we have

$$\left(\frac{1}{2}\right)^2 : 12^2 : : 10 \text{ lb.} : 5760 \text{ lb.} = \text{the upward pressure on } W.$$

In this machine, as in the mechanical powers, the power applied multiplied by the space it passes through is equal to the weight raised multiplied by the space *it* passes through. For P being the pressure applied to the area b, and B being the area of a section of the cylinder, we have for the weight raised, $W=\frac{B}{b}P$. Now the column of water of height H which is forced out of the tube and into the cylinder, and which therefore increases the cylindrical column by a height h, has for its volume

$$bH=Bh, \therefore \frac{B}{b}=\frac{H}{h}, \therefore W=\frac{H}{h}P, \therefore Wh=PH.$$

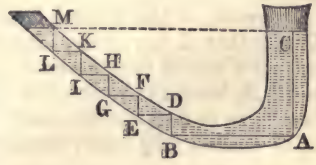
488. *The upper surface of a fluid at rest is always horizontal.*—Observation sufficiently shows this to be the case: it is a necessary consequence of the equal transmission of pressure in all directions. If we could conceive any portion of the still surface of a fluid to be raised up higher than the surrounding parts, the pressure of this protuberant part could not be transmitted laterally or horizontally, for as it is unsupported horizontally, such lateral pressure would prevail, and the heap of fluid would spread itself horizontally: the fact that all fluids press laterally necessitates the

consequence that their surfaces when at rest must present a horizontal level.

The pressures here spoken of are of course entirely due to gravity, which, although acting upon all masses in a strictly vertical direction causing weight, yet in the case of fluids this weight or pressure is transmitted equally on all sides. The top particles, in the heap of fluid above supposed, press with their weight the particles beneath them in a downward direction, but these latter are equally pressed in a horizontal direction, and meeting with no resistance, they necessarily yield to this lateral pressure, and continue to spread till the horizontal pressures are balanced, or, which is the same thing, till the vertical pressures on the particles constituting any horizontal layer of particles are all equal, that is, till the vertical distances of these several particles from the upper surface are all equal, which cannot happen till this upper surface becomes also horizontal.

489. Not only is the upper surface horizontal when that surface is continuous, but the two separate surfaces of a fluid, in a bent receptacle containing it, are both in the same horizontal plane. In the annexed figure let A, B, be the points at the extremities of the horizontal line AB, lying wholly in the channel of communication between the two ascending portions of the bent tube, these portions being of any relative bulk whatever.

Then since every particle in the line AB is at rest, the horizontal pressures at A, B, must be equal: the pressure on A is the weight of the vertical column of particles CA: the pressure on B is equal to the pressure on D (or the reaction of this pressure), augmented by the weight of the column of particles DB. But the pressure on D is the same as that on E, the line of particles DE being at rest. The pressure on E, however, is equal to that on F, augmented by the weight of the column of particles FE, $\therefore$ the pressure on B is equal to the pressure on F, augmented by the weight of the two columns DB, FE. Proceeding in this way, and observing that there is no pressure on the point M at the surface, we see that the pressure on B is equal to the weight of the five vertical columns of particles DB, FE, HG, KI, ML. And since the pressures on A, B, are equal, it follows that these five vertical lines are together equal to the single vertical line CA. Consequently every point in the two surfaces at C, M, is at the same vertical distance from the horizontal plane passing through the two points A, B: these two surfaces are therefore themselves in one horizontal plane (*De Launay:* "Cours Elémentaire de Mécanique.")



In the figure, the arm AC of the tube has been considered to be upright, so that C is at the surface of the liquid: but if this arm were inclined like the arm BM, it might be proved, as in the case of BM, that the pressure on A is still that due to a vertical column of particles whose height is AC= the height of M.

It will be noticed that what is called a horizontal plane is a flat surface perpendicular to the vertical line drawn from any point in it: on account of the remoteness of the centre of the earth, all the verticals from the surface of a fluid, enclosed in a vessel, being undistinguishable from

parallel lines, that surface is undistinguishable from a strictly plane surface. But in extensive sheets of fluid—as, for instance, the surface of the ocean—the condition of equilibrium would be that that surface be spherical, every point in it being equidistant from the centre of the earth. If any portion of the water were farther distant, there would be lateral pressure without lateral resistance, which would necessarily result in motion. In the actual case, however, the sphericity of the surface of the ocean is modified by a force opposed to that of gravity:—the opposing component, namely, of the centrifugal force, brought into action by the earth's rotation (p. 441).

490. **Levelling Instrument.**—From the property that the two surfaces of a fluid contained in a bent receptacle are always in the same horizontal plane, the levelling instrument derives its value. The common level consists of a bent tube, as in the figure, open at the two ends. This tube is nearly filled with a fluid—generally mercury—which supports two floats bearing sights, with a slender wire across each, the wires being equidistant from the surface of the fluid. When the instrument is held in the hand, the two surfaces bearing the floats are necessarily horizontal, however the tube itself may be inclined, and ∴ the wires are always in the same horizontal plane; and whatever objects which, seen through the sights, are on the same level as the wires, must be in the same horizontal plane as those wires.



Suppose, in the second figure at p. 179, the levelling instrument is at C, and that A is a distant object seen to be in the plane of C's horizon: the chief use of the instrument is to enable us to find the situation of the point B, beneath A, which is on the same spherical level as C, that is, to enable us to find the *depression* AB. Calling the distance CA, d, which is, of course, very small in comparison with the rad. R of the earth, and putting h for the depression, we know (p. 179) that $h=\frac{d^2}{2R}$. It is thus found that the depression for $\frac{1}{8}$ of a mile is $\frac{1}{8}$ of an inch, taking the radius of the earth to be about 4000 miles. And in general,

$$\text{since } \frac{d^2}{2R} \textit{ miles} = \frac{5280}{8000}\, d^2 \textit{ feet} = \cdot 66 d^2, \therefore h = \frac{2}{3} d^2 \text{ very nearly,}$$

where h is the number of *feet* in the depression, and d the number of *miles* in the distance CA. Hence, if d be 1 mile, the depression is 8 inches; if it be 2 miles, the depression is 32 inches; and so on.

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491. **Symbols used in Hydrostatics.**—The symbols employed in this subject, and the meanings attached to them, are the same as those already explained at p. 443, of the Dynamics, with a single additional one, standing for a term not hitherto introduced:—the term *specific gravity*. If a body weigh S times as much as a mass of distilled water of equal volume, the number S is said to denote the *specific gravity* of the substance of which that body consists. The term, therefore,

merely signifies the ratio of the weight of any volume of the substance, whether solid or fluid, to the weight of an equal volume of distilled water. This ratio (S) is the same abstract number as the ratio D, for the *density* of the substance:—the former ratio, however, is that of *weights*, the latter is that of *quantities of matter*:—of *masses*.

In treating of liquids, the symbol g will always stand for the *weight* impressed by gravity upon a mass of distilled water, of which the volume is 1 cubic foot, that is, it will denote 1000 ounces. And just as, at p. 446, the weight W of any substance of volume V was represented by $W=DVg$, so here it will be equally expressed by $W=SVg$, D and S being the same abstract number: the only concrete quantity concerned in the following investigations is *weight* or *pressure*.

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492. Proposition I.—If a body be immersed in a fluid, the pressure on any portion A of its surface, is equal to the weight of a column of the fluid whose horizontal base is equal to the area A, and whose altitude is the depth of the centre of gravity of A below the surface.

Let a_1 be the length of any linear column of fluid particles, P_1 being the point of the surface pressed. Regarding this point as a small area, the weight of the column will be SP_1a_1g, where S is the specific gravity of the fluid. Similarly, for the weight of another column of the fluid of length a_1, pressing upon another point P_2, we have SP_2a_2g; and so on,

$$\therefore \text{ Pressure on surface } A=S(P_1a_1+P_2a_2+\ldots+P_na_n)g=Sg\Sigma(Pa),$$

where $\Sigma(Pa)$ stands for the sum of P_1a_1, P_2a_2, &c.

Let now G denote the depth of the centre of gravity of the assemblage of points P_1, P_2, &c., that is, of the proposed area A, then by [1], p. 384, $\Sigma(Pa)=\Sigma(P)G=AG$,

$$\therefore \text{ Pressure on surface } A=SAGg=SVg=\text{weight of } V,$$

AG being the volume V of a column of fluid of horizontal base A, and of altitude G. If the fluid be water, then $S=1$, and Pressure$=Vg=W$.

If any side of a vessel incline inwards, the pressure on that side is of course an *upward* pressure, the amount of which is calculated as above. This pressure, reacting downwards, adds to the pressure on the bottom arising from the weight of the fluid: these two pressures together amount to the weight which the bottom would sustain, if the inclined side were replaced by a vertical side. On the other hand, if the sides of a vessel incline outwards, as in the figure at p. 464, the pressures on those sides are all downwards, and the bottom sustains a pressure *less* than that due to the weight of the fluid; inasmuch as, in *all* cases, the pressure on the bottom is the weight of a *vertical* column of fluid having that bottom for its base, and which reaches upwards to the surface (p. 457).

493. It follows from the preceding proposition that if a given surface immersed in a fluid rotate round its centre of gravity, the pressure on that surface will be the same in every position of it. Also, if the surface be a rectangle, the pressure upon it, when it forms the bottom of a vessel, will be double the pressure upon it when it forms one of the vertical sides of the same vessel, supposing the fluid to fill the vessel, because the centre of gravity in the former case is twice the depth of the centre of gravity in the latter: hence the pressure sustained by the four upright

sides of a cubical vessel, filled with any liquid, is equal to twice the pressure on the base, that is, to twice the *weight* of the liquid.

This proposition enables us readily to calculate the pressure against a rectangular dam, or against a pair of flood-gates, for we have only to multiply the area of the dam, or flood-gate, by half the depth of water to get the volume of water, the weight of which will be the pressure: thus, suppose the water is 8 feet deep, and the breadth of each flood-gate 5 feet; then the area of the surface pressed is 40 feet, and the depth of the centre of gravity of it is 4 ft.,

$\therefore$ Pressure$=40\times4=160$ cubic ft. of water$=160000$ oz. $=10000$ lbs.

Pressures on straight lines.—The centre of gravity of a straight line being at its middle point, if two straight lines a_1, a_2, be placed vertically in a fluid, the upper extremities of each being at the surface, we have

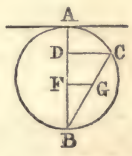
$$\text{Pressure on } a_1 : \text{Pressure on } a_2 :: a_1\frac{a_1}{2} : a_2\frac{a_2}{2}, \text{ or } a_1^2 : a_2^2,$$

so that the pressures are as the squares of the lengths of the lines. But if the lines are inclined to the surface of the fluid at the angles α_1, α_2, respectively, then the depths of the centre of gravity are

$$\frac{a_1 \sin \alpha_1}{2}, \text{ and } \frac{a_2 \sin \alpha_2}{2}; \text{ so that the pressures } \textit{then} \text{ are as}$$

$$a_1\frac{a_1 \sin \alpha_1}{2} : a_2\frac{a_2 \sin \alpha_2}{2}, \text{ or as } a_1^2 \sin \alpha_1 : a_2^2 \sin \alpha_2.$$

PROP. II. A circle being just covered by vertical immersion in a fluid, to draw from its lowest point that chord on which the pressure shall be the greatest.



Let BC be the chord, and G its centre of gravity, or middle point:

Put $BF=x$, then $BD=2x$, and $BC^2=BA \,.\, BD=4rx$,

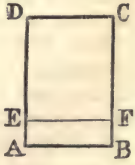
also $AF=2r-x=$ the depth of G below the surface:

hence the pressure on BC is $(2r-x)\sqrt{4rx}=$a *maximum*, or squaring and dividing by $4r$, $x^3-4rx^2+4r^2x=$ a maximum,

$$\therefore \text{ (p. 148), } 3x^2-8rx+4r^2=0, \therefore x=\frac{2}{3}r.$$

PROP. III. To divide the perpendicular face of a rectangular embankment, or of a flood-gate, into component rectangles, so that the pressure upon each may be the same.

Let the division be into n rectangles, of which AF is the lowest. Then since the centre of gravity of a rectangle is at its middle point, the depth of the centre of gravity of AC is $\frac{1}{2}DA$, and the depth of the centre of gravity of EC is $\frac{1}{2}DE$: the pressures on these rectangles are therefore as



$$\frac{1}{2}DA \,.\, DA : \frac{1}{2}DE \,.\, DE, \text{ or as } DA^2 : DE^2.$$

Now since the pressure on AC is to be divided into n equal pressures, and

that on EC into $n-1$ of these, the pressures on the two rectangles must be to each other as n to $n-1$,

$$\therefore DA^2 : DE^2 :: n : n-1, \therefore DE = DA\sqrt{\frac{n-1}{n}},$$

$$\therefore EA = DA - DE = DA\left(1-\sqrt{\frac{n-1}{n}}\right).$$

Suppose the rectangle AB is to be divided into two rectangles, such that the pressure on the upper may be equal to that on the lower:

$$\text{then } n=2, \text{ and } DE = \frac{DA}{\sqrt{2}} = \frac{DA\sqrt{2}}{2}.$$

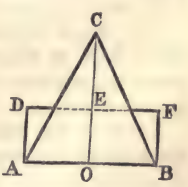
Hence, the proper degree of strength at the bottom of the embankment being secured, it would be waste of material to make it equally strong throughout: the strength upwards may diminish as the square of the depth diminishes.

PROP. IV. To determine the pressure upon the interior surface of a hollow sphere filled with a fluid of given specific gravity S.

The centre of the sphere is the centre of gravity of the surface: the distance of this from the surface of the fluid is the radius r, and the area of the spheric surface being $4\pi r^2$ (326), the pressure is equal to that of a vertical column of the fluid of base $4\pi r^2$, and altitude r, the weight of which column is $4\pi r^3 Sg$. This, therefore, expresses the amount of pressure sustained by the interior surface of the sphere. But since the volume of the sphere or of the contained fluid is $\frac{4}{3}\pi r^3$, the weight of it is only $\frac{4}{3}\pi r^3 Sg$: hence the internal pressure of the fluid is three times the weight of it.

PROP. V. A hollow cone rests with its base on a smooth horizontal plane, and water is poured in at an orifice at the top; to what height will the water rise before it lifts the cone off its support and escapes?

Here we have to find the height of water, the upward pressure of which against the interior surface of the cone is just equal to the weight or downward pressure of the cone itself. Let E be the point in the axis of the cone to which the water must rise: then the pressure on the base is made up of the weight of the water above it, and of the reaction of the upward pressure against the interior curve surface. But the pressure on the base is the same as it would be if a cylindrical vessel of water DB stood on that base (p. 457): consequently the pressure on the interior curve surface must be equal to the weight of that portion of the water in the circumscribing cylinder DB, which is *exterior* to the cone. This weight therefore must be equal to that of the conical shell.



Put $OC=a$, $OA=b$, $CE=x$: then the vol. of the cone CAB is $\frac{1}{3}\pi b^2 a$,

and since similar cones are as the cubes of their altitudes, the vol. of the cone whose alt. is CE is

$$\frac{1}{3}\pi b^2 a.\frac{x^3}{a^3}.$$

The difference of these volumes is the vol. of the water in the cone, that is,

$$\text{vol. of water in cone} = \frac{1}{3}\pi b^2 a\left(1-\frac{x^3}{a^3}\right),$$

$$\therefore \text{ the weight} = \frac{1}{3}\pi b^2 a\left(1-\frac{x^3}{a^3}\right)g.$$

Again, replacing the conic frustum by the cylinder of water AF, we have

$$\text{vol. of water in cylinder} = \pi b^2(a-x), \ \therefore \text{ the weight} = \pi b^2(a-x)g.$$

The difference of these weights is the weight of the cone w, that is,

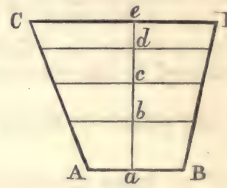
$$\pi b^2\left\{(a-x)-\frac{1}{3}a\left(1-\frac{x^3}{a^3}\right)\right\}g=w,$$

$$\therefore \frac{1}{3}\left(\frac{x}{a}\right)^3-\frac{x}{a}+\frac{2}{3}=\frac{w}{\pi b^2 ag}, \text{ or } \left(\frac{x}{a}\right)^3-3\left(\frac{x}{a}\right)+2=\frac{3w}{\pi b^2 ag}.$$

The solution of this cubic equation will give the value of x.

PROP. VI. To determine the pressure on the horizontal base of a vessel containing different fluids that do not mix.

Let the thicknesses ab, bc, cd, &c., of the several layers of fluid be p_1, p_2, p_3, &c., and let S_1, S_2, S_3, be their respective specific gravities. Then the pressures of these layers on the base AB will be (492) $AB \,.\, p_1 S_1 g$, $AB \,.\, p_2 S_2 g$, &c., and $\therefore$ the whole pressure on the base will be



$$\text{Pressure} = \text{Base} \times (p_1 S_1 + p_2 S_2 + p_3 S_3 + \ldots + p_n S_n)g.$$

Hence, if the area of the base be multiplied by the sum of the products arising from multiplying the thickness of each layer by its specific gravity, the numerical result will be the number of thousands of ounces of pressure on the base. If the thickness be the same for each layer, then the pressure will be found by multiplying the area of the base by the common thickness, and the product by the sum of the specific gravities, and by 1000 oz.

For ex. A cylinder is filled with mercury to half its height, and the other half is filled with water: taking 14 for the specific gravity of mercury, required the pressure on the base, and on the concave surface of the cylinder.

Let h be the height of the cylinder, and r the radius of its base, then the area of the base will be πr^2,

$$\text{and since } S_1=14, \text{ and } S_2=1, \text{ and } p_1=p_2=\frac{1}{2}h, \text{ we have}$$

$$\text{Pressure on base} = \frac{1}{2}\pi r^2 h(14+1)\,1000 \text{ oz.} = \frac{15}{2}\pi r^2 h \times 1000 \text{ oz.}$$

Now disregarding the water, and measuring downwards from the surface of the mercury, the depth of the middle section, and therefore the depth of the centre of gravity of the surface pressed by the mercury,

$$\text{is } \frac{1}{2} \text{ of } \frac{1}{2}h, \text{ that is, } \frac{1}{4}h, \ \therefore \text{ since, generally, Pressure} = SAGg \ (492), \text{ we have}$$

Pressure on lower half of cylinder by mercury alone $=14 \cdot 2\pi r \frac{1}{2} h \cdot \frac{1}{4} hg.$

Pressure on surface of mercury by the water $= 1 \cdot 2\pi r \frac{1}{2} h \cdot \frac{1}{2} hg.$

Pressure on upper half of cylinder by water alone $= 1 \cdot 2\pi r \frac{1}{2} h \cdot \frac{1}{4} hg.$

Sum of the Pressures $= \frac{17}{4} \pi r h^2 g = \frac{17}{4} \pi r h^2 \times 1000$ oz.

Examples for Exercise.

(1) The height of a closed cylindrical vessel is 1 foot, its diameter 6 inches; the diameter of a tube fixed to the top (see the fig. at p. 457) is 1 inch: it is required to find the length of this tube, so that when the cylinder and tube are filled with a liquid, the pressure on the bottom may be 9 times the weight of the liquid.

(2) If the area of a horizontal section of a tube, inserted, as in last example, into the top of a closed cylindrical vessel, be 1 square inch, and the area of a section of the cylinder 1 square yard: required the upward pressure on the top of the cylinder, when the height of the water in the tube is 18 feet from the top of the vessel.

(3) If an equilateral triangle be immersed in a fluid so that the extremity of one side may be on the surface, and the side itself perp. to the surface, prove that the pressures on the sides are as the numbers 1, 2, 3.

(4) If one of the vertical sides of a vessel filled with fluid be a rectangle, and if this rectangle be divided by a diagonal into two triangles, prove that the pressure on one triangle will be double that on the other.

(5) If a triangle be immersed at any inclination in a fluid, with its vertex downwards, and its base on the surface of the fluid, prove that the pressures on two horizontal lines in it, one at the same distance from the base that the other is from the vertex, will be equal.

(6) If any number of fluids that do not mix remain at rest in a bent tube, the sum of the products of the vertical heights, or thicknesses of the layers, into the specific gravities of the fluids in one branch of the tube, will be equal to the sum of the corresponding products in the other branch.

(7) A semicircle, whose radius is r, is immersed vertically in a fluid, with its diameter upon the upper surface: on which of the chords parallel to the diameter will the pressure be the greatest, supposing the density of the fluid to increase as the depth?

(8) Required the chord as in last example when the figure is a parabola, whose height is h.

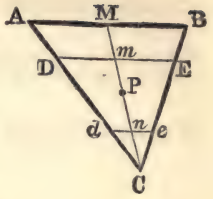
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494. **Centre of Pressure.**—The centre of pressure of a fluid, on a plane surface immersed in it, is the point to which if a single force, equal to the resultant of the pressures, were applied in the opposite direction, it would keep the plane at rest.

Problem I. To find the centre of pressure on a plane triangle immersed, at any inclination, in a fluid, and whose base is on the upper horizontal surface of the fluid.

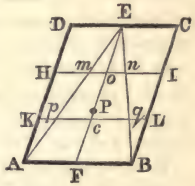
Let ABC be the immersed triangle, AB being at the surface of the fluid. Draw CM bisecting AB, and let DE, de, parallels to AB, be such that $Mm = Cn$: then (ex. 5, above) the pressure on these lines will be equal; and it is evident that the centre of pressure on each

is at its middle point: hence the extremities m, n, of the line mn, may be regarded as pressed by equal forces, the resultant of which is the sum of both acting at the middle point P, which point is also the middle of CM. Hence P is the point of application of the resultant of all the pressures upon pairs of parallels equidistant from P; but the whole pressure on the triangle may be regarded as made up of all these linear pressures: P, therefore, is the centre of pressure on the triangle.



Prob. II. To find the centre of pressure on a parallelogram, immersed at any inclination, one of whose sides is on the surface of the fluid.

Let AC be the immersed parallelogram, and let EF bisect the opposite sides DC, AB: then since the pressures upon the two equal parallelograms DF, CF, must be equal, the centre of pressure on DB must be in the line EF. Draw EA, EB, and any horizontal lines HI, KL, &c.: these lines being equal, the pressures upon them are as their depths (492), and their depths are as Eo to Ec, that is, as mn to pq. If, then, the pressure on AB be represented in magnitude by that line itself, the pressures on HI, KL, &c., will be correctly represented by the lines mn, pq, &c. If, therefore, the fluid pressures on the parallelogram were all removed, and then a pressure equal to that originally on AB were applied to that line, and also pressures applied to every parallel, pq, mn, &c., in the inscribed triangle—each pressure being as the parallel pressed—the parallelogram DB would still be at rest.



Now the resultant of all the pressures or weights, thus uniformly diffused over the triangle, must pass through the centre of gravity of the triangle: this point $\therefore$ is the centre of fluid pressure on the parallelogram, that is, the centre of pressure is on the bisecting line EF, and at a distance EP, from the surface, equal to $\frac{2}{3}EF$.

From this proposition it is easy to ascertain the centre of pressure on a parallelogram, when its upper horizontal side is below the surface of the fluid: thus, let HI be the upper side, DC being at the surface of the fluid. Then, as shown above, the pressure on HC may be replaced by a pressure uniformly diffused over the triangle Emn, and the pressure on AC by the extension of the pressure on Emn uniformly over the triangle EAB: hence the pressure on the parallelogram AI may be replaced by a pressure uniformly spread over the trapezoid $AmnB$: the centre of pressure is therefore the centre of gravity of that trapezoid.

Since the centre of pressure is the point about which all the pressures balance, it is the point where the opposing pressure should be more especially applied to prevent the surface pressed from giving way. We see from the above that the staves of vats and casks, which differ but little from so many rectangles, should be more especially strengthened at one-third their lengths from the bottom.

495. **Resultant of Fluid Pressures.**—When a solid body is immersed in a fluid, and is kept at rest there, the pressure of the sur-

rounding fluid on its surface must be the same, of whatever material the body is formed: if, therefore, the body were removed, and *solidified fluid*, exactly filling its place, were substituted for it, this would sustain the same pressure as the original body; and being a portion of the fluid itself, it would of course remain at rest, without requiring any support beyond that from the surrounding fluid. The only forces, therefore, that keep the solidified mass at rest are its own weight, acting vertically downwards at its centre of gravity, and the resultant of the fluid pressures; which resultant must ∴ act vertically upwards, through the centre of gravity of the mass. Hence the resultant of all the pressures upon the surface of a body immersed in a fluid, and kept at rest, is equal to the *weight* of the mass of fluid displaced, and is directed *upwards* through the centre of gravity of that mass. The same is true if the body be only partially immersed, that is, if it *float:* the forces acting upon it are still its own weight, acting vertically downwards, through its centre of gravity, and the resultant of the pressures of the fluid in contact with the body: this resultant must ∴ act vertically upwards through the centre of gravity of the mass of fluid displaced. And it is plain, in both cases, since the body does not turn round, that the vertical through the centre of gravity of the body must coincide with that through the centre of gravity of the displaced fluid: this latter centre is sometimes called the *centre of buoyancy*. Since the upward pressure is thus just sufficient to sustain the mass of water displaced, and since it is also just sufficient to sustain the floating body, it follows that the weight of the fluid displaced is equal to that of the body. If, therefore, a body be immersed in a fluid—either wholly or partially—it loses just so much of its weight as is equal to the weight of the fluid it displaces.

The matter may be considered somewhat differently. Imagine verticals to be drawn all round the body, touching it at its extreme boundaries: the immersed part will thus be imbedded as it were in a vertical column of fluid: all the fluid in this column which is above the *upper* surface of the body, exerts a downward pressure equal to its weight: all the fluid in the column (supposed *wholly* fluid), which is above the *under* surface of the body, exerts an upward pressure equal to its weight. The difference between these two opposing weights is, of course, the weight of the mass of fluid, displaced by the body, acting upwards.

Since no part of the weight of a body acts horizontally, the horizontal pressures of the fluid upon an immersed body at rest must themselves equilibrate; that is, their resultant is zero.

496. **Equilibrium of a Floating Body.**—In order that a body may remain at rest when immersed wholly or partially in a fluid, we have just seen that the following two conditions are necessary:—

1. The weight of the body must be equal to the weight of the volume of fluid displaced by the immersed portion of it.

2. The centre of gravity of the body, and the centre of gravity of the fluid displaced, must both be in the same vertical.

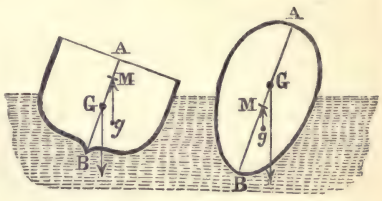
If these conditions have not place, the body will either sink or rise; or else turn round, and will come to a state of rest only when the conditions are attained.

If it be heavier than the volume of displaced fluid, it will sink lower, because the upward pressure is inadequate to support a weight greater than that of the fluid displaced. If it be lighter, it will rise, because the

upward pressure, being equal to the weight of the fluid displaced, will then exceed the weight of the body.

When a partially-immersed body at rest is disturbed by any impulse which, without altering the volume of fluid displaced, changes its situation, and causes the centre of gravity of the body, and that of the fluid displaced, to be in different verticals, then the upward and downward pressures, though still equal, will not be directly opposed: they will form a *couple* (416), and the body will in consequence return, when the disturbance has ceased, to its original position, or will else continue to rotate, in the direction of the disturbance, and will either overset, or find a new position of equilibrium:—the first of these alternatives will evidently happen when the resultant of the fluid pressures, in the disturbed position of the body, meets the line which originally joined the centres of gravity of the displaced fluid and the body (which line the disturbance has turned out of the vertical) *above* the latter point; and either the second or third result will take place if this new resultant of fluid pressures meet the aforesaid line *below* the centre of gravity of the body. This point of meeting, for a slight angular disturbance of the body, is called the *metacentre* for that amount of angular disturbance: when it is above the centre of gravity of the body, the equilibrium is stable: when it is below, the equilibrium is unstable.

In the annexed diagrams G is the centre of gravity of the body, g is the centre of gravity of the fluid *after* displacement; the centre of gravity of the equal volume of fluid *before* displacement being in the line AB, which line, before displacement, was the common vertical through both centres: the point M, where the vertical from g meets this line, is the metacentre, corresponding to the angular disturbance. In the first diagram, the body, before disturbance, was in stable equilibrium: in the second, the equilibrium was unstable: in the former case, the equal forces acting at G, M, in opposite directions, tend to restore the body to its upright position; in the latter case, these same forces tend to incline the body more and more *from* that position.



Should it happen that *the line of support* gM (the direction of the upward pressure) passes through G, then the body will remain in its new position as securely at rest as before the disturbance operated: this kind of equilibrium is called indifferent, or *neutral* equilibrium:—it evidently has place in a floating sphere, or cylinder.

In the construction and loading of ships, it is of the utmost importance that the centre of gravity of the whole body be so low, that the metacentre may always be above that point, even for a considerable disturbance. In a gale of wind the centre of gravity is sometimes elevated above its safe position by the shifting of the cargo:—when it is in this way raised above the metacentre, the vessel must capsize.

497. When a given body is sufficiently light to float upon a given fluid, the volumes of it above and below the plane of the fluid, called the *plane of floatation*, may be found thus.

Let V_1 be the volume of the body, and V the volume of the portion of

it beneath the plane of floatation: let also W be the weight of the body in ounces, and D the density of the fluid.

Then $DVg=W$, $\therefore\ V=\frac{W}{Dg}=\frac{W}{1000\,D}$ cubic feet ...[1].

If the uniform density of the floating body be D_1, then $D_1V_1g=W$,

$\therefore\ DV=D_1V_1$...[2], $\therefore\ V : V_1 :: D_1 : D$, $\therefore\ V : V_1-V :: D_1 : D-D_1$.

Ex. 1. A cone of wood of density D_1 floats on water with its vertex downwards: what portion of its height h is immersed?

Let x be depth of the vertex: then, since similar cones are as the cubes of their altitudes, the volumes of the whole cone and the cone immersed, may be represented by kh^3 and kx^3, and the density of the water being 1, we have [1],

$$kx^3=D_1kh^3,\ \therefore\ x=h\sqrt[3]{D_1},\ \therefore\ \frac{x}{h}=\sqrt[3]{D_1}.$$

2. What portion of the height will be immersed when the cone floats with its base downwards?

Let y be the height of the portion of the cone above the water: then the volume of this portion being ky^3, and the volume of the whole cone kh^3, we have for the volume of the immersed part,

$k(h^3-y^3)$; hence [1], $k(h^3-y^3)=D_1kh^3$, $\therefore\ y=h\sqrt[3]{(1-D_1)}$,

$\therefore\ h-y=h\{1-\sqrt[3]{(1-D_1)}\}$, the portion of the height immersed.

3. A hemispherical bowl, of given weight W, floats upon a fluid with $\frac{1}{3}$ of its vertical radius r below the surface: required the weight W' that must be put into it to make it float with $\frac{2}{3}$ of its radius below the surface.

Putting h for the height of the segment immersed, we have (p. 273)

$V=\frac{\pi}{6}(6r-2h)h^2$, so that when the heights are $\frac{1}{3}r$, and $\frac{2}{3}r$,

the respective volumes immersed are $\frac{8}{81}\pi r^3$, and $\frac{28}{81}\pi r^3$:

hence, if D be the density of the fluid, we have for the weights of these volumes, by [1],

$W=\frac{8\,D}{81}\pi r^3g$, and $W+W'=\frac{28\,D}{81}\pi r^3g$. Eliminating D from this,

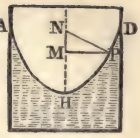
$W+W'=\frac{7}{2}W$, $\therefore\ W=\frac{5}{2}W=2\frac{1}{2}W$, the weight required.

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498. **Equilibrium of a Rotating Fluid.**—A vessel containing water, or any other liquid, revolves round its vertical axis, with a given angular velocity: to determine the form which the interior surface of the fluid will assume.

Let AHD be the form assumed by the surface of the fluid when all the

forces acting upon it balance, P being any point on that surface. Draw PM perp. to the axis of rotation, and PN perp. to the curve at P. The forces which act upon P are—



1. Gravity, in the vertical direction NM.

2. The centrifugal force, in the horizontal direction MP.

3. The reaction of the pressure in the direction PN.

These three forces, acting as pressures, keep P from all independent motion.

Let ω be the angular velocity of rotation per second: then the circular arc of radius $MP=y$, described by P in one second, will be ωy, that is to say, the velocity of P will be $v=\omega y$.

$$\text{Now the centrifugal force on } P \text{ is } \frac{v^2}{y} \text{ (p. 440), } \therefore \text{ centrif. force} = \omega^2 y \text{ ;}$$

and since P has no independent motion, we have by the triangle of forces (see NOTE below),

$$NM : MP :: g : \text{centrif. force, that is, } NM : y :: g : \omega^2 y, \therefore NM = \frac{g}{\omega^2}.$$

Hence the curve AHD is such that the subnormal NM of any point P is constant:—a property that belongs exclusively to the parabola (p. 345), so that the surface of equilibrium is that generated by the rotation of a parabola about its axis in a vertical position, the vertex of the curve being downwards: the surface thus generated is a *paraboloid*.

Suppose, for example, that the vessel makes 10 rotations in a second; then the angular velocity is

$$\omega = 2\pi \times 10 = 3{\cdot}1416 \times 20 = 62{\cdot}832, \therefore \omega^2 = 3947{\cdot}86,$$

$$\therefore NM = \frac{32{\cdot}2}{3947{\cdot}86} \text{ feet} = \frac{386{\cdot}4}{3947{\cdot}86} \text{ inches.}$$

NOTE.—The dynamical effects of pressures, upon the same particle or body, are obviously proportional to their statical effects.

---

499. **Specific Gravities of Bodies.**—The specific gravity of a body, as already defined, is the ratio of its weight to the weight of an equal volume of distilled water; or it is the number by which the weight of a volume of distilled water must be multiplied to give the weight of an equal volume of the proposed substance.

Putting S_1 for the specific gravity of the fluid in which a body floats, and S for that of the body itself, also V_1 for the part immersed, and V for the whole volume, the weight of which is W, we have

$$(492),\ W = SVg, \text{ and also } (495),\ W = S_1 V_1 g,$$

$$\therefore \frac{V}{V_1} = \frac{S_1}{S}, \text{ the same as at (497), for densities: hence}$$

$$\frac{\text{The whole body}}{\text{Part immersed}} = \frac{\text{Sp. gr. of fluid}}{\text{Sp. gr. of body}}.$$ And for *volumes* we may evidently put *weights*.

Suppose now the same body to be placed in another fluid of specific

gravity S', the part of the volume immersed being V'; then as before we shall have $\frac{V}{V'}=\frac{S'}{S}$. Dividing this by the former, $\frac{V_1}{V'}=\frac{S'}{S_1}$, so that the volumes immersed are inversely as the specific gravities of the fluids.

PROB. I. To find the specific gravity of a substance when a fluid of less specific gravity is employed.

Let W be the weight of any portion of the substance in air (or in a vacuum), and W' the weight when it is immersed; then the weight of the fluid displaced is

$$W-W' \text{ (p. 461)}=S_1Vg, \text{ also } W=SVg, \therefore \frac{S}{S_1}=\frac{W}{W-W'}, \text{ that is,}$$

weight lost : weight of body : : sp. gr. of fluid : sp. gr. of body.

PROB. II. To find the sp. gr. of a substance when a fluid of greater sp. gr. S_1 is employed.

In order that the given body of weight W may sink, let a weight w (called a *sinker*) be attached to it which weighs w' in the fluid. Let the compound body weigh W_1, W'_1, in air (or in a vacuum), and in the fluid: then, S representing the required sp. gr., we have

$$\begin{aligned}
&\text{Weight of fluid displaced by the compound} = W_1-W'_1,\\
&\quad\text{,,}\qquad\text{,,}\qquad\text{,,}\quad\text{sinker} = w-w',\\
&\quad\text{,,}\qquad\text{,,}\qquad\text{,,}\quad\text{body} = (W_1-W'_1)-(w-w'),\\
&\therefore \text{ as above, } \frac{S}{S_1}=\frac{W}{(W_1-W'_1)-(w-w')}, \therefore S=\frac{W}{(W_1-W'_1)-(w-w')}S_1,
\end{aligned}$$

where S_1 is known.

Since air is itself a fluid, a body in air loses a portion of its weight equal to the weight of the air it displaces: the weight in air is thus called the *apparent* weight of the body. In very delicate experiments the weight of the displaced air must therefore be added to the apparent weight to give the true weight, or the weight the body would have in a vacuum. But the true weight may be found from finding what the body weighs in two fluids of known specific gravities. Let W be the true weight, and W_1, W_2, the weights in two fluids whose specific gravities are S_1, S_2: then the respective weights lost are

$$W-W_1,\ W-W_2, \therefore \frac{W}{W-W_1}=\frac{S}{S_1}, \frac{W}{W-W_2}=\frac{S}{S_2},$$

$$\therefore \frac{W-W_1}{W-W_2}=\frac{S_1}{S_2}, \therefore W=\frac{W_2S_1-W_1S_2}{S_1-S_2}.$$

Also, from the preceding equation, we have the following general principle for bodies wholly immersed in fluids:—

The wt. lost in one fluid : wt. lost in another : : sp. gr. of former : sp. gr. of latter,

a truth nearly self-evident, and which suffices for the particular problems considered above: thus, the conclusion from the first problem is that

$$\frac{W}{W-W'}=\frac{S}{S_1}, \text{ and that from the second, that } \frac{W}{(W_1-W'_1)-(w-w')}=\frac{S}{S_1}.$$

Now here W is the weight lost in a fluid, of which the sp. gr. is S, that is, the whole weight would be lost in such a fluid, and $W-W'$, and

$(W_1 - W'_1) - (w - w')$ are the weights lost by W in a fluid whose sp. gr. is S_1.

PROB. III. To determine the volume of any small body, however irregular, when its weight and specific gravity are known.

Let W be the number of ounces it weighs, and let S be its sp. gr.; then since

$$W = SVg, \ \therefore \ V = \frac{W}{1000\,S} \text{ cubic feet.}$$

In a similar way may the capacity of an irregular vessel be ascertained. Let the weight of water that will fill the vessel be w ounces, then the capacity of the vessel will be $V = \frac{w}{1000}$ cubic ft. In general it is easier to ascertain the weight of a minute body than to measure its dimensions: the following therefore is the method usually adopted for finding the diameter of a small sphere of ascertained weight, in grains troy, and of a substance of known specific gravity. Let d be the diameter in parts of an inch, w the number of grains troy in the weight of the sphere, and S the sp. gr. of the substance of which it is formed: then the troy weight of a cubic inch of pure water being 253·1808 grains, and the volume of a sphere of 1 inch diameter being ·523598 cubic inch, we have

$$1 : {\cdot}523598 : : 253{\cdot}1808 \text{ grains} : 132{\cdot}5648 \text{ grains} = \text{wt. of sph. of water of 1 in. diam.}$$

And since the volumes and $\therefore$ the weights of spheres, of the same material, are as the cubes of their diameters, we have

$$1^3 : d^3 : : 132{\cdot}5648 : 132{\cdot}5648 d^3,$$

the number of grains in a sphere of water of diameter d: hence the weight w of the proposed sphere is

$$w = 132{\cdot}5648 d^3 S \text{ grains}, \ \therefore \ d = \sqrt[3]{\frac{w}{132{\cdot}5648\,S}} = {\cdot}19612 \sqrt[3]{\frac{w}{S}} \text{ inches.}$$

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500. **The Hydrostatic Balance.**—Specific gravities of substances are ascertained by means of the hydrostatic balance, to the under part of one of the scale-pans of which a hook is fixed. The body to be experimented upon is suspended by a fine thread to this hook, and then immersed in a fluid of known specific gravity, as distilled water at a temperature of 60° of Fahrenheit's thermometer. The weight of the body in the fluid being thus found, and its weight in air, or in a vacuum, being already known, its specific gravity is determined as in Prob. I.; or if the body be so light as to require a sinker, it is determined as in Prob. II.



If the substance consist of loose materials, as a powder, its specific gravity is found by imbedding a known weight of it in wax, or some such substance, employed as a sinker, and then computing S by Prob. II. For determining the specific gravities of liquids, an instrument called the Hydrometer is employed.

501. **The Common Hydrometer.**—In its simplest form the

hydrometer consists of two hollow spheres, usually of glass, a larger and a smaller one below it, as in the annexed diagram. Into the lower sphere mercury or small shot is introduced, in order that the spheres may sink in the liquid. The axis of the stem D is a straight line through the centres of the spheres, so that when the instrument rests in the fluid the stem may be vertical.



Suppose that when placed in distilled water, the stem sinks to some point A, and that when placed in another liquid L, it sinks to the point B: then by the property (499), namely,

$$\frac{S'}{S_1}=\frac{V_1}{V'},\text{ we have }\frac{\text{Specific gravity of } L}{1}=\frac{\text{Volume of } AC}{\text{Volume of } BC}=\frac{\text{Vol. of } (DC-DA)}{\text{Vol. of } (DC-DB)}.$$

Let the whole volume of the instrument be considered to consist of 4000 equal parts, and the stem be divided so that each portion is one of those parts. Suppose the stem to contain 50 of these parts, numbered from D downwards, and that it sinks to 30 in one liquid L, and to 10 in another liquid L', then

$$\frac{\text{Specific gravity of } L}{\text{Specific gravity of } L'}=\frac{4000-10}{4000-30}=\frac{3990}{3970}.$$

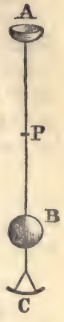
And in this way the specific gravities of different liquids may be compared.

When used for the purpose of testing the strength of alcoholic liquors, as by officers of the Excise, the point on the stem to which the hydrometer sinks in distilled water is marked 1·000, and the point to which it sinks in a lighter fluid of known specific gravity, say ·900, is marked ·900. The interval between these points being then divided into 100 equal parts, the instrument becomes adapted to show by inspection the specific gravities of all liquids whose sp. gravities are between these limits. And the scale may in like manner be extended in either direction, and thus be available for other purposes. Proof spirits consists of half alcohol and half water: its sp. gravity is ·920, which point is marked "*Proof*" on the Exciseman's scale: if the liquor be below proof the instrument will not sink so far, and if it be above proof, it will sink farther.

502. *The Specific Gravity Bottle.*—But there is another way of determining the specific gravity of a liquid. A bottle, made to contain 1000 grains or 500 grains of distilled water, is accompanied with a weight which is the exact counterpoise of the empty bottle and its stopper: this is filled with the liquid, and the weight which, in addition to the counterpoise, balances the full bottle, is of course the weight of the liquid. This weight, divided by 1000 grs., or by 500 grs., according to the size of the bottle, is the sp. gr. of the liquid: it is therefore expressed by the *number* of grains in the weight of the liquid, converted into a decimal, or into a whole number and a decimal, by advancing the decimal point in that number three places forward for a 1000 grain-bottle, and by doing the same with double that number for a 500 grain-bottle. There must of course be liquid sufficient to fill the bottle.

503. **Nicholson's Hydrometer.**—By means of this instrument

the specific gravities may be compared, not only of two fluids but of two solids, or of a solid and a fluid. The stem of this instrument is a slender wire of hardened steel, having a dish at each end, and a hollow globe or cylinder between the two, as in the figure. The adjustment of the instrument is such that when a given weight is placed in the upper dish A, it will sink, in distilled water, to the point P, in the middle of the stem AB: it is used as follows:—



1. *To compare the specific gravities of two liquids.*—Let W be the known weight of the instrument, w the weight which must be placed in the dish A to sink the hydrometer to P in the liquid L, w_1 the weight required to sink it to P in the liquid L': then

Weight of the vol. of L displaced $=W+w$: Weight of vol. of L' displaced $=W+w_1$,

$$\therefore \frac{\text{Specific gravity of } L}{\text{Specific gravity of } L'}=\frac{W+w}{W+w_1}.$$

2. *To compare the specific gravities of a solid and a liquid.*—Let w be the weight which must be placed in the dish A to sink the instrument to P. Replace w by the solid, and let w_1 be the additional weight necessary to sink the instrument again to P. Remove the solid to the lower dish C, and let w_2 be the additional weight to be placed in the upper dish to sink the instrument to P: then we shall have

Weight of the solid in air $=w-w_1$
" " in the fluid $=w-w_2$ $\Big\}$ $\therefore$ wt. of fluid displaced $=w_2-w_1$,

$$\therefore \frac{\text{Specific gravity of solid}}{\text{Specific gravity of fluid}}=\frac{w-w_1}{w_2-w_1}.$$

Example 1. A body whose sp. gr. is $S=\cdot6$ floats with $\frac{2}{3}$ of its volume below the surface: required the sp. gr. S_1 of the fluid.

By the principle at (499), the whole body $(=1)\times$ sp. gr. of the body $=$ the part immersed $\left(=\frac{2}{3}\right)\times$ sp. gr. of the fluid, that is, $\cdot6=\frac{2}{3}S_1$, $\therefore S_1=\cdot9$.

2. A piece of cork weighing 6 oz. is attached to 24 oz. of brass, of which the sp. gr. is 8·7: the weight of the whole in water is 2·24 oz.: required the sp. gr. of the cork.

Here the weight lost by the compound is 30 oz. $-2\cdot24$ oz. $=W_1-W'_1$ (p. 471).

The weight lost by the brass sinker is $\frac{24}{8\cdot7}$ oz. $\quad=w-w'$.

The difference of these is the weight lost by the cork alone; and since the sp. gr. of the fluid is $S_1=1$, we have for the sp. gr. of the cork (p. 471)

$$S=\frac{6}{27\cdot76-\frac{24}{8\cdot7}}=\frac{6}{27\cdot76-2\cdot76}=\frac{6}{25}=\cdot24.$$

3. A mass of gold being immersed in a cylinder of water caused the surface to rise a inches: a mass of silver of the same weight W, caused it to rise b inches; and a mass, still of the same weight, but composed of

gold and silver, caused it to rise c inches: required the proportion of gold and silver in the composition. Let x be the volume of the gold, and y that of the silver; then $x+y$ is the volume of the compound, and since the elevations of the surface of the water are proportional to the volumes immersed, we have

$$c : a :: x+y : \frac{a}{c}(x+y) = \text{the volume of the gold mass of weight } W;$$

$$c : b :: x+y : \frac{b}{c}(x+y) = \quad ,, \quad ,, \quad \text{silver mass} \quad ,,$$

$$\therefore \frac{a}{c}(x+y) : x :: W : \frac{c}{a}\cdot\frac{x}{x+y}W = \text{weight of gold in the compound.}$$

$$\frac{b}{c}(x+y) : y :: W : \frac{c}{b}\cdot\frac{y}{x+y}W = \quad ,, \quad \text{silver} \quad ,,$$

$$\therefore \frac{cx}{a(x+y)}W + \frac{cy}{b(x+y)}W = W, \ \therefore bcx+acy = ab(x+y),$$

$$\therefore (bc-ba)x = (ab-ac)y, \ \therefore x : y :: a(b-c) : b(c-a).$$

This gives the ratio of the *volumes* of gold and silver. To find the ratio of their *weights* let S be the sp. gr. of the gold, and S' that of the silver; then the weights of x and y will be to one another as Sx to $S'y$,

$$\therefore \text{wt. of the gold : wt. of the silver} :: a(b-c)S : b(c-a)S' :: \frac{a(b-c)}{b(c-a)}\frac{S}{S'} : 1.$$

It was probably in some such way as this that Archimedes discovered the proportion of gold and silver in the crown fabricated for Hiero, king of Syracuse, who had ordered the workmen to make it of a given weight of *pure gold*.

Examples for Exercise.

(1) A body floats in a fluid of 13·5 sp. gr., and $\frac{2}{3}$ of its volume is above the surface: required the sp. gr. of the body.

(2) If a piece of gold weigh 15 lb. in air, and 44 lb. in mercury, and the sp. gr. of the gold be 19, required the sp. gr. of the mercury.

(3) A piece of cork, whose sp. gr. is ·25, floats in water: how much of it will be above the surface?

(4) A block of wood weighs 3 lb. in air, a piece of lead weighing 6 lb. in water, is attached to it as a sinker: the compound weighs 5 lb. in water: required the sp. gr. of the wood.

(5) A body whose sp. gr. is $\frac{2}{5}$ floats in water: required the ratio of the parts of it in and above the water.

(6) A cubic foot of wood, of sp. gr. $\frac{12}{13}$, is put into water: with how much more wood must it be loaded, in order that its upper surface may sink to the level of the water?

(7) The specific gravities of two bodies weighing W, and W', are S and S': show that the specific gravity of the united mass is $\frac{W+W'}{WS'+W'S}SS'$.

(8) A piece of cast iron, of sp. gr. 7·425, weighed 40 oz. in air, and 35·61 oz. in a fluid: required the sp. gr. of the fluid.

(9) The sp. gr. of a composition of 1 cwt. of tin of sp. gr. 7·32, and of copper of sp. gr. 9, has for sp. gr. 8·784: what are the weights of tin and copper in the mixture?

(10) 120 grains of brass, of sp. gr. 8, are counterpoised in air by a copper weight of sp. gr. 9; both, still hanging to the balance, are then immersed in water · what weight must be applied to the ascending arm of the balance to restore the equilibrium, the additional weight not being immersed?

(11) 37 lb. of tin loses 5 lb. in water, and 23 lb. of lead loses 2 lb. in water: a mixture of tin and lead, weighing 120 lb., is found to lose 14 lb. in water: how much of each metal is there in the mixture?

(12) A man, whose weight is 168 lb., can just float in water when a certain quantity of cork is attached to him. Assuming the sp. gr. of the man to be 1·12, and that of the cork ·24, what quantity of cork is necessary?

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504. **Elastic Fluids.**—The properties of fluids established in the foregoing articles belong to fluids in general: *elastic* fluids, however, have additional properties peculiar to themselves, and these, therefore, demand a distinct examination. Of such fluids, the atmosphere which surrounds us is of the most importance; and it is in reference to this, that the science, which treats of the mechanical properties of elastic fluids, is called *Pneumatics*:—from a Greek term, signifying the air we breathe. In the following articles atmospheric air will be the only elastic fluid considered: but the investigations would have been the same, and the conclusions similar, if any other elastic fluid had been the subject of examination.

505. **Weight of a Cubic Foot of Common Air.**—The air which surrounds the earth, like every fluid, has weight, as may be proved by many conclusive experiments. We may, in fact, take any definite volume of it, as we would take so much water, and actually weigh it in a balance, as follows:—Let a vessel, the neck of which is provided with a stop-cock, have the capacity of a cubic foot, that is, let it be capable of holding $62\frac{1}{2}$ lb., or 1000 ounces of distilled water. This vessel may be exhausted of its air by means of an air-pump, a machine which will be presently described. Thus emptied, and the stop-cock closed so as to prevent the re-admission of air, let the vessel be accurately weighed in a very sensible balance. The exact weight of the empty vessel being thus ascertained, let the stop-cock be opened and the air admitted, and when thus filled with a cubic foot of air, let the vessel be once more weighed: it will be found necessary to put about 536 grains of additional weight into the other scale-pan to restore the balance, thus showing that the weight of a cubic foot of air at the surface of the earth is about 536 grains. It is thus found that on the average the weight of any portion of air at the earth's surface is about $\frac{1}{816}$ of the weight of an equal volume of distilled water.

Taking the cubic foot for the unit of volume, that bulk of air, when the thermometer stands at 60°, and the barometer at 30 inches, weighs 536 grains: hence the pressure communicated by gravity to a cubic foot of air in this state is 536 grains, which is therefore the value of g in reference to elastic fluids, just as 1000 oz. was the value of g in reference to liquids.

506. **Weight of the Atmosphere.**—If a tube about three feet

in length, and closed at one end, be filled with mercury, and the open end be then closed with the finger, and the tube inverted in a basin of mercury, the finger being removed while the mouth of the tube is below the surface, it will be found that the column of the fluid, originally 3 feet in height, will fall only six or seven inches, and that 29 or 30 inches above the surface will remain suspended in the tube. Hence the surface of the mercury in the basin which surrounds the inserted tube must, on every portion of it equal in area to a horizontal section of the tube, sustain a pressure from the atmosphere which it requires a column of 29 or 30 inches of mercury, on the same area, to balance (486).

Mercury is about $13\frac{1}{2}$ times as heavy as distilled water, so that the pressure of the atmosphere on any given area of the earth's surface is the same as the pressure of a column of water on that area, and of height about

$$13\tfrac{1}{2} \times 30 = 405 \text{ inches} = 33\tfrac{3}{4} \text{ feet.}$$

The truth of this conclusion was actually put to an experimental test by Pascal in 1647. He procured glass tubes, closed at one end, of the great length of 40 feet, and found that when filled with common water in a deep river, and then raised vertically with the open end downwards, the fluid ceased to fall when the water in the tube stood at about $32\frac{1}{4}$ feet above the surface of the river. Had the water been distilled water, which is somewhat lighter, a still higher column would have been supported. Since 30 cubic inches of mercury weigh about 15 lb., it follows that every square inch of the surface of the earth at the level of the sea sustains, on the average, an atmospheric pressure of 15 lb. If the atmosphere were to be removed, and to be replaced by its weight of mercury, the depth of the mercurial coating would be about 30 inches. If R denote the radius of the earth in feet, and r the proper depth of mercury, S being its specific gravity, we shall have for the weight W of the enveloping coat of mercury

$$W = \left(\frac{4\pi(R+r)^3}{3} - \frac{4\pi R^3}{3}\right) Sg = 4\pi r\left(R^2 + Rr + \frac{1}{3}r^2\right) S \times 1000 \text{ oz.}$$

which weight the celebrated Cotes calculated to be equal to the weight of a globe of lead of 60 miles in diameter, or upwards of

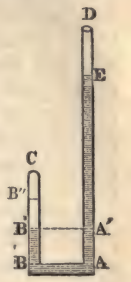
$$77{,}670{,}000{,}000{,}000 \text{ tons.}$$

Still this coating of air is comparatively thin: the thickness of it is estimated at not more than 50 miles, so that, taking the average radius of the earth at 4000 miles, the thickness of the surrounding fluid is only $\frac{1}{80}$th of the radius: on a twelve-inch globe this thickness would be only about $\frac{1}{13}$th of an inch.

507. **Law of Mariotte and Boyle.**—It was proved experimentally by both these philosophers at about the same time (1662), and independently of each other, that—

The elastic force of air (at a given temperature) is directly proportional to its density, or, which is the same thing, inversely proportional to its volume.

Let DAC be a bent tube of uniform bore, open at both ends C, D; and let mercury be poured in so as to fill the bend of the tube, AB. Let the end C be now closed, and such a quantity of mercury be poured into the tube at D as will cause the surface originally at B, to rise to B' halfway between B and C. It will be found upon measurement that the column of mercury AE thus poured in will always be such that the portion of it $A'D$ above the level of B' will be exactly equal to the mercurial column that balances the atmospheric pressure at the time. We thus see that when the surface A sustained the pressure of only the column of air above it, the air in the shorter leg occupied the space BC, the upward pressure upon it at B being merely the weight of the air which enters the tube at D; but that when *twice* this pressure acts down the tube, and is communicated upwards to the air in the shorter leg, that air is then compressed into half its former volume, $B'C$.



Let now another column of mercury (29 or 30 inches high, according to the state of the atmosphere) be poured into the tube at D: the air in $B'C$ will be found to become compressed by this additional pressure into a volume $B''C$ exactly equal to one-third of its original volume BC, and so on. Similar results are found to follow, whatever fractional part of a column of mercury equal to $A'E$—which represents the whole atmospheric pressure—be poured into the tube, the general conclusion being that at whatever points B', B'', the columns CB', CB'', of compressed air may terminate,

$$\text{Pressure on } B''C : \text{Pressure on } B'C :: \text{Vol. } B'C : \text{Vol. } B''C.$$

And since these pressures are balanced by the downward reaction, or force of elasticity of the condensed air, we have

$$\text{Elastic force of } B''C : \text{Elastic force of } B'C :: \text{Vol. } B'C : \text{Vol. } B''C,$$

and because the density increases as the volume diminishes, the last two terms of either proportion may be replaced by Density of $B''C$: Density of $B'C$, so that the pressure divided by the density is constant; and the same proportions hold for other elastic fluids.

[The foregoing Law of Mariotte is that for the same elastic fluid $VP = V'P'$ (representing the pressure and corresponding volume by symbols), provided the temperature remain unchanged. The experiments of Gay-Lussac and Dalton further show that if the temperatures of the two volumes V, V', are T, T', then the equation is

$$\frac{VP}{1+\alpha T} = \frac{V'P'}{1+\alpha T'}, \text{ where } \alpha \text{ is constant,}$$

at least for the same gas, and $= \cdot 00366$ for air when the degrees of temperature are measured on the Centigrade Thermometer. The constant is called the coefficient of dilatation. Suppose a volume V of air, under a pressure P, has the temperature T, and it be required to find to what temperature T' it must be raised in order that it may occupy a volume V'. The preceding equation gives

$$T' = \left(\frac{1}{\alpha} + T\right)\frac{V'P'}{VP} - \frac{1}{\alpha}.$$

Let, for instance, the volume V' be double V, and the pressure P' two-thirds of P, the first temperature being $T=10°$ (Centigrade): then since $\alpha=\cdot00366$, we find the required temperature to be $T'=104\cdot4°$].

508. As in inelastic fluids the standard of density is pure water, so in elastic fluids it is atmospheric air taken at the ordinary temperature of 60°, and when the mercurial column measuring its weight or pressure is 30 inches:—the density of such air is 1. When by compression a portion of this air is reduced to half its bulk, the density of it is 2, and the corresponding pressure on it = two atmospheres, and when compressed to $\frac{1}{n}$ of its bulk, its density is n, and the pressure upon it = n atmospheres.

PROP. I. If altitudes increase in arithmetical progression, the corresponding pressures of the atmosphere decrease in geometrical progression.

Conceive a vertical column of the atmosphere to be divided into equal strata, so thin that each stratum may be regarded as of uniform density throughout. Let p_1 be the pressure on the base of the column, that is, on the bottom of the first stratum, and δ_1 the density of that stratum. In like manner, let p_2 be the pressure on the bottom of the second stratum, and δ_2 the density of that stratum, and so on: then putting A for the area of the base of the column, and τ for the thickness of each stratum, the difference between the pressures on the bottom and top of the nth stratum, that is, between the pressure on the bottom of the nth stratum, and that on the bottom of the $(n+1)$th, must be the weight $A\tau\delta_n g$ of that nth stratum (446), that is,

$p_n-p_{n+1}=A\tau\delta_n g$. Now $\frac{p_n}{\delta_n}$ is constant (p. 478): Call it k,

then $\delta_n=\frac{p_n}{k}$, $\therefore \frac{p_n-p_{n+1}}{p_n}=A\frac{\tau g}{k}$...[1].

In like manner, $\frac{p_{n-1}-p_n}{p_{n-1}}=A\frac{\tau g}{k}$,

$\therefore \frac{p_n-p_{n+1}}{p_n}=\frac{p_{n-1}-p_n}{p_{n-1}}$, $\therefore p_{n-1}\,p_{n+1}=p_n{}^2$.

Height in Miles.	Times Rarer.
0	1
3½	2
7	4
14	16
21	64
28	256
35	1024
42	4096

Hence the pressures p_1, p_2, p_3, ..., and consequently the densities, are in geometrical progression; the heights 0, τ, 2τ, 3τ, ... being in arithmetical progression. From observation it is found that at 3½ miles above the surface of the sea the air is only about half as dense, that is, it is twice as rare: hence the construction of the small table annexed, which, however, can be regarded only as an approximation to the truth, since the difference of temperature of the various strata, and the diminution of the force of gravity, have been neglected.

PROP. II. To investigate a formula for finding the difference of the heights of two stations above the surface of the sea by means of the barometer.

Let z be the height in feet of the upper station above the surface of the sea, and z' the height of the lower station. Then dividing the intervening air into strata of 1 foot thick, the pressure may be taken as uniform throughout the same stratum; taking then $\tau=1$ foot, and also $A=1$, that

is, regarding the area of the base of the atmospheric column as 1 sq. foot, we have [1]

$$1-\frac{p_{n+1}}{p_n}=\frac{g}{k'} \therefore \text{the ratio } \frac{p_{n+1}}{p_n}=1-\frac{g}{k}\ldots[2].$$

Now let the mercury in the tube for measuring the atmospheric pressure, that is, the height of the barometer, be h at the upper station, and h' at the lower; then these heights being as the pressures on the zth and z'th strata, they must be as the corresponding powers of the ratio [2] by Prob. I., that is,

$$\frac{h}{h'}=\left(1-\frac{g}{k}\right)^z \div \left(1-\frac{g}{k}\right)^{z'}=\left(1-\frac{g}{k}\right)^{z-z'},$$

$$\therefore \log \frac{h}{h'}=(z-z')\log\left(1-\frac{g}{k}\right), \therefore z-z'=\log\frac{h}{h'} \div \log\left(1-\frac{g}{k}\right) \text{ feet.}$$

From the small value which observations assign to the fraction $\frac{g}{k}$, powers of it may be neglected; and since, by the logarithmic theorem (112),

$$\log_e\left(1-\frac{g}{k}\right)=-\left\{\frac{g}{k}+\frac{1}{2}\left(\frac{g}{k}\right)^2+\ldots\right\},$$

we have, by disregarding all the small terms after the first,

$$z-z'=\log_e\frac{h}{h'} \div \left(-\frac{g}{k}\right)=\log_e\frac{h'}{h} \div \frac{g}{k}=\frac{k}{g}\log_e\frac{h'}{h} \text{ feet.}$$

The constant fraction $\frac{k}{g}$ is determined thus: Let H' be the height of the barometer at a foot above the sea-level, and H its height at a known altitude of a feet above that station, then by the formula, since $z-z'=a$, we have

$$a=\frac{k}{g}\log_e\frac{H'}{H}, \therefore \frac{k}{g}=a \div \log_e\frac{H'}{H} \text{ ft.}$$

And thus the constant fraction in the general formula may be determined by experiment: it has been in this manner found to be 64,020, when the Napierian are converted into common logs, so that dividing by 6, the practical formula for the computation of any altitude A, above the level of the sea, would be

$$A=10670 \log \frac{\text{Height of barom. at alt. } A}{\text{Height of barom. at the sea-level}} \text{ fathoms,}$$

provided the temperature of the air were uniformly 55° Fahr., for which temperature the constant was determined.

But experiment has shown that the altitude of a place, as deduced from this formula, will vary by about $\frac{1}{435}$th of its whole value for every degree, by which the mean of the temperatures at the two stations differs from 55°: this variation to be added when the mean exceeds 55°, and to be subtracted in the contrary case. In order, therefore, to convert the constant multiplier into the more convenient number 10,000, we proceed thus. The difference between 10,670 and 10,000 is 670, and the difference between 10,670 and the value it would have for 1° less of temperature, is

$\frac{10670}{435}$=24·5 : hence to find for how many degrees less of temperature the difference is 670, we have the proportion

$$24{\cdot}5 : 670 :: 1° : 27°, \therefore 55°-27°=28°.$$

At this temperature ∴ the altitude A in fathoms is

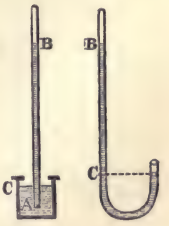
$$A=10000 \log \frac{\text{Height of barom. at alt. } A}{\text{Height of barom. at the sea-level}},$$

and this value must be increased or diminished by the $\frac{1}{435}$th part of itself for every degree which half the sum of the temperatures at the two stations exceeds or falls short of 28°.

We are thus furnished with a very simple practical rule, but it must be regarded only as a tolerably close approximation: when great accuracy is required corrections must be made for the variations of gravity depending upon the height of the station and the latitude of the place, for the dilatation of the mercury in the barometer which lengthens to the extent of about $\frac{1}{10000}$th part of the whole for every additional degree of temperature, and for the state of the atmosphere at the time of observation in reference to the moisture suspended in it. These corrections are made by the help of special *Tables*, constructed conformably to the results of observation: those which are at present regarded as the most accurate are the tables of M. Delcros, published in the "Annuaire Météorologique" for 1849, and those of M. Plantamour, in the same periodical for 1852. By the former set of tables the height of Mont Blanc, calculated from very careful barometric and hygrometric observations, made on the 29th of August, 1844, by MM. Bravais, and Martins, was concluded to be 4814·5 *metres*. By the latter tables, the height determined was 4811·7 metres. [The metre is 39·37 English inches.]

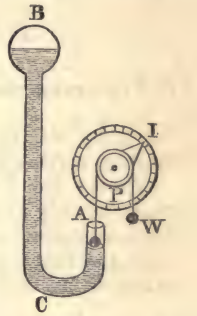
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509. **The Barometer.**—This instrument is so well known that in the preceding articles we have presumed the reader to have at least a general acquaintance with its object. It may be regarded as a balance to determine the weight of a column of air on a given base, and reaching vertically upwards to the extreme limit of the atmosphere, the counterpoise being a column of mercury upon an equal base. The tube containing the mercury is either straight or bent, as in the annexed figures: when straight, its lower extremity, which is unclosed, is inserted in a small cistern, or box, or bag, of mercury; the pressure of a column of air downwards on an area $=A$ of the surface of the mercury in the cistern, is communicated upwards at A balancing the weight of the column AB in the tube previously exhausted of air. Every other area equal to A, of the surface surrounding the tube, is of course equally pressed, and equally communicates that pressure upwards, but the area covered by the tube BA is the only one of these areas upon which an equal downward column of air does not press. The mer-



curial column AB, therefore, is equal in weight to the weight of the atmospheric column on an equal area A: the height of the mercury in the tube above the surface of that in the cistern always ranges between 28 and 31 inches, and scales are attached for reading off the height at any time.

The bent tube is the form used for the *Wheel Barometer*, or Weather Glass, as it is sometimes called: the end A is open to the air, the other end B is closed, and usually enlarged into a globe, the upper part of which, not filled by the mercury, being a vacuum. Upon the surface of the mercury at A, is an iron ball, connected with another rather lighter, by a string passing over a pulley P. As the ball at A rises and falls with the mercury, the string turns the pulley by its friction, and an index, turning with it, points to the different degrees marked on the surrounding circle. A very slight variation in the height of the surface at A will make a considerable difference in the position of the index I, if this surrounding circle be tolerably large: but the friction of the pulley is an impediment.



510. **The Thermometer.**—The lower end of the tube in this instrument is expanded into a bulb, and no part of the fluid within is in contact with the air. As in the barometric tube the space above the fluid is a vacuum, and the fluid itself is either alcohol or mercury. The bore of the tube is so small that a very slight expansion of the fluid which occupies it makes a sensible difference in the length of the fluid column, and thus renders it a fit measurer of the heat of any fluid in which it may be plunged (the surrounding air for instance), or of any body with which it may be brought in contact, since bodies in general expand by the application of heat, and contract by cold:—water, however, being an exception, within certain limits.

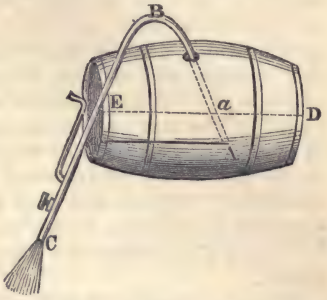
In the thermometer of Fahrenheit, the point at which the mercury stands when the instrument is plunged into melting snow, and allowed to remain there till the mercury will fall no lower, is marked 32°:—this is the *freezing* point. The point at which the mercury stands when the instrument is immersed in the vapour of boiling water, is marked 212°:—this is the *boiling* point; and the intermediate degrees are marked on the attached scale.

In the Centigrade Thermometer the freezing point is marked 0°, and the boiling point 100°:—whence its name. In Reaumur's, these points are 0° and 80° respectively. Distinguishing these three kinds of degrees by $F°$, $C°$, and $R°$—the initials of the names they bear—we may convert the number of degrees which marks any temperature on one scale into the number which marks the same temperature on either of the other scales, from the relations

$$36\,C°=45\,R°=20(F-32°),\text{ or }\frac{C°}{5}=\frac{R°}{4}=\frac{F°-32°}{9}.$$

511. **The Syphon.**—This is a bent tube ABC, with unequal legs AB, BC, used for the purpose of transferring liquids from one vessel to another. The end of the shorter leg is inserted in the liquid, and the air in the tube withdrawn by the application of the mouth to a small suction-pipe, communicating with the tube near the end C of the longer leg.

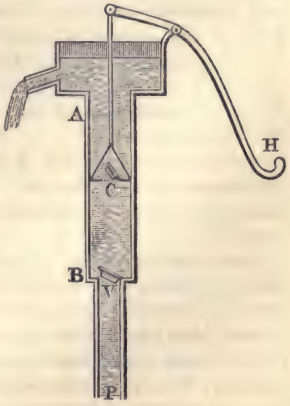
There is a stop-cock between the insertion of the suction-pipe and the open end *C*, to cut off all communication of the atmosphere with the interior of the tube, preparatory to the exhaustion of air within it. This exhaustion having been effected, the pressure of the air on the surface *DE* of the liquid forces the liquid up the leg *AB*, the highest point *B* of which is much below the height to which the liquid *would* ascend but for the bend: it is ∴ driven downwards by the reaction of *B* with a pressure equal to the weight of the atmosphere pressing on *A*, diminished by the weight of the sustained column *aB*, *a* being at the surface. If the leg *BC* terminated at *E*, on a level with *a*, the two columns *aB*, *EB* would balance, since the upward pressure of the atmosphere at *E*, diminished by the weight of the vertical column of liquid of the height of *B*, would be equal to the upward pressure at *a*, diminished by the weight of a like vertical column of liquid, in which case no liquid would flow out; but if the end *C* be below the level *DE*, the equilibrium is destroyed:—the atmospheric pressures upward at *a*, *C*, are still the same, but there is more weight of liquid pressing downward at *C*, than there is sustained at *a*; and since the passage is free, the liquid must be forced out by this excess of pressure, and must continue to flow till the surface of it in the vessel falls to a level with *C*, when it will stop.



If the syphon have no stop-cock, it must be inverted and filled with liquid, and both ends being closed, and the instrument turned to its proper position, the shorter end must be inserted in the liquid, and *then* both ends unclosed.

If the liquid be water, *B* must not be higher above its surface than 33 or 34 feet (p. 477). If it be mercury, it must not be higher than about 29 or 30 inches (p. 477).

512. **The Common Pump.**—The annexed figure represents a section of the common suction pump. *AB* is the pump-barrel, communicating by the pipe below it (called the *suction* pipe), with the water in the well. *C* is the air-tight piston (the *sucker*), in which there is a valve opening upwards: there is another valve *V*, also opening upwards, which, when shut, closes the communication between the barrel and the pipe. The piston *C* is worked in the barrel by aid of a lever *H*:—the pump-handle.



Suppose no water to be as yet in the pipe: the pump-handle is raised, and the piston connected with the pump-rod descends, and as *V* is closed, the air in the barrel below *C* is thus condensed, and by its upward pressure opens the valve in *C*, and escapes: the handle is then pulled down, and *C* raised, the valve in it being closed by its own weight, and by the down-

ward pressure of the atmosphere above, while it leaves a vacuum or a partial vacuum in the barrel below it: this vacuum is immediately filled up by the expansion of the air in the pipe, the upward pressure of which forces open the valve V, the downward pressure upon V having been wholly, or at least partially, removed. Instead, therefore, of the barrel and pipe being filled, as at first, with atmospheric air of the same density as that which presses upon the surface of the well, the pump contains only rarefied air; so that the pressure down the pipe is less than the pressure up it from the weight of the external air, and therefore water is forced up the pipe by this excess of external pressure. By another stroke of the pump-handle another barrel-full of air is pumped out, and the water in the pipe, being thus relieved still more of downward pressure upon it, ascends still higher; and after a few strokes enters the barrel, and it is then lifted up by the piston and pumped out; more water following till the barrel is full, when the water can be made to flow out in a continuous stream.

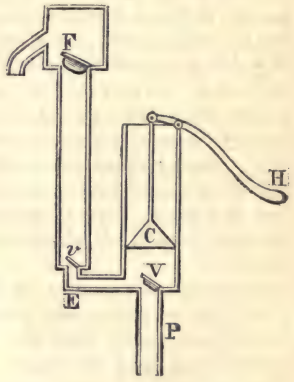
It will be observed that whenever the upward pressure against either of the valves does not exceed the downward pressure the valve closes by its own weight.

Since the pressure of the atmosphere on any area is equal to that of a column of water on that area of only 33 or 34 feet (p. 477), it is plain that the water cannot be made to pass the valve V, into the barrel, if V be not *less* than at this height from the surface of the well: nor can the barrel be filled to the spout if the spout exceed this height. Supposing the barrel to be constantly full, the quantity discharged at the spout, at each stroke of the handle, is a column of water, the base of which is the horizontal section of the piston, and the altitude the height to which the piston is raised—called the length of the stroke. Let the radius of the section be r feet, and the length of the stroke l feet: then—

Quantity discharged $= 3{\cdot}1416r^2l$ cubic feet $= 3{\cdot}1416r^2l \times 6\frac{1}{4}$ gallons,

Since a cubic foot of water or about 1000 oz. $= 62\frac{1}{2}$ lbs. avoirdupois.

513. **The Forcing Pump.**—In order that water may be pumped out of a well, in a continuous stream, the spout must not be higher than 33 or 34 feet from the surface of the water in the well, if the common pump be employed: but the *forcing* pump will deliver water continuously at any height. The annexed diagram is a section of this pump. As in the common pump, V is a valve opening upwards at the junction of the suction-pipe and barrel, or cylinder AV. The piston C is air-tight as before, but has no value:—it is solid.



Conceive the piston C to be raised to its highest point, the valves V and v, both opening upwards, being closed. As C descends the air below it presses V closer down, opens v, and rushing up the pipe vF, escapes: the valve v then closes by its own weight.

As C ascends, a vacuum beneath it would be formed; but the upward pressure of the air in P forces V open, and by its expansion fills VC, V

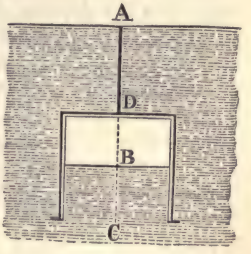
then closing by its own weight; v also remains closed from the superior downward pressure of the atmosphere: water, however, ascends in P, on account of the diminished pressure down the tube.

In this way every descent of C compresses the air between it and V till the elasticity of it becomes sufficient to force open v, when another portion of the air, originally in CP, escapes. And every ascent of C is followed by the air, still in P, forcing open the valve V, to fill up the vacuum VC, and by a further ascent of water in P, in consequence of the further diminution of pressure down P.

At length the water, by thus rising higher and higher, passes through V, when the next descent of C will force it through v, which instantly closes upon the withdrawal of the upward pressure, that is, upon the ascent of C; and the return of the water in vF is thus prevented. The next downward stroke of C forces more water through v, till at length the column rises to F; and it may then be forced out in a continuous stream.

The Fire Engine is a combination of two forcing pumps, the pistons of which are worked by a lever, having its fulcrum at its middle; and the power which raises and depresses the piston-rods is applied at each end alternately. The water from the fire-plug is pumped into a central receptacle (with which the two cylinders containing the pistons communicate), called the *air-vessel*, the air in which being forced to contract the more as more water is pumped into it by the descending pistons, exercises the greater pressure upon the water, and forces it with considerable velocity out of the delivery pipe.

514. **The Diving Bell** is so called, because in its original construction a bell-shaped figure was given to it. At present, it is a heavy iron chest, open at the bottom, with a seat in the interior sufficiently high to enable persons sitting on it to keep their heads above the water, which gradually rises in the bell as it descends, but which can never completely fill it on account of the inclosed air, which as gradually becomes more and more condensed. A flexible pipe, communicating through the top (or bottom) of the bell, and furnished with stop-cocks, allows of the escape of the air unfit for respiration, and of fresh air being pumped in. We may find the space into which the air, originally in the bell, will be compressed, when at any depth below the surface, thus: Let AB be the depth of the surface of the water in the bell below the surface of the water above. Then taking 33 feet for the height of a column of water, the pressure of which would be equal to that of the atmosphere, the condensed air in the bell sustains a pressure equal to 33 feet of water (the original pressure) + the pressure of another column of water of height AB. Calling this height h, and remembering that the spaces into which air is condensed are inversely as the condensing pressures (507), we have



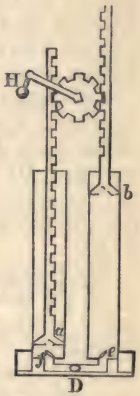
$$\frac{\text{Vol. of condensed air}}{\text{Vol. of original air}} = \frac{DB}{DC} = \frac{33}{33+h},$$

$\therefore$ putting $DC=a$, the depth $AD=d$, and $DB=x$, $\frac{x}{a} = \frac{33}{33+d+x}$,

$$\therefore x^2+(33+d)x=33a,$$

a quadratic which has only one positive root; and this gives the thickness $x=DB$ of the layer of air above B.

515. **The Air Pump.**—This is a machine constructed for the purpose of exhausting a vessel of as much of the air contained in it as possible. There are several forms given to it, but that best known is the double-barrelled pump of Hawksbee, or the common air pump. The vessel to be emptied of its air, and which is called the *receiver*, communicates by means of a pipe inserted at D, with two smooth barrels, in which two closely-fitting pistons with valves are moved by rack-work. The four valves a, b, e, f, all open upwards; and the results produced by working the handle H are as follows:—



As one of the pistons descends, its valve is forced open, and the air in the barrel passes through: as this goes on, the other piston ascends, with its valve closed, emptying the barrel up which it moves of its air; the vacuum being instantly filled from the air in the receiver, through the pipe at D.

Reversing the motion of the handle, the first piston now ascends, emptying of its air the barrel up which it moves; and, as before, fresh air from the receiver rushes in.

These alternate ascents and descents of the pistons so rarefy the air in the receiver, that at length its elastic force is too feeble to raise the valves, and thus to pass into the barrels: when, therefore, this stage is arrived at, the exhaustion cannot be carried on any further.

Prob. I. To find the density of the air in the receiver after n strokes, or turns of the wheel.

Let R be the capacity of the receiver, and B that of each barrel: then regarding the density of the atmospheric air as 1, let δ_1, δ_2, ..., δ_n, be the densities of the air in the receiver after 1, 2, ..., n, strokes, or turns of the wheel. Then since at every turn a barrel full of air is withdrawn, the volume R of air is expanded into $R+B$; hence (507),

$$\begin{aligned}
\delta_1(R+B)&= R, &\therefore \delta_1&= \frac{R}{R+B};\\
\delta_2(R+B)&= \delta_1 R, &\therefore \delta_2&=\left(\frac{R}{R+B}\right)^2;\\
\delta_3(R+B)&= \delta_2 R, &\therefore \delta_3&=\left(\frac{R}{R+B}\right)^3;\\
&\vdots &&\vdots\\
\delta_n(R+B)&=\delta_{n-1}R, &\therefore \delta_n&=\left(\frac{R}{R+B}\right)^n.
\end{aligned}$$

Consequently the density decreases in geometrical progression. The rarefied air of volume R, which remains in the receiver after n strokes, is equal only to a volume $\delta_n R$ of the original air; and since, as just shown, $\delta_n R=\left(\frac{R}{R+B}\right)^n R=\frac{R^{n+1}}{(R+B)^n}$, it follows that this last fraction denotes the part of the original volume of air which still remains.

The quantity of original air diminishes, therefore, in geometrical progression; and since a decreasing geometrical progression may be continued to infinity, it follows that the receiver could never be completely emptied of air by any number of strokes of the air pump, even should the air, however rarefied, be always sufficiently elastic to open the valves.

Prob. II. To find what number of strokes are requisite to reduce the air in the receiver to a given density.

Let δ be the given density, that of atmospheric air being 1, and put n for the required number of strokes: then by last problem,

$$\delta=\left(\frac{R}{R+B}\right)^n, \quad \therefore \log\delta=n\{\log R-\log(R+B)\},$$

$$\therefore n=\frac{\log\delta}{\log R-\log(R+B)}.$$

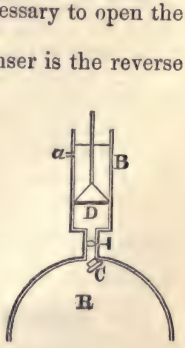
Since both numerator and denominator of this fraction are negative, δ being always less than 1, the fraction itself is positive. As inferred above, so here, if $\delta=0$, that is, if the receiver is to be completely exhausted, n must be infinite, since $\log 0=-\infty$.

516. **Smeaton's Air Pump.**—This consists of but a single barrel AB, communicating with the receiver through the pipe C. There are three valves:—one at B, one in the piston D, and one in the top plate A of the barrel:—all opening upwards. As the piston ascends from the bottom of the barrel, D is the only valve that closes; for A is forced open by the upward pressure against it, and D closed by the same pressure downwards; and the under portion DB of the barrel being exhausted of air by the stroke, the valve B is forced open by the air rushing in from the receiver. If, as before, R, B, represent the capacities of the receiver and barrel, we shall have the same expression for the density δ_n after n strokes, namely, $\delta_n=\left(\frac{R}{R+B}\right)^n$, the original density being 1.



Although the density after n strokes is thus the same in both machines, yet a greater degree of rarefaction may be obtained by Smeaton's than by Hawksbee's machine; for in Smeaton's the valve of the descending piston is relieved from the downward pressure of the atmosphere, and therefore will open, and allow air to escape, for a much less upward pressure than is necessary to open the valves a, b, in the older air pump of Hawksbee.

517. **The Condenser.**—The object of the condenser is the reverse of that of the air pump: it consists of a receiver R, a portion of which is represented in the annexed diagram. Into this vessel, containing at first only atmospheric air in its ordinary state, additional air is forced by the descent of a solid piston moving in the cylinder B, which communicates with the receiver through the valve C opening with a spring downwards, immediately above which is a stop-cock. Near the top of the cylinder is an aperture a, which, when the piston is at its greatest elevation, is immediately below it. Suppose that from this elevation the piston is forced down:—the barrel full of air beneath, becomes more and more condensed, till its pressure at length opens the valve, and the air in the barrel is thus driven into the receiver. The piston having reached its greatest depression, the air immediately under it, and that in the receiver, must have the same density, so that the valve C closes by the action of its spring. Again, raising the piston above the aperture a, through which a second barrel full of air rushes, this further quantity is



forced into the receiver as before: and so on, till the upward pressure of the condensed air is so great against the valve, that the downward pressure of the condensed barrel full of common air is insufficient to overcome it and force the valve open.

Since a volume B of common air is forced into the receiver at every stroke, the quantity of common air condensed into the volume R, after n strokes, must be $R+nB$. Putting δ for its density, that of common air being 1, we have (507),

$$(R+nB)\delta=R, \ \therefore \ \delta=1+\frac{nB}{R}.$$

Hence the density increases in arithmetical progression, the increase being $\frac{B}{R}$ times the density of the external air at every stroke.

Examples for Exercise.

(1) How much of the original air will be left in the receiver of an air pump after four strokes of the piston, the capacity of the receiver being ten times that of the barrel?

(2) Required the pressure per square inch, on the interior of the receiver in the last example, after two strokes of the piston; the pressure of the atmosphere being $14\frac{2}{3}$ lb. per sq. inch.

(3) Required the pressure per sq. inch on the interior of the receiver of a condenser, after two strokes of the piston, the atmospheric pressure being $14\frac{2}{3}$ lb. per sq. inch.

(4) When the mercurial barometer stands at 30 inches, at what height will a water barometer stand, the density of mercury being 13·5?

(5) At what height will a water barometer stand when the atmospheric pressure is 15 lb. per sq. inch?

(6) A diving-bell is sunk till its top is 45 feet below the surface of the water: what height of air is there then in the bell, the barometer outside standing at 30 in., and the mercury being 13·5 times the density of the water, the height of the bell being 5 ft.?

(7) What number of degrees of Fahrenheit correspond to $-3°$, and to 49° Centigrade? And what degree of Reaumur corresponds to 39° Fahr.?

(8) A barometer which exposed to the atmosphere would stand at 30 in., is placed under the receiver of Smeaton's air pump: the barrel and receiver have equal capacities: what will the height of the barometer be after three strokes of the piston?

(9) A cylinder 36 inches high, and 24 inches in diameter, is filled with atmospheric air when its pressure is 12 lb. on a square inch: to what depth will an air-tight piston of 3 tons weight sink in the cylinder?

(10) A spherical shell or balloon is to be formed of material of which the thickness is k, and its density, relatively to the surrounding air, d; and it is to be filled with gas of density δ: prove that, in order that it may ascend in the air, its exterior radius must exceed $k \div \left\{1-\sqrt[3]{\frac{d-1}{d-\delta}}\right\}$.

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518. **Scholium.**—On the subject of the foregoing short treatise, the student may consult with advantage the neat little "Manual" of Professors Galbraith and Haughton, and the larger works of Bland and Webster. But equally as in the other branches of the mixed sciences, its complete development requires the aid of the Differential and Integral Calculus, a department of pure mathematics upon which we shall now enter.

End of the Hydrostatics.

IX. THE DIFFERENTIAL AND INTEGRAL CALCULUS.

519. The symbols employed in the Higher Calculus—the principles of which we are now about to explain—are the same as those always used in the operations of common algebra, with the addition of merely one or two, peculiar to the present subject. But it is more especially necessary here to class the quantities which those symbols represent under two distinct heads:—*constant* quantities, and *variable* quantities; for no analytical investigation into which variable quantities do not enter, or in which a change of value may not be allowably assumed, can ever involve the processes of the differential calculus, which is essentially the science of variable quantities.

The distinction between constants and variables has been sufficiently exemplified in the Analytical Geometry: the former have values absolutely fixed and determinate, like the figures of common arithmetic:—the latter may take any values we please to give them, consistently, of course, with the condition or equation which fixes—not their values—but their relation of dependency on each other. In ordinary algebra the earlier letters of the alphabet, a, b, c, &c., are used to represent known quantities, and the final letters, z, y, x, &c., to denote unknown (though constant) quantities. Here, however, the leading letters will be employed for constant quantities, whether known or unknown, and the final letters for variables.

520. **Differentiation.**—Before we can explain the meaning of this term, it will be necessary to examine into the effect produced upon a function of a variable x, by changing it from one value to another.

(1) Let the function be ax^2+b: we shall represent it by y, writing $y=ax^2+b$, and shall observe what change takes place in the function y by changing x into $x+h$. Putting y' for the changed function, we have

$$y'=a(x+h)^2+b=(ax^2+b)+2axh+h^2,$$

from which, if we subtract the original function $y=ax^2+b$, we shall have $y'-y=2axh+h^2$ for remainder; that is, using the customary form of expression, if h be the *increment* of the variable, $2axh+h^2$ will be the *increment* of the proposed function of it, whether h be positive or negative. In the latter case, h would more properly be a *decrement:* but it is usual to call it a positive increment in the one case, and a negative increment in the other. The ratio of the increment of the function to the increment of the variable in the case before us is

$$\frac{y'-y}{h}=2ax+h.....[1].$$

(2) As a second example let the function be $y=ax^2+bx-c$. Giving to the variable x the increment h, the increment $y'-y$ of the function will be

$$y'-y=\{a(x+h)^2+b(x+h)-c\}-(ax^2+bx-c)=(2ax+b)h+ah^2,$$

and the ratio of the increment of the function to that of the variable will be

$$\frac{y'-y}{h}=2ax+b+ah.....[2].$$

(3) Suppose, lastly, that the function is $y=ax^3-bx-c$, then we have

$$y'-y=\{a(x+h)^3-b(x+h)-c\}-(ax^3-bx-c)=(3ax^2-b)h+3axh^2+ah^3,$$

$$\therefore \frac{y'-y}{h}=3ax^2-b+3axh+ah^2\ldots\ldots[3].$$

The increment, or difference h between the original value x of the variable and the new value $x+h$, is frequently represented by the notation Δx, where Δ stands not for a quantity, or factor, but for the words "difference of," or "increment of;" and, in like manner, for $y'-y$, the increment of the function, the analogous symbol Δy is employed; so that the three expressions marked above may be written thus:—

$$(1)\ \frac{\Delta y}{\Delta x}=(2ax)+\Delta x. \qquad (2)\ \frac{\Delta y}{\Delta x}=(2ax+b)+a\Delta x.$$

$$(3)\ \frac{\Delta y}{\Delta x}=(3ax^2-b)+3ax\Delta x+a(\Delta x)^2\ldots[1].$$

Each of these is of course an identical equation: neither implies any condition: the first member merely *indicates* what in the second is actually performed, that is, the second member merely interprets the meaning of the first.

It is of importance to notice, in reference to the three preceding results for $\frac{\Delta y}{\Delta x}$, that the quantity within the vinculum in each case—and which we may regard as the first term of each result—is independent of Δx or h, so that this first term will remain unchanged, however we alter the value of h: suppose, then, that assuming any particular value k for h, we conceive that value to continuously diminish from $h=k$ down to $h=0$, the aforesaid first term, which all along will remain undisturbed, will then express the complete value of $\frac{\Delta y}{\Delta x}$, or $\frac{y'-y}{h}$; for all that follows it will vanish. In all these instances, therefore, this first term accurately denotes the *limit* of the ratio $\frac{\Delta y}{\Delta x}$, or that final or ultimate value of it to which it tends as Δx or h continuously diminishes, and at which it actually arrives when these continual diminutions bring h down to its ultimate value $h=0$. It is this limiting value of the general ratio $\frac{\Delta y}{\Delta x}$, in any particular case, that the calculus selects for its operations.

In the foregoing examples we have arrived at this limiting value of $\frac{\Delta y}{\Delta x}$ by first getting the general value, namely, that value which is independent of all restriction as to the value of the increment Δx, and then introducing the hypothesis of $\Delta x=0$. The calculus will furnish rules for enabling us to obtain the limiting value of the ratio $\frac{\Delta y}{\Delta x}$, whatever function of x y may be, without first developing its general value; that is to say, if for distinction, we use the notation $\frac{dy}{dx}$ for the limiting value of the general ratio $\frac{\Delta y}{\Delta x}$, the calculus will enable us at once to find $\frac{dy}{dx}$, without encumbering the

expression with the terms involving Δx, $(\Delta x)^2$, &c. (as above), which are afterwards to be rejected. Employing the notation here proposed, the limits of the ratios [1] are severally

$$(1)\ \frac{dy}{dx}=2ax. \quad (2)\ \frac{dy}{dx}=2ax+b. \quad (3)\ \frac{dy}{dx}=3ax^2-b \ldots [2].$$

It is plain that these are only so many values of distinct vanishing fractions:—the values, in fact, of the fractions

$$\frac{2axh+h^2}{h}, \ \frac{(2ax+b)h+ah^2}{h}, \ \frac{(3ax^2-b)h+3axh^2+ah^3}{h}, \text{ when } h=0,$$

each of which fractions, in the proposed hypothesis, *before* the division by h is executed, assumes the form $\frac{0}{0}$, and it is this form that the more significant notation $\frac{dy}{dx}$ replaces. We say this is a more significant notation, because it clearly implies that the general fraction or ratio, of which this denotes a single specific case, was $\frac{\Delta y}{\Delta x}$, the numerator of which stands for a certain fixed and definite algebraic expression of the form $F(x+\Delta x)-F(x)$, where Δx is of arbitrary value: the notation indicates, in fact, what we may call the pedigree of the particular $\frac{0}{0}$ we are thus symbolizing.

521. From what has now been said it is presumed that the student will clearly understand what is meant by the expression *the limit of the ratio of the increment of a function to the increment of the variable.* If, as above, x be the variable, and y be put for the function, or the proposed algebraic expression involving that variable, the limit here spoken of is represented by $\frac{dy}{dx}$, which is called the DIFFERENTIAL COEFFICIENT, derived from the function: thus in [2] above, $2ax$ is the differential coefficient derived from the function ax^2; $2ax+b$ is that derived from ax^2+bx+c; and $3ax^2-b$ is the differential coefficient derived from ax^3-bx+c. The relations [2] necessitate the relations

$$(1)\ dy=2ax.dx. \quad (2)\ dy=(2ax+b)dx. \quad (3)\ dy=(3ax^2-b)dx \ldots [3].$$

The *zeros* dx, dy, are called *differentials*:—the former, the differential of x, and the latter, the differential of y: and hence the propriety of calling $\frac{dy}{dx}$ a *differential coefficient:* it is always the coefficient of the differential dx in the expression for dy. The derivation of dy from y in all cases (to be hereafter taught) is called *differentiation.*

522. [It is sometimes said, in reference to such equations as those marked [2], that if dy, dx, are each zero, we have divided 0 by 0, and have therefore no more right to put any one thing for the quotient than any other, for $\frac{0}{0}$ means anything. But the results [2] have not been obtained by any such division of 0 by 0: they have been arrived at, in each

case, by dividing $F(x+\Delta x)-F(x)$ by Δx, and by this operation only. The result is a general expression, over the form of which—implying a law that all the particular cases of it must conform to—we have no control; but over the particular cases themselves, so far as their values depend upon the value of Δx, we have complete control, inasmuch as Δx is not fixed by any condition, but is free to take any value we may please to give it. Our aim has then been to select from the innumerable series of possible or admissible values—all conforming to one and the same law—that which would be the last, or ultimate value in this series, if it were formed by giving first to Δx some finite value k, as small as we please, and then continuously diminishing k down to $k=0$: the value of the ratio thus obtained terminates the entire series of values, and continuously unites with the individuals of that series in as strict obedience to the law of the general expression.

It has been objected, however, that this is to change the original hypothesis:—to make that (namely Δx), 0 at the end of an operation, which at the beginning was a finite quantity. But there was no "original hypothesis" as to the value of Δx: this quantity Δx (or h) was designedly left free from *all* restriction as to value at the outset, *in order* that at any subsequent stage we might, without violating any previous restriction, give to it any value we chose, and with a view, of course, to its *ultimate* value, 0. It must not be overlooked that the fundamental operations of algebra—though called by arithmetical names—are quite free from arithmetical restrictions: $(x^2+ax)\div x$ is $x+a$, and would still be $x+a$, though there were no such thing as arithmetic in existence: as it is thus universally true, it is true when the symbols stand for numbers—any whatever; so that when $x=0$, $(x^2+ax)\div x=a$, whatever number is put for a. We thus see that the differential coefficient $\frac{dy}{dx}$, derived from any function y of x, is not so derived by dividing 0 by 0: it is derived by dividing the algebraic quantity Δy by the algebraic quantity Δx, and *then* giving to Δx the arithmetical interpretation $\Delta x=0$. Having *thus* found the true value (say $f(x)$) of the vanishing fraction $\frac{dy}{dx}$, we are justified *then* in writing $\frac{dy}{dx}=f(x)$, and consequently in writing $dy=(fx)dx$. This is more than stating the truism $0=0$:—it is conveying the information that the zero dy comes to be zero by multiplying the zero dx by the factor $f(x)$, and in no other way.

Those who object to regard dy, dx, as *zeros*, call them *infinitesimals*; which in reality is only using a long word for a short one: an infinitesimal has no finite value; and this is all that can be said of zero, which, however, is the preferable term, because it is more precise (see article 191)].

523. In the illustrations just given, and in every similar case, $y=F(x)$, not only is x a variable, but y is also a variable, as is evident; but as the variation of the entire function y is consequent upon the variation of x, this latter is called the *independent variable*, and y the *dependent variable*. Moreover, when, as here, the function that y is of x is actually exhibited, y is said to be an *explicit* function of x, by which is meant that y is at once expressed in terms of x and constants; but when the expression for y in terms of x can be obtained only by solving an equation in which both

variables are combined together, y is said to be an *implicit* function of x: thus, in each of the equations

$$ax^2+by^2+cxy+m=0,\ x^3-2x^2y+y=4y^3-3x,\ \&c.,$$

y is an implicit function of x, since the expression for y in terms of x can be discovered only by solving the equation in which the two variables enter. And generally, in the equation $F(x,\ y)=0$, y is an implicit function of x.

524. The student will not fail to have observed that, in the difference $F(x+h)-F(x)$, between one state $F(x)$ of the function and another state $F(x+h)$ of the same function, consequent upon giving to x the increment h, whatever constants may have entered the primitive function $F(x)$—provided they have entered only as additive or subtractive quantities—will have disappeared. In the first example $y=ax^2+b$ for instance, there is no trace of the additive constant b, in the difference $y'-y$; and in like manner, there is no trace of the subtractive constant c in the difference $y'-y$ of either of the two following examples. Such constants must of necessity always be eliminated in taking the difference, because the changing of x into $x+h$ cannot affect *them:* they remain as additive or subtractive quantities equally in $F(x+h)$ and in $F(x)$, and must therefore disappear in subtracting one of these from the other.

525. In terminating these preliminary remarks, we have only to press upon the student's special attention that the variable quantities, considered in the calculus upon which he is now entering, are all *continuously* varying quantities: that is, if a be an admissible value of a variable, then there must be another admissible value b, such that every one of the continuous series of values $a\ldots\ldots b$ must *also* be an admissible value of the variable. It must be remembered that the quantity $\frac{dy}{dx}$ has significance only as the *limiting* value of $\frac{\Delta y}{\Delta x}$; or the value at which this arrives, when, by continuous diminution, Δx is at length exhausted, and becomes zero.

We have thought it necessary to impress this consideration thus early upon the mind of the student, because, in the applications of the calculus, even writers themselves have sometimes inadvertently overlooked it. In treating of the higher curves, a case is not unfrequently met with where the equation of the curve is satisfied for values of x and y, which belong to an isolated point:—a point entirely detached from the curve. Such a point is called a *conjugate point;* and it has been attempted to apply the calculus to *it:* but as it is not one of a continuous series of points, any such application is a violation of fundamental principles.

NOTE.—In studying the processes by which the following fundamental rules for the differentiation of expressions whose variation, or change of value, is due to the variation of the quantity of which the proposed expression is a function, it must be borne in mind that—

1. Constants, which enter merely as additive or subtractive quantities, disappear from the difference of the function, and therefore from the differential.

2. Constants, which enter as factors or divisors of a variable, are always preserved as such in the difference, and in the differential.

Thus, if $u=ay$, then, whether y be the independent variable, or whether it be itself a function of x, and changes its value only in consequence of

a change in x, the increment or *difference* of u must be a times the increment or difference of y; that is,

$$\Delta u = a\Delta y, \text{ and } \therefore du = ady.$$

Also, if the change in y is due to a change in x, of which y is a function, then

$$\frac{\Delta u}{\Delta x} = a\frac{\Delta y}{\Delta x}, \text{ and } \therefore \frac{du}{dx} = a\frac{dy}{dx}.$$

3. As constants never vary, they have no difference; that is, the difference, and $\therefore$ the differential coefficient is always 0.

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526. **Investigation of Rules.**—1. To differentiate the product of two or more functions of the same variable.

Let y, z, be two functions of the variable x in the expression $u=yz$: then changing the x in each of the functions y, z, into $x+\Delta x$, and putting Δy, Δz, Δu, for the increments which y, z, u, take in consequence of that change, we have

$$u+\Delta u = (y+\Delta y)\,(z+\Delta z) = yz + y\Delta z + z\Delta y + \Delta y\Delta z.$$

$$\text{But } u = yz, \therefore \Delta u = y\Delta z + z\Delta y + \Delta y\Delta z, \therefore \frac{\Delta u}{\Delta x} = y\frac{\Delta z}{\Delta x} + z\frac{\Delta y}{\Delta x} + \frac{\Delta y}{\Delta x}\Delta z.$$

This expresses the general ratio of the increment of the function u to the increment of the variable x. Passing then from the general to the limiting ratio, that is, putting differentials for differences, we have

$$\frac{du}{dx} = y\frac{dz}{dx} + z\frac{dy}{dx} + \frac{dy}{dx}dz; \text{ but } dz = 0, \therefore \frac{du}{dx} = y\frac{dz}{dx} + z\frac{dy}{dx},$$

$$\therefore du = ydz + zdy \ldots [1].$$

Hence to differentiate the product of two functions of the same variable the Rule is this: Multiply each function by the differential of the other, and add the results.

Having thus got a rule for differentiating the product of two functions, it is easy to extend it to the product of three, or of any number of functions: thus, let $u=wyz$ be the product of three functions of x: then putting v for wy, the expression is

$$u = vz, \therefore [1],\ du = vdz + zdv : \text{ but } v = wy, \therefore [1],\ dv = wdy + ydw,$$

$$\therefore \text{ by substitution, } du = wydz + zwdy + zydw \ldots [2].$$

And it is plain that in this way the differential may be found, be the factors ever so many; the general rule, therefore, is this:

Rule.—Multiply the differential of each factor by the product of all the other factors, and add the results.

2. To differentiate a power of any function of a single variable.

As a *positive* integral power is a product of equal factors, the differential of such a power is immediately inferred from the general rule above: thus, let $u=y^n$, y being a function of the independent variable x; then since $y^n = y.y.y \ldots$ to n factors, we have by the rule referred to,

$$du = y^{n-1}dy + y^{n-1}dy + \ldots \text{ to } n \text{ terms; that is, } du = ny^{n-1}dy \ldots [3],$$

$$\therefore \frac{du}{dx} = ny^{n-1}\frac{dy}{dx}. \quad \text{If } y=x, \text{ this of course is } \frac{du}{dx} = nx^{n-1}.$$

But if the exponent of the power be a positive fraction instead of a positive integer, namely, $u=y^{\frac{m}{n}}$, then, since $u^n=y^m$, we have, from the preceding case,

$$nu^{n-1}du=my^{m-1}dy,\ \therefore\ du=\frac{m}{n}.\frac{y^{m-1}}{u^{n-1}}dy\ ;\ \text{but}\ u=y^{\frac{m}{n}},$$

$$\therefore\ du=\frac{m}{n}.\frac{y^{m-1}}{y^{m-\frac{m}{n}}}dy=\frac{m}{n}.y^{\frac{m}{n}-1}dy,\ \therefore\ \frac{du}{dx}=\frac{m}{n}.y^{\frac{m}{n}-1}\frac{dy}{dx}.$$

Let now the exponent be negative, that is, let

$$u=y^{-\frac{m}{n}}:\ \text{then}\ u^n=\frac{1}{y^m},\ \therefore\ u^ny^m-1=0,$$

$$\therefore\ [1],\ u^nd(y^m)+y^md(u^n)=0,\ \text{that is}\ [3],\ mu^ny^{m-1}dy+ny^mu^{n-1}du=0,$$

$$\therefore\ du=-\frac{m}{n}.\frac{u^ny^{m-1}}{u^{n-1}y^m}=-\frac{m}{n}uy^{-1}dy,\ \text{but}\ u=y^{-\frac{m}{n}},$$

$$\therefore\ du=-\frac{m}{n}y^{-\frac{m}{n}-1}dy,\ \therefore\ \frac{du}{dx}=-\frac{m}{n}y^{-\frac{m}{n}-1}\frac{dy}{dx}\ldots[4].$$

Hence, generally, to differentiate a power the rule is this :—

RULE.—Multiply together the exponent of the power, the power itself when its exponent is diminished by unit, and the differential of the root. This general rule might have been deduced at once from the binomial theorem, thus: whatever be p in $u=y^p$, we have

$$u+\Delta u=(y+\Delta y)^p=y^p+py^{p-1}\Delta y+\frac{p(p-1)}{2}y^{p-2}(\Delta y)^2+\ldots,$$

$$\therefore\ \frac{\Delta u}{\Delta y}=py^{p-1}+\frac{p(p-1)}{2}y^{p-2}\Delta y+\ldots,\ \text{and this, when}\ \Delta y\ \text{becomes 0, is}\ \frac{du}{dy}=py^{p-1},$$

$$\therefore\ \frac{du}{dx}=py^{p-1}\frac{dy}{dx},\ \text{whether}\ p\ \text{be whole or fractional, positive or negative.}$$

3. To differentiate a fraction.—Let $u=\frac{y}{z}$ where y, z, are functions of x,

$$\therefore\ uz=y,\ \therefore\ [1],\ udz+zdu=dy,\ \therefore\ du=\frac{dy-udz}{z},\ \text{that is,}$$

putting $\frac{y}{z}$ for u in the second member, $du=\frac{zdy-ydz}{z^2}$, $\therefore\ \frac{du}{dx}=\frac{1}{z^2}\left(z\frac{dy}{dx}-y\frac{dz}{dx}\right)$.

Otherwise. $\Delta u=\frac{y+\Delta y}{z+\Delta z}-\frac{y}{z}=\frac{z\Delta y-y\Delta z}{z(z+\Delta z)}$, $\therefore\ \frac{\Delta u}{\Delta x}=\frac{z\frac{\Delta y}{\Delta x}-y\frac{\Delta z}{\Delta x}}{z(z+\Delta z)}$. Hence when the differences become 0, $\frac{du}{dx}=\frac{1}{z^2}\left(z\frac{dy}{dx}-y\frac{dz}{dx}\right)$, $\therefore\ du=\frac{zdy-ydz}{z^2}\ldots[5]$.

RULE.—From the product of the denominator and differential of the numerator, subtract the product of the numerator and differential of the denominator, and divide the remainder by the square of the denominator.

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527. **Algebraic Functions.**—We shall now give a few examples of the application of the foregoing general rules to particular cases of algebraic functions.

To differentiate $y=f(x)$, the function $f(x)$ having the following forms:—

(1) $y=3x^4-2x^3+6x$.

By the rule for powers (526),

the differential of $3x^4$ is $12x^3dx$,

,, of $-2x^3$,, $-6x^2dx$,

,, of $6x$,, $6dx$,

$\therefore dy=(12x^3-6x^2+6)dx$,

$$\therefore \frac{dy}{dx}=12x^3-6x^2+6.$$

(2) $y=(3x^2+2x-4)^3$.

By the rule for powers (526),

$dy=3(3x^2+2x-4)^2d(3x^2+2x-4)$.

But the differential of $3x^2+2x-4$ is $(6x+2)dx$,

$\therefore dy=6(3x^2+2x-4)^2(3x+1)dx$,

$$\therefore \frac{dy}{dx}=6(3x^2+2x-4)^2(3x+1).$$

(3) $y=(x^3+a)(3x^2+b)$.

By the rule for a product (526), we have

$dy=(x^3+a)d(3x^2+b)+(3x^2+b)d(x^3+a)$.

And by that for powers,

$d(3x^2+b)=6xdx$, $d(x^3+a)=3x^2dx$,

$\therefore dy=(x^3+a)6xdx+(3x^2+b)3x^2dx$,

$$\therefore \frac{dy}{dx}=15x^4+3bx^2+6ax.$$

(4) $y=(a+bx^m)^n$. The differential of the root, or expression raised to the power n, is $mbx^{m-1}dx$, hence, by the rule for powers,

$$dy=n(a+bx^m)^{n-1}mbx^{m-1}dx,$$

$$\therefore \frac{dy}{dx}=bmn(a+bx^m)^{n-1}x^{m-1}.$$

(5) $y=\sqrt{(a+bx^2)}=(a+bx^2)^{\frac{1}{2}}$.

The differential of the expression under the radical is $2bxdx$, hence, by the rule for powers,

$$dy=\frac{1}{2}(a+bx^2)^{-\frac{1}{2}}2bxdx,$$

$$\therefore \frac{dy}{dx}=\frac{bx}{\sqrt{(a+bx^2)}}.$$

(6) $y=\dfrac{(x+3)^3}{(x^2-2)^2}$. By the rule for fractions,

$$dy=\frac{(x^2-2)^2d(x+3)^3-(x+3)^3d(x^2-2)^2}{(x^2-2)^4}.$$

By powers, $d(x+3)^3=3(x+3)^2dx$,

$d(x^2-2)^2=2(x^2-2)\times 2xdx$,

$$\therefore \frac{dy}{dx}=\frac{3(x^2-2)^2(x+3)^2-4x(x+3)^3(x^2-2)}{(x^2-2)^4}$$

$$=\frac{\{3(x^2-2)-4x(x+3)\}(x+3)^2}{(x^2-2)^3}$$

$$=-\frac{(x^2+12x+6)(x+3)^2}{(x^2-2)^3}.$$

(7) $y=a+\sqrt{\left(b+\dfrac{c}{x^2}\right)}$.

$$dy=\frac{1}{2}\left(b+\frac{c}{x^2}\right)^{-\frac{1}{2}}d\left(b+\frac{c}{x^2}\right).$$

By the rule for fractions,

$$d\left(b+\frac{c}{x^2}\right)=\frac{x^2\times 0-2cx}{x^4}dx=-\frac{2c}{x^3}dx,$$

$$\therefore \frac{dy}{dx}=-\frac{c}{x^3\sqrt{\left(b+\dfrac{c}{x^2}\right)}}.$$

(8) $y^2=ax^2+b$, $\therefore 2ydy=2axdx$,

$$\therefore \frac{dy}{dx}=\frac{ax}{y}=\frac{ax}{\sqrt{(ax^2+b)}}.$$

Examples for Exercise.

(1) $y=2x^3-4x^2+3x-2$.

(2) $y=a-3x^2$.

(3) $y=(2+3x)x^3$.

(4) $y=a+\dfrac{b}{x^2}$.

(5) $y=\dfrac{x^2}{a^2+x^2}$.

(6) $y=\dfrac{x^2}{(a+x^3)^2}$.

(7) $y=(a+bx)^{\frac{5}{4}}$.

(8) $y=\dfrac{x}{\sqrt{(1-x^2)}}$.

(9) $y=a\sqrt{x}+\dfrac{b}{x}$.

(10) $y=a+\dfrac{4\sqrt{x}}{3+x^2}$.

(11) $y=\{x+\sqrt{(x^2-1)}\}^2$.

(12) $y=\dfrac{1}{a^2}\{x^2+x\sqrt{(a^2+x^2)}\}$.

528. **Applications to Geometry.**—The general equation of a secant passing through two points (x', y'), (x'', y''), in any plane curve, is (352), (367), &c.

$$y-y'=\frac{y'-y''}{x'-x''}(x-x'), \text{ or } y-y'=\frac{\Delta y'}{\Delta x'}(x-x'), \text{ whatever be the value of } \Delta x'.$$

When $\Delta x'$, by continuously diminishing becomes $=0$, and consequently $\Delta y'=0$, the secant becomes a tangent at the point (x', y'): the equation of the tangent is therefore

$$y-y'=\frac{dy'}{dx'}(x-x').....[1],$$

so that if from the equation of the curve we find the general value of $\frac{dy}{dx}$, and then in the result we put the particular values of x, y, belonging to the proposed point, [1] will become the algebraic equation of the tangent at that point.

For ex. Let the curve be the ellipse

$$a^2y^2+b^2x^2=a^2b^2, \text{ or } a^2y^2=a^2b^2-b^2x^2.$$

$$\text{Differentiating, we have } 2a^2y\frac{dy}{dx}=-2b^2x, \therefore \frac{dy}{dx}=-\frac{b^2}{a^2}\cdot\frac{x}{y}.$$

If, therefore, (x', y') denote the point of contact with the tangent, [1] gives

$$y-y'=-\frac{b^2}{a^2}\cdot\frac{x'}{y'}(x-x'),$$

the equation of the tangent where a, b, are any semi-conjugates (see p. 320).

Again: let it be required to find the general equation of the tangent to a line of the second order, the equation of the line being

$$ay^2+bxy+cx^2+ey+kx+p=0.....[2].$$

$$\text{Differentiating, } 2ay\frac{dy}{dx}+b\left(x\frac{dy}{dx}+y\right)+2cx+e\frac{dy}{dx}+k=0,$$

$$\therefore \frac{dy}{dx}=-\frac{2cx+by+k}{2ay+bx+e}, \therefore y-y'=-\frac{2cx'+by'+k}{2ay'+bx'+e}(x-x')$$

is the equation of the tangent at the point (x', y') of the curve [2].

We thus see that in the application of the calculus to curves, the differential coefficient $\frac{dy}{dx}$ always denotes the *coefficient of inclination* of a tangent to the curve at the point (x, y). In the straight line too, namely,

$$y=ax+b, \text{ we also have } \frac{dy}{dx}=a.$$

When the axes to which the curve is referred are rectangular, $\frac{dy}{dx}$ is the trig. tangent of the angle which the linear tangent makes with the axis of x: in reference, therefore, to such axes, the general equation of the normal, at any point, (x', y') of a curve, is

$$y-y'=-\frac{1}{\frac{dy'}{dx'}}(x-x')\ldots\ldots[2].$$

And from these equations [1], [2], it is easy to deduce, as in the Analytical Geometry, expressions for the lengths T of the tangent, N of the normal, T_1 of the subtangent, and N_1 of the subnormal, whatever be the curve: thus

$$T_1=-y'\div\frac{dy'}{dx'},\ N_1=-y'\frac{dy'}{dx'},$$

$$T=\sqrt{(y'^2+T_1^2)}=y'\sqrt{\left\{1+\frac{1}{\left(\frac{dy'}{dx'}\right)^2}\right\}},\ N=\sqrt{(y'^2+N_1^2)}=y'\sqrt{\left\{1+\left(\frac{dy'}{dx'}\right)^2\right\}}.$$

Let the curve be the ellipse: then, as above,

$$\frac{dy'}{dx'}=-\frac{b^2}{a^2}\cdot\frac{x'}{y'};\ \text{and the subtangent is}$$

$$T_1=y'\times\frac{a^2y'}{b^2x'}=\frac{a^2y'^2}{b^2x'};\ \text{but } y'^2=\frac{b^2}{a^2}(a^2-x'^2),\ \therefore T_1=\frac{a^2-x'^2}{x'},\ \text{as at p. 322;}$$

and similarly of the other expressions. These few illustrations of the geometrical meaning of the differential coefficient $\frac{dy}{dx}$ will serve to give the student an insight into the use of this important symbol in the higher geometry: further examples of its application will be given hereafter.

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529. **Logarithmic and Exponential Functions.**—1. *To differentiate log x, of base a.*

Put $y=\log x$, then $\Delta y=\log(x+\Delta x)-\log x$,

$$\therefore\frac{\Delta y}{\Delta x}=\frac{\log\frac{x+\Delta x}{x}}{\Delta x}=\frac{\log\left(1+\frac{\Delta x}{x}\right)}{\Delta x}.$$

Now (110), $\log\left(1+\frac{\Delta x}{x}\right)=M\left\{\frac{\Delta x}{x}-\frac{1}{2}\left(\frac{\Delta x}{x}\right)^2+\frac{1}{3}\left(\frac{\Delta x}{x}\right)^3-\frac{1}{4}\left(\frac{\Delta x}{x}\right)^4+\ldots\right\}$,

$$\therefore\frac{\Delta y}{\Delta x}=\frac{\log\left(1+\frac{\Delta x}{x}\right)}{\Delta x}=M\left\{\frac{1}{x}-\frac{1}{2}\frac{\Delta x}{x}+\frac{1}{3}\left(\frac{\Delta x}{x}\right)^2-\frac{1}{4}\left(\frac{\Delta x}{x}\right)^3+\ldots\right\},$$

which, when Δx becomes 0, is $\frac{dy}{dx}=M\frac{1}{x}=\frac{1}{\log_e a}\cdot\frac{1}{x}$, $\therefore dy=\frac{1}{\log_e a}\cdot\frac{dx}{x}$.

In analysis, Napierian logs are almost exclusively employed, and for these, since $M=1$, we have $d\log_e x=\frac{dx}{x}$: hence—

The differential of the Napierian log of a variable is equal to the differential of the variable divided by the variable itself.

2. *To differentiate the exponential function a^x.*

Put $y=a^x$, then $\log_e y=x\log_e a$, $\therefore d\log_e y=\log_e a\,.\,dx$,

that is, $\frac{dy}{y}=\log_e a\,.\,dx$, $\therefore\ dy=\log_e a\,.\,a^x dx$.

If $y=e^x$, the differential coefficient is the same as the original function, that is, $\frac{dy}{dx}=e^x$. The rule, therefore, is this :—

To differentiate an exponential, *multiply together the Nap. log of the base, the exponential itself, and the differential of the variable exponent.*

(1) $y=\log_e \frac{x}{\sqrt{(a^2+x^2)}}=$

$\log_e x-\frac{1}{2}\log_e(a^2+x^2)$.

By the rule (1), $d\log_e x=\frac{dx}{x}$, and

$d\log_e(a^2+x^2)=\frac{2xdx}{a^2+x^2}$: hence

$\frac{dy}{dx}=\frac{1}{x}-\frac{x}{a^2+x^2}=\frac{a^2}{(a^2+x^2)x}$.

(2) $y=x(a^2+x^2)\sqrt{(a^2-x^2)}$.

Taking logs, $\log_e y=$

$\log_e x+\log_e(a^2+x^2)+\frac{1}{2}\log_e(a^2-x^2)$,

$\therefore \frac{dy}{y}=\frac{dx}{x}+\frac{2xdx}{a^2+x^2}-\frac{xdx}{a^2-x^2}=$

$\frac{a^4+a^2x^2-4x^4}{x(a^2+x^2)(a^2-x^2)}dx$, $\therefore \frac{dy}{dx}=$

$\frac{a^4+a^2x^2-4x^4}{x(a^2+x^2)(a^2-x^2)}y$; or putting for

y its value, $\frac{dy}{dx}=\frac{a^4+a^2x^2-4x^4}{\sqrt{(a^2-x^2)}}$.

(3) $y=x^n\sqrt{-1}$,

$\therefore \log_e y=n\sqrt{-1}\,.\log_e x$,

$\therefore \frac{dy}{y}=n\sqrt{-1}\,.\frac{dx}{x}$, $\therefore \frac{dy}{dx}=n\frac{y}{x}\sqrt{-1}$,

that is, $\frac{dy}{dx}=(n\sqrt{-1})x^{(n\sqrt{-1})-1}$.

A result which shows that the rule for powers equally applies when the exponent is imaginary.

(4) $y=\frac{e^x}{e^x+1}$, $\therefore dy=$

$\frac{(e^x+1)de^x-e^xde^x}{(e^x+1)^2}=\frac{(e^x+1-e^x)de^x}{(e^x+1)^2}$,

$\therefore \frac{dy}{dx}=\frac{e^x}{(e^x+1)^2}$.

(5) $y=x^z$, $\therefore \log_e y=z\log_e x$,

$\therefore \frac{dy}{y}=zd\log_e x+\log_e x dz$,

$=z\frac{dx}{x}+\log_e x dz$.

Multiplying by $y=x^z$,

$dy=\left(zx^{z}+x^z\log_e x\frac{dz}{dx}\right)dx$,

$\therefore \frac{dy}{dx}=zx^{z-1}+x^z\log_e x\frac{dz}{dx}$.

(6) $y=a^{x^x}$. Put $x^x=z$, $\therefore y=a^z$.

By rule (2), $dy=\log_e a\,.\,a^z dz$...[1].

We have, therefore, to differentiate $z=x^x$.

Taking logs, $\log_e z=x\log_e x$,

$\therefore$ (Rule 1), $\frac{dz}{z}=xd\log_e x+\log_e x dx$,

$=(1+\log_e x)dx$,

$\therefore dz=z(1+\log_e x)dx$.

Substituting this in [1], we have

$dy=\log_e a\,.\,a^{x^x}.x^x.(1+\log_e x)dx$.

(7) $y=\log_e(\log_e x)$. Put $\log_e x=z$, then

$y=\log_e z$, $\therefore dy=\frac{dz}{z}$: but $dz=\frac{dx}{x}$,

$\therefore dy=\frac{dx}{xz}=\frac{dx}{x\log_e x}$, $\therefore \frac{dy}{dx}=\frac{1}{x\log_e x}$.

NOTE.—Although in the above examples the subscribed e has been introduced to imply Napierian logs, yet in future it will often be omitted, Nap. logs being always understood whenever the base is not indicated.

EXAMPLES FOR EXERCISE.

(1) $y=x\log x$.

(2) $y=x\div\log x$.

(3) $y=a^{cx}$.

(4) $y=\log\frac{x}{a+\sqrt{(a^2+x^2)}}$.

(5) $y=\log\{\sqrt{(x^2\pm a^2)}+x\}$.

(6) $y=\log\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$.

(7) $y=a^{\log x}$.

(8) $y=e^{x^n}$.

(9) $y=x^{x^x}$.

(10) $\log y=\frac{\sqrt{(1+x^2)}}{x}$.

530. **Trigonometrical Functions.**—1. *To differentiate sin x.*

Put $y=\sin x$, then $\Delta y=\sin(x+\Delta x)-\sin x=$(page 200), $2\cos(x+\frac{1}{2}\Delta x)\sin\frac{1}{2}\Delta x$,

$$\therefore \frac{\Delta y}{\Delta x}=\cos(x+\tfrac{1}{2}\Delta x)\frac{\sin\frac{1}{2}\Delta x}{\frac{1}{2}\Delta x}.$$ Now when $\frac{1}{2}\Delta x=0$, $\frac{\sin\frac{1}{2}\Delta x}{\frac{1}{2}\Delta x}=1$ (p. 210),

$$\therefore \frac{dy}{dx}=\cos x, \ \therefore dy=\cos x dx.$$

2. *To differentiate cos x.*

Put $y=\cos x$, then $\Delta y=\cos(x+\Delta x)-\cos x=-2\sin(x+\frac{1}{2}\Delta x)\sin\frac{1}{2}\Delta x$ (p. 200),

$$\therefore \frac{\Delta y}{\Delta x}=-\sin(x+\tfrac{1}{2}\Delta x)\frac{\sin\frac{1}{2}\Delta x}{\frac{1}{2}\Delta x},$$ but when $\frac{1}{2}\Delta x=0$, $\frac{\sin\frac{1}{2}\Delta x}{\frac{1}{2}\Delta x}=1$,

$$\therefore \frac{dy}{dx}=-\sin x, \ \therefore dy=-\sin x dx.$$

Otherwise. $\cos^2 x=1-\sin^2 x$, $\therefore 2\cos x d\cos x=-2\sin x\cos x dx$,

$$\therefore d\cos x=-\sin x dx.$$

Since $\text{versin}\, x=1-\cos x$, $\therefore d\,\text{versin}\, x=-d\cos x=\sin x dx$.

3. *To differentiate tan x.*

$$\text{Since } \tan x=\frac{\sin x}{\cos x}, \ \therefore d\tan x=\frac{\cos x d\sin x-\sin x d\cos x}{\cos^2 x}=$$

$$\frac{\cos^2 x+\sin^2 x}{\cos^2 x}dx=\frac{1}{\cos^2 x}dx, \text{ that is, } d\tan x=\sec^2 x dx.$$

4. *To differentiate cot x.*

$$\text{Since } \cot x=\frac{\cos x}{\sin x}, \ \therefore d\cot x=\frac{\sin x d\cos x-\cos x d\sin x}{\sin^2 x},$$

$$\therefore \frac{d\cot x}{dx}=-\frac{\sin^2 x+\cos^2 x}{\sin^2 x}=-\frac{1}{\sin^2 x}=-\text{cosec}^2 x.$$ Or since $\cot x=\tan(90°-x)$,

$$\therefore d\cot x=d\tan(90°-x)=\sec^2(90°-x)\times -dx=-\text{cosec}^2 x dx.$$

5. *To differentiate sec x.*

$$\text{Since } \sec x=\frac{1}{\cos x}, \ \therefore d\sec x=d\frac{1}{\cos x}=\frac{\sin x}{\cos^2 x}dx, \text{ that is, } d\sec x=\tan x\sec x dx.$$

6. *To differentiate cosec x.*

$$\text{Here } d\,\text{cosec}\, x=d\frac{1}{\sin x}=-\frac{\cos x}{\sin^2 x}dx=-\cot x\,\text{cosec}\, x dx.$$

Collecting these results together, we have

$d\sin x=\cos x dx.$	$d\sec x=\tan x\sec x dx.$
$d\cos x=-\sin x dx.$	$d\,\text{cosec}\, x=-\cot x\,\text{cosec}\, x dx.$
$d\,\text{versin}\, x=\sin x dx.$	
$d\tan x=\sec^2 x dx.$	These forms it will be necessary to keep
$d\cot x=-\text{cosec}^2 x dx.$	in remembrance.

531. **Inverse Functions.**—In the preceding article, the independent variable was the angle; the sine, cosine, &c., being functions of it. We have now to consider the *inverse* functions, the angle being regarded as a function of the sine, cosine, &c. If $y=F(x)$ be any direct function, then the inverse function will be denoted by $x=F^{-1}(y)$. Thus,

$y=\log x$, $y=\sin x$, $y=\cos x$, &c., being direct functions, their inverses are $x=\log^{-1} y$, $x=\sin^{-1} y$, $x=\cos^{-1} y$, as already stated at p. 207.

(1) To differentiate $y=\sin x$. Taking the direct function, we have $\sin y=x$, and differentiating, $\cos y dy=dx$, $\therefore \frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{(1-x^2)}}$.

(2) To differentiate $y=\cos^{-1} x$, $\therefore \cos y=x$, $\therefore -\sin y dy=dx$, $\therefore \frac{dy}{dx}=\frac{-1}{\sqrt{(1-x^2)}}$.

(3) To differentiate $y=\text{versin}^{-1} x$, $\therefore \text{versin } y=x$, $\therefore \sin y dy=dx$, $\therefore \frac{dy}{dx}=\frac{1}{\sqrt{(2-x)x}}$.

(4) To differentiate $y=\tan^{-1} x$, $\therefore \tan y=x$, $\therefore \sec^2 y dy=dx$, $\therefore \frac{dy}{dx}=\frac{1}{1+x^2}$.

(5) To differentiate $y=\cot^{-1} x$, $\therefore \cot y=x$, $\therefore -\text{cosec}^2 y dy=dx$, $\therefore \frac{dy}{dx}=\frac{-1}{1+x^2}$.

(6) To differentiate $y=\sec^{-1} x$, $\therefore \sec y=x$, $\therefore \tan y \sec y dy=dx$, $\therefore \frac{dy}{dx}=\frac{1}{x\sqrt{(x^2-1)}}$.

(7) To differentiate $y=\text{cosec}^{-1} x$, $\therefore \text{cosec } y=x$, $\therefore -\cot y \text{ cosec } y dy=dx$, $\therefore \frac{dy}{dx}=\frac{-1}{x\sqrt{(x^2-1)}}$.

(8) To differentiate $y=\log x$, $\therefore \log y=x$, $\therefore \frac{dy}{y}=dx$, $\therefore \frac{dy}{dx}=\log^{-1} x$.

This last example is the same as $y=e^x$, since e^x is the number whose Nap. log is x, that is, it is $\log^{-1} x$.

The preceding forms, like those in the former article, being in frequent request, should be kept in the memory. But $\frac{dy}{dx}$ here is always the reciprocal of $\frac{dy}{dx}$ for the direct functions, when y is put for x.

(1) $y=\sin^n x$, $\therefore dy=n \sin^{n-1} d \sin x$,
$=n \sin^{n-1} \cos x dx$,
$\therefore \frac{dy}{dx}=n \sin^{n-1} \cos x$.

(2) $y=\cos x+\cos 2x+\cos 3x+\ldots$,
$\therefore \frac{dy}{dx}=-(\sin x+2 \sin 2x+3 \sin 3x+\ldots)$.

(3) $u=y \tan x^2$, y a function of x,
$du=yd \tan x^2+\tan x^2 dy$.
But $d \tan x^2=\sec^2 x^2 d.x^2=2x \sec^2 x^2 dx$,
$\therefore \frac{du}{dx}=2yx \sec^2 x^2+\tan x^2 \frac{dy}{dx}$.

(4) $y=xe^{\cos x}$,
$\therefore dy=xde^{\cos x}+e^{\cos x} dx$,
$\therefore \frac{dy}{dx}=e^{\cos x}(1-x \sin x)$.

(5) $y=x \sin^{-1} x^2$,
$\therefore dy=xd \sin^{-1} x^2+\sin^{-1} x^2 dx$.
But $d \sin^{-1} x^2=\frac{2x}{\sqrt{(1-x^4)}} dx$,
$\therefore \frac{dy}{dx}=\frac{2x^2}{\sqrt{(1-x^4)}}+\sin^{-1} x^2$.

(6) $y=\tan^{-1} \frac{x}{2}$,
$\therefore \frac{dy}{dx}=\frac{\frac{1}{2}}{1+\left(\frac{x}{2}\right)^2}=\frac{2}{4+x^2}$.

(7) $y=\cot^{-1}(a+bx)^2$,
$\therefore dy=\frac{-1}{1+(a+bx)^4} d(a+bx)^2$.
But $d(a+bx)^2=2(a+bx)bdx$,
$\therefore \frac{dy}{dx}=-\frac{b(a+bx)}{1+(a+bx)^4}$.

(8) $y=(\sin^{-1} x)^2$,
$\therefore dy=2 \sin^{-1} x d \sin^{-1} x$,
$\therefore \frac{dy}{dx}=\frac{2}{\sqrt{(1-x^2)}} \sin^{-1} x$.

Examples for Exercise.

(1) $y=\cos ax$.
(2) $y=\log(xe^{\cos x})$.
(3) $y=xe^{\tan x}$.
(4) $u=\cot x^y$.
(5) $y=\sin^{-1} ax$.
(6) $y=\sec^{-1} x^n$.
(7) $y=\text{versin}^{-1} e^x$.
(8) $y=\text{cosec}^{-1} mx^2$.
(9) $y=\sin^{-1}\frac{1-x^2}{1+x^2}$.
(10) $y=\tan^{-1}\frac{2x}{1-x^2}$.
(11) $y=\cos^{-1}\frac{1}{\sqrt{(1+x^2)}}$.
(12) $y=\log\sqrt{\frac{1+\sin x}{1-\sin x}}$.

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532. In the preceding articles rules have been given for differentiating any elementary function, and by the combination of these rules, functions however complicated may be differentiated, as several of the examples worked out above sufficiently show. Whatever be the function $F(x)$ therefore, we can always find the limit of the ratio $\frac{F(x+h)-F(x)}{h}$.

If $F(x+h)$ be developable in a series of terms, proceeding according to the positive integer powers of h, that is, if

$$F(x+h)=A+Bh+Ch^2+\ldots,$$

the limiting ratio spoken of will always be the coefficient of the first power of h in that development, that is, it will be B. For the first term A must be $=F(x)$, since in the above identity, if $h=0$, we must have

$$F(x)=A, \text{ so that } F(x+h)-F(x)=Bh+Ch^2+\ldots,$$

and therefore the limiting ratio must be $\frac{dF(x)}{dx}=B$. Hence, whenever $F(x+h)$ is developable in the above form, we may be sure that that development is always

$$F(x+h)=F(x)+\frac{dF(x)}{dx}h+Ch^2+\ldots\ldots$$

If page 116 be referred to, it will be seen that what was there called the limiting equation, derived from the algebraic equation $F(x)=0$, is no other than what is now shown to be the differential coefficient derived from $F(x)$ equated to 0, that is, the limiting equation is $\frac{dF(x)}{dx}=0$. In like manner, in Newton's method of approximation (p. 135), the correction $x'=-\frac{f(r)}{f_1(r)}$, is no other than the original polynomial, divided by its differential coefficient, when in each r is put for x. And similarly in the theory of vanishing fractions (p. 144)

$$\frac{F_1(x)}{f_1(x)}, \frac{F_2(x)}{f_2(x)}, \text{ &c., merely imply that num. and den. of } \frac{F(x)}{f(x)}$$

are each to be *differentiated*, then the num. and den. of the resulting fraction, and so on.

533. The student must not fall into the mistake of supposing that he may apply to an algebraic or numerical equation the operation called differentiation, as he would apply the ordinary operations of algebra, and thus get a new equation which holds simultaneously with the original:

the equation $F(x)=0$, does not imply that $F_1(x)=0$ $\left(\text{where } F_1(x) \text{ is put for } \frac{dF(x)}{dx}\right)$

unless $F(x)$ fulfils a special condition:—unless, in fact, it involves equal factors, that is, unless $F(x)=0$ has equal roots. It must be always remembered that the differential calculus is exclusively occupied with quantities which, without any violation of the conditions to which they may otherwise be subjected, may be regarded as *varying continuously*. If the conditions be such as to forbid this continuous variation, the calculus becomes inapplicable. In the algebraic equation $F(x)=0$, x has certain determinate values:—the *roots* of the equation, to the exclusion of all other values; that is to say, x has, not *variable*, but *constant*, values. If, however, we remove the restriction that no values of x are admissible, except those which make $F(x)=0$, and take the expression $F(x)$ independently of all conditions, *then*, of course, we may treat x as a continuous variable and freely apply the calculus. Thus, putting $x+h$ for x, we may write, as above,

$$F(x+h)=F(x)+F_1(x)h+Ch^2+\ldots\ldots,$$

and are at full liberty to impose the conditions $F(x)=0$, $F_1(x)=0$; the consequence of which would be that for every value a of x which satisfied them both, we should have

$$F(a+h)=Ch^2+\ldots\ldots$$

Now, by giving a sufficiently small value to h, the series on the right, for that small value, and for all values still smaller, would, as a whole, have the same sign as C, whether that small value of h be positive or negative (p. 150). But we know that if $F(a+h)$ and $F(a-h)$ have the same sign, then between the numbers $a+h$ and $a-h$ there must be an even number of roots of the equation $F(x)=0$, or else no roots at all. Now *here* the condition is that one root, namely, a, necessarily lies between $a+h$ and $a-h$: hence two roots, at least, must so lie, and that however small be the interval $a+h$, $a-h$, $\therefore$ the equation must have *two* roots at least equal to a, provided the condition $F_1(x)=0$ have place simultaneously with $F(x)=0$. This accords with what has been already established in the theory of equations, and we advert to that subject here only for the purpose of showing thus early, as at art. (528), that differential coefficients have interesting and important bearings upon geometrical and algebraic inquiries.

Having thus established the fundamental rules and principles of the differential calculus, we shall postpone for a while the further development of those principles; and shall, in like manner, now unfold the leading elementary operations of the Integral Calculus.

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534. **Integration.**—This is the name applied to the operation which is the inverse of differentiation. When a function is given, its differential may be found by the preceding rules: when, on the contrary, the differential is given, the corresponding function—called the *integral* of that

differential—must be found by the rules of the *integral calculus*. The symbol used to indicate that the integral of any proposed differential is to be taken, is $\int$, placed before the expression; so that the symbols d and $\int$ are symbols of operation the reverse of each other:—both prefixed to a function, leave that function in its original state: for instance, $\int dF(x)$ is $F(x)$, and $d\int F_1(x)dx$ is $(F_1x)dx$. The following three principles are at once suggested by the direct process:—

1. Since $daF(x)$ is the same as $adF(x)$, namely, $aF_1(x)dx$, it follows that in the reverse operation, $\int aF_1(x)dx$ is the same as $a\int F_1(x)dx$; so that, whether in differentiating or integrating, any constant factor or divisor may be taken from under the sign of operation and placed before it: thus, daX is the same as adX, and $\int aXdx$ the same as $a\int Xdx$.

2. Since the differential of the sum of any number of functions is the same as the sum of their several differentials, it follows that, when we have to integrate the sum of any number of differentials, as for instance,

$$Xdx+X_1dx+X_2dx+\&c.,$$

the integral will be equally indicated, whether the sign $\int$ be prefixed to the sum as a whole, or to each individual term; that is,

$$\int(Xdx+X_1dx+X_2dx+\&c.)=\int Xdx+\int X_1dx+\int X_2dx+\&c.$$

3. Since a constant connected with a function, by addition or subtraction, disappears in the differential, it follows that in the result of the reverse operation of returning from the differential to the primitive function, the constant should reappear. But as the differential remains just the same, whatever the constant in the primitive or integral may have been, we cannot possibly know, from the mere differential being given, what particular constant has thus been discarded. All that we can do, therefore, is to integrate as if no additive or subtractive constant at all entered the primitive we are returning to, and then to annex to the bare integral thus found, a symbol C, standing for a constant of indeterminate value. If, however, we can discover what particular value the constant ought to take, for any one particular value of the variable, then, since it *is* constant, that so-called particular value of it is the *only* value it can have.

We shall now proceed to the elementary rules for integration; and, in general, shall frequently write for brevity Fx, instead of $F(x)$.

535. **Integration of the Form** $(Fx)^n d(Fx)$.—It is plain that this differs from the differential of $(Fx)^{n+1}+a$, only in this,—that it is not multiplied by the factor $(n+1)$, for (p. 495),

$$d[(Fx)^{n+1}+a]=(n+1)(Fx)^n d(Fx),\ \therefore\ \int(Fx)^n d(Fx)=\frac{(Fx)^{n+1}}{n+1}+C.$$

Rule.—Increase the exponent of the function under the vinculum by unity: divide the power thus increased by the increased exponent, and annex the arbitrary constant C.

(1) Integrate ax^3dx.

$$a\int x^3dx=\frac{ax^4}{4}+C.$$

(2) Integrate $\frac{3dx}{5x^2}$.

$$\frac{3}{5}\int x^{-2}dx=\frac{3}{5}\frac{x^{-1}}{-1}+C=-\frac{3}{5x}+C.$$

(3) Integrate $(a+x^2)^3xdx$.

Here the differential without the vinculum, namely, xdx, would be the complete differential of the expression within, and therefore the above general form would be complied with, if it were multiplied by 2, seeing that of $a+x^2$ the differential is not

xdx, but $2xdx$: hence, introducing the wanting factor 2, and dividing also by 2, we have

$$\int (a+x^2)^3 xdx = \frac{1}{2}\int (a+x^2)^3 . 2xdx =$$

$$\frac{1}{2}\frac{(a+x^2)^4}{4} + C = \frac{(a+x^2)^4}{8} + C.$$

(4) Integrate $(b+cx^n)^m ax^{n-1}dx$.

Here it is easily seen that the differential without the vinculum requires to be multiplied by $\frac{nc}{a}$, $\therefore \frac{a}{nc}\int (b+cx^n)^m ncx^{n-1}dx =$

$$\frac{a(b+cx^n)^{m+1}}{nc(m+1)} + C.$$

(5) Integrate $(2ax-x^2)^{\frac{5}{2}}(a-x)dx$.

Multiplying the differential by 2,

$$\frac{1}{2}\int (2ax-x^2)^{\frac{5}{2}}(2a-2x)dx =$$

$$\frac{(2ax-x^2)^{\frac{7}{2}}}{7} + C.$$

NOTE.—In some cases it may not be easy to discover by inspection whether a factor, and if it exist, what factor, is necessary to render the expression without the vinculum the differential of that within it: but we may always settle the matter thus. Taking the last example, assume its integral $= A(2ax-x^2)^{\frac{7}{2}} + C$, then differentiating, we must have

$$(2ax-x^2)^{\frac{5}{2}}(a-x)dx =$$

$$\frac{7}{2}A(2ax-x^2)^{\frac{5}{2}}(2a-2x)dx,$$

$\therefore 7A(a-x) = a-x$, $\therefore 7Aa = a$, $7Ax = x$,

which conditions agree in giving $A = \frac{1}{7}$,

so that the integral is $\frac{1}{7}(2ax-x^2)^{\frac{7}{2}} + C$.

But if the example had been

$$(2ax-x^2)^{\frac{5}{2}}(5a-x)dx,$$

then, as before, assuming the integral to be $A(2ax-x^2)^{\frac{7}{2}} + C$, and differentiating it, we should have $(2ax-x^2)^{\frac{5}{2}}(5a-x) =$

$\frac{7}{2}A(2ax-x^2)^{\frac{5}{2}}(2a-2x)$, $\therefore 5 = 7A$, $1 = 7A$,

which are contradictory: we infer, therefore, that the differential is not convertible into the proposed form. We shall add but one example more.

(6) $\int (a+bx)^2 dx = \frac{1}{b}\int (a+bx)^2 bdx =$

$\frac{(a+bx)^3}{3b} + C$. But we may proceed thus:

$$\int (a+bx)^2 dx =$$

$$\int (a^2dx + 2abxdx + b^2x^2dx) =$$

$a^2x + abx^2 + \frac{b^2x^3}{3} + C$. Now there is an apparent discrepancy here; for, as it is easy to see, $\frac{(a+bx)^3}{3b}$ is not equal to $a^2x + abx^2 + \frac{b^2x^3}{3}$, and yet, whichever of these unequal expressions we differentiate, the result will be the proposed differential. This fact alone would suggest to us that the two results can differ only by an additive or subtractive constant, and accordingly we find that they differ only by the constant $\frac{a^3}{3b}$. Hence, in the second mode of integrating, the constant C differs from the constant in the first result by $\frac{a^3}{3b}$. In either result the constant is arbitrary; but if its value be fixed in one, it is fixed in the other, in virtue of this constant difference between them.

It thus behoves the student, when he arrives at an integral different from the integral of the same differential as obtained by another person, not to conclude that either is incorrect:—both will be equally correct if they differ only by a constant.

EXAMPLES FOR EXERCISE.

(1) $\int \frac{a}{x^3}dx =$

(2) $\int (a - \frac{b}{x^3} + x^{\frac{3}{2}})dx =$

(3) $6\int \sqrt{(4x^2+3)}xdx =$

(4) $\int \frac{3x^4}{x^5 \pm a}dx =$ [See (536)]

(5) $\int \frac{2adx}{x\sqrt{(2ax-x^2)}} =$

(6) $\int \frac{xdx}{(2ax-x^2)^{\frac{3}{2}}} =$

(7) $\int \frac{x^8 dx}{\sqrt{(a^9+6x^9)}} =$

(8) $\int \frac{\sqrt{(a^2+x^2)}}{x^4} dx =$

(9) $\int \frac{1-x^n}{1-x} dx =$

(10) $\int \frac{6x^5(x^6-1)}{1-x} dx =$

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536. There is one case, coming under the general form above, which is not the differential of any power, and to which, therefore, the foregoing rule for the integration of it does not apply: it is the case in which $n=-1$, the form being $\frac{dX}{X}$, which we know to be the differential not of a power of X but of $\log X$ (529): hence

$$\int \frac{dX}{X} = \log X + C = \log cX,$$ putting c for the number whose log is C.

And this formula is always to be employed when the proposed differential is a fraction whose numerator is the differential of the denominator, thus:—

(1) $\int \frac{3x^2 dx}{x^3+a^2} = \log C(x^3+a^2).$

(2) $\int \frac{a dx}{a+bx} = \frac{a}{b}\int \frac{b dx}{a+bx} =$

$\frac{a}{b} \log C(a+bx).$

(3) $\int \frac{5x^3 dx}{3x^4+7} = \frac{5}{12} \log C(3x^4+7).$

(4) $\int \frac{(x-a)^4 dx}{x^2} =$

$\frac{x^2}{2} - 3ax + \frac{a^3}{x} + 3a^2 \log x + C.$

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537. Integration of the Forms $\frac{dx}{\sqrt{(a^2-b^2x^2)}}$, $\frac{dx}{a^2+b^2x^2}$,

$$\frac{dx}{x\sqrt{(b^2x^2-a^2)}}, \quad \frac{dx}{\sqrt{(a^2x-b^2x^2)}}.$$

Since (531) $d \sin^{-1} x = \frac{dx}{\sqrt{(1-x^2)}}$, $\therefore \frac{1}{b} d \sin^{-1} \frac{b}{a} x = \frac{dx}{\sqrt{(a^2-b^2x^2)}}$,

$$\therefore \int \frac{dx}{\sqrt{(a^2-b^2x^2)}} = \frac{1}{b} \sin^{-1} \frac{b}{a} x.$$

,, $d \cos^{-1} x = \frac{-dx}{\sqrt{(1-x^2)}}$, $\therefore \frac{1}{b} d \cos^{-1} \frac{b}{a} x = \frac{-dx}{\sqrt{(a^2-b^2x^2)}}$,

$$\therefore \int \frac{-dx}{\sqrt{(a^2-b^2x^2)}} = \frac{1}{b} \cos^{-1} \frac{b}{a} x.$$

,, $d \tan^{-1} x = \frac{dx}{1+x^2}$, $\therefore \frac{1}{ab} d \tan^{-1} \frac{b}{a} x = \frac{dx}{a^2+b^2x^2}$,

$$\therefore \int \frac{dx}{a^2+b^2x^2} = \frac{1}{ab} \tan^{-1} \frac{b}{a} x.$$

,, $d \cot^{-1} x = \frac{-dx}{1+x^2}$, $\therefore \frac{1}{ab} d \cot^{-1} \frac{b}{a} x = \frac{-dx}{a^2+b^2x^2}$,

$$\therefore \int \frac{-dx}{a^2+b^2x^2} = \frac{1}{ab} \cot^{-1} \frac{b}{a} x.$$

Since (531) $d\sec^{-1}x=\frac{dx}{x\sqrt{(x^2-1)}}$, $\therefore \frac{1}{a}d\sec^{-1}\frac{b}{a}x=\frac{dx}{x\sqrt{(b^2x^2-a^2)}}$,

$$\therefore \int\frac{dx}{x\sqrt{(b^2x^2-a^2)}}=\frac{1}{a}\sec^{-1}\frac{b}{a}x.$$

,, $d\operatorname{cosec}^{-1}x=\frac{-dx}{x\sqrt{(x^2-1)}}$, $\therefore \frac{1}{a}d\operatorname{cosec}^{-1}\frac{b}{a}x=\frac{-dx}{x\sqrt{(b^2x^2-a^2)}}$,

$$\therefore \int\frac{-dx}{x\sqrt{(b^2x^2-a^2)}}=\frac{1}{a}\operatorname{cosec}^{-1}\frac{b}{a}x.$$

,, $d\operatorname{versin}^{-1}x=\frac{dx}{\sqrt{(2x-x^2)}}$, $\therefore \frac{1}{b}d\operatorname{versin}^{-1}\frac{2b^2}{a^2}x=\frac{dx}{\sqrt{(a^2x-b^2x^2)}}$,

$$\therefore \int\frac{dx}{\sqrt{(a^2x-b^2x^2)}}=\frac{1}{b}\operatorname{versin}^{-1}\frac{2b^2}{a}x.$$

,, $d\operatorname{coversin}^{-1}x=\frac{-dx}{\sqrt{(2x-x^2)}}$, $\therefore \frac{1}{b}d\operatorname{coversin}^{-1}\frac{2b^2}{a^2}x=\frac{-dx}{\sqrt{(a^2x-b^2x^2)}}$,

$$\therefore \int\frac{-dx}{\sqrt{(a^2x-b^2x^2)}}=\frac{1}{b}\operatorname{coversin}^{-1}\frac{2b^2}{a}x.$$

Besides these forms, there are the following, deducible like them from the elementary differentials already given, namely,

Since $da^x=\log_e a\,.\,a^x dx$, $\therefore \int a^x dx=\frac{a^x}{\log_e a}$, $\therefore \int e^x dx=e^x$.

,, $d\sin x=\cos x dx$, $\therefore \int\cos x dx=\sin x$.

,, $d\cos x=-\sin x dx$, $\therefore \int\sin x dx=-\cos x$.

,, $d\tan x=\frac{dx}{\cos^2 x}$, $\therefore \int\frac{dx}{\cos^2 x}=\tan x$.

,, $d\cot x=\frac{dx}{\sin^2 x}$, $\therefore \int\frac{dx}{\sin^2 x}=\cot x$.

,, $d\sec x=\tan x\sec x dx$, $\therefore \int\tan x\sec x dx=\sec x$.

,, $d\operatorname{cosec}x=-\cot x\operatorname{cosec}x dx$, $\therefore \int\cot x\operatorname{cosec}x dx=-\operatorname{cosec}x$.

NOTE.—It will be seen in the table of the differentials of the circular or angular functions (p. 501) that the differential of an angle is the same as the differential of the complementary angle with opposite sign: thus,

$$d\sin^{-1}x=-d\cos^{-1}x,\ d\tan^{-1}x=-d\cot^{-1}x,\ \&c.$$

Such ought evidently to be the case, because *every* increment or increase of an angle θ is so much decrement or diminution of the complement $\frac{\pi}{2}-\theta$. In returning, however, from the differentials to their integrals, since we get first

$$\int\frac{dx}{\sqrt{(1-x^2)}}=\sin^{-1}x, \text{ and then, } \int\frac{dx}{\sqrt{(1-x^2)}}=-\cos^{-1}x,$$

the result of the operation might seem to imply that $\sin^{-1}x=-\cos^{-1}x$: but it must be remembered that every integral to be complete must have an arbitrary constant appended to it, so that, in their complete state, the integrals just written would be $\sin^{-1}x+C$, and $-\cos^{-1}x+C'$, where the

constants C, C', must be such as to render these identical; that is, such that

$$\sin^{-1}x+\cos^{-1}x+C-C'=0.$$

Now $\sin^{-1}x+\cos^{-1}x$=one or other of the values $\left(2n+\frac{1}{2}\right)\pi=C'-C$,

$$\therefore \sin^{-1}x+C=-\cos^{-1}x+\left(2n+\frac{1}{2}\right)\pi+C,\ \therefore \sin^{-1}x=\left(2n+\frac{1}{2}\right)\pi-\cos^{-1}x,$$

the left-hand member being always identical with one or other of the values of the right-hand member.

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538. **Integration of the Forms** $\frac{dx}{\sqrt{(x^2\pm a^2)}}$, $\frac{dx}{x^2\sim a^2}$, $\frac{dx}{x\sqrt{(a^2\pm x^2)}}$, $\frac{dx}{\sqrt{(2ax+x^2)}}$.

1. Put $x^2\pm a^2=u^2$, $xdx=udu$, $\therefore \frac{dx}{u}=\frac{du}{x}=\frac{dx+du}{x+u}=\frac{d(x+u)}{x+u}$,

$$\therefore \int\frac{dx}{u}=\int\frac{dx}{\sqrt{(x^2\pm a^2)}}=\int\frac{d(x+u)}{x+u}=\log(x+u)=\log\{x+\sqrt{(x^2\pm a^2)}\}.$$

2. Since $\frac{1}{x^2-a^2}=\frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)$, $\therefore \int\frac{dx}{x^2-a^2}=\frac{1}{2a}\left\{\int\frac{dx}{x-a}-\int\frac{dx}{x+a}\right\}$,

that is, $\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\frac{x-a}{x+a}$, $\therefore$ putting $-x$ for x, $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\frac{a+x}{a-x}$.

3. Put $x=\frac{a}{u}$, $\therefore \log x=\log a-\log u$, $\therefore \frac{dx}{x}=-\frac{du}{u}$.

Also $a^2\pm x^2=a^2\pm\frac{a^2}{u^2}=a^2\left(1\pm\frac{1}{u^2}\right)$, $\therefore \int\frac{dx}{x\sqrt{(a^2\pm x^2)}}=\int\frac{-du}{au\sqrt{\left(1\pm\frac{1}{u^2}\right)}}=$

$$\int\frac{-du}{a\sqrt{(u^2\pm1)}},\ \text{which, by the first form, }=-\frac{1}{a}\log\{u+\sqrt{(u^2\pm1)}\},$$

$$\therefore \int\frac{dx}{x\sqrt{(a^2\pm x^2)}}=-\frac{1}{a}\log\left\{\frac{a}{x}+\sqrt{\left(\frac{a^2}{x^2}\pm1\right)}\right\}=\frac{1}{a}\log\frac{x}{a+\sqrt{(a^2\pm x^2)}}.$$

4. Since $2ax+x^2=(x+a)^2-a^2$, $\therefore \int\frac{dx}{\sqrt{(2ax+x^2)}}=\int\frac{d(x+a)}{\sqrt{\{(x+a)^2-a^2\}}}$,

which, by the first form, is $=\log\{x+a+\sqrt{(2ax+x^2)}\}$.

These four integrals, we see, are arrived at by the aid of such algebraic expedients as are fitted to bring them under the known form $\frac{dX}{X}$ for a logarithm; and by means of similar artifices many other differentials to which the preceding rules are not immediately applicable may be transformed into equivalent differentials, which are integrable by the forms already given: but no general precepts can be laid down for bringing this conformity about. The following, however, is a principle of wide application.

539. From the differential of a product, namely, $d(uv)=udv+vdu$, we have

$$uv=\int udv+\int vdu, \quad \therefore \int udv=uv-\int vdu \ldots [1],$$

so that we can always integrate udv, provided we can integrate vdu.

If for u we put $\frac{1}{u}$ the formula is $\int \frac{dv}{u}=\frac{v}{u}+\int \frac{v}{u}\frac{du}{u}$.

The following are examples of *integration by parts*, as this method is called.

(1) To integrate $xe^x dx$. Put x for u in [1], and $e^x dx$ for dv, $\therefore e^x=v$, and $\int vdu=\int e^x dx=e^x$; and we shall have $\int xe^x dx=\int udv=uv-\int vdu=xe^x-e^x$.

(2) To integrate $x \log x dx$. Put $\log x=u$, and $xdx=dv$, $\therefore du=\frac{dx}{x}$, and

$$v=\frac{1}{2}x^2: \text{ also } \int vdu=\frac{1}{2}\int xdx=\frac{1}{4}x^2: \text{ hence}$$

$$\int x \log x dx=\int udv=uv-\int vdu=\frac{1}{2}x^2 \log x-\frac{1}{4}x^2.$$

(3) To integrate $x^m \log x dx$. Put $\log x=u$, $x^m dx=dv$, $\therefore du=\frac{dx}{x}$, $v=\frac{x^{m+1}}{m+1}$;

$$\text{also } \int vdu=\int \frac{x^m}{m+1}dx=\frac{x^{m+1}}{(m+1)^2},$$

$$\therefore \int x^m \log x dx=\int udv=uv-\int vdu=\frac{x^{m+1}\log x}{m+1}-\frac{x^{m+1}}{(m+1)^2}=\frac{x^{m+1}}{m+1}\left\{\log x-\frac{1}{m+1}\right\},$$

which becomes the former case when $m=1$.

(4) To integrate $x^m(\log x)^n dx$. Put $(\log x)^n=u$, and $x^m dx=dv$,

$\therefore du=\frac{n(\log x)^{n-1}dx}{x}$, and $v=\frac{x^{m+1}}{m+1}$; also $\int vdu=\frac{n}{m+1}\int x^m(\log x)^{n-1}dx$: hence,

$$\int x^m (\log x)^n dx=\int udv=uv-\int vdu=\frac{x^{m+1}(\log x)^n}{m+1}-\frac{n}{m+1}\int x^m (\log x)^{n-1}dx.$$

This last can be integrated by the above case if $n=2$, $\therefore$ the integration can be effected when $n=3$, $n=4$, &c.: we thus have

$$\int x^m (\log x)^2 dx=\frac{x^{m+1}}{m+1}\left\{(\log x)^2-\frac{2}{m+1}\log x+\frac{2.1}{m+1}\right\},$$

$$\int x^m (\log x)^3 dx=\frac{x^{m+1}}{m+1}\left\{(\log x)^3-\frac{3}{m+1}(\log x)^2+\frac{3.2}{(m+1)^2}\log x-\frac{3.2.1}{(m+1)^2}\right\},$$

$$\vdots$$

$$\int x^m (\log x)^n dx=$$

$$\frac{x^{m+1}}{m+1}\left\{(\log x)^n-\frac{n}{m+1}(\log x)^{n-1}+\frac{n(n-1)}{(m+1)^2}(\log x)^{n-2}-\ldots\pm\frac{n(n-1)\ldots 2.1}{(m+1)^n}\right\}.$$

If $m=-1$, this cannot be computed, as all the denominators are then 0; but the formula is not required in this case, for then

$$\int (\log x)^n \frac{dx}{x}=\int (\log x)^n d\log x=\frac{(\log x)^{n+1}}{n+1}.$$

(5) To integrate $\frac{x^2dx}{(1-x^2)^2}$. This is the same as $-\frac{1}{2}x.\frac{d(1-x^2)}{(1-x^2)^2}$. Put $\frac{1}{2}x=u$,

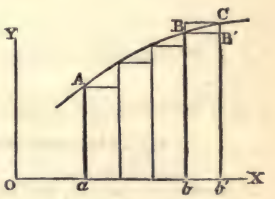
and $\frac{d(1-x^2)}{(1-x^2)^2}=dv$, $\therefore du=\frac{1}{2}dx$, and (535) $v=\frac{1}{1-x^2}$; also

$$\int vdu=\frac{1}{2}\int\frac{dx}{(1-x^2)}=\frac{1}{4}\int\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx=\frac{1}{4}\log\frac{1+x}{1-x},$$

hence $$\int\frac{x^2dx}{(1-x^2)^2}=\int udv=uv-\int vdu=\frac{1}{2}.\frac{x}{1-x^2}+\frac{1}{4}\log\frac{1+x}{1-x}.$$

NOTE.—It must not be overlooked that in all these integrations the arbitrary constant is suppressed: it must never be forgotten that the introduction of such a constant is always necessary to complete the integral, and to justify us in equating integrals of the same differential arrived at in different ways, and appearing under different forms, as noticed at p. 505. Thus, in the first of the integrals at (538), if we make $a=0$, it becomes $\int\frac{dx}{x}$, which, without the constant, is log x, whereas the general integral without the constant, when $a=0$, is log $2x$. But, supplying the omitted constants, the former integral in a complete form is log Cx, and the latter log $2C'x$, or log Cx, so that there is no contradiction.

540. **Areas of Curves: Definite Integrals.**—Let AB be a portion of any curve referred to rectangular axes OX, OY, and let it be required to find an expression for the area included between the ordinates aA, bB, which we shall consider as the first and nth ordinates, and shall denote by y_1, y_n. Let ab be divided into $n-1$ equal parts, each part being denoted by Δx, and take another part bb' also equal to Δx, so that there may be n equal parts in ab'. Then, drawing the parallels as in the figure, there will be n rectangles all of equal breadth between aA, and $b'B'$, and the area of the polygonal figure aB', which is the sum of these, will be



$$\text{Area of Polygon}=(y_1+y_2+y_3+\ldots\ldots+y_n)\Delta x.$$

It is plain that if Δx continuously diminish, the rectangles thus becoming narrower and narrower as they increase in number, the polygon will continuously approach nearer and nearer to coincidence with the curvilinear area, which latter will be the limit of the polygonal areas: hence,

$$\text{Area of Curve}=(y_1+y_2+y_3+\ldots\ldots+y_n)dx\ldots[1].$$

Again: complete the rectangle BC, the point C being on the curve: then corresponding to the increment Δx, the increment of the area of the curve is the curvilinear space bC; and taking Δx, or bb', sufficiently small, this increment will be between the rectangles bB', bC, in magnitude, for that and for all smaller values of Δx, and y_n being *any* ordinate,

the ratio of these rectangles is $\frac{y\Delta x}{(y+\Delta y)\Delta x}=\frac{y}{y+\Delta y}$, which in the limit, that is,

when $\Delta x=0$, and $\therefore \Delta y=0$, is $\frac{y}{y}=1$. Hence Δ Area being always intermediate between

$y\Delta x$ and $(y+\Delta y)\Delta x$, the ratio of Δ Area to either of these rectangles must be 1 in the limit; that is, in the limit, $\frac{\Delta \text{ Area}}{y\Delta x}=1$, or $\frac{d \text{ Area}}{ydx}=1$, $\therefore$ Area$=\int ydx$...[2].

541. If, instead of being rectangular, the axes are oblique, ϕ being their angle of inclination, then it is plain that we must replace y above by $y \sin \varphi$, so that

for rectangular axes, Area$=\int ydx$: *for oblique axes*, Area$=\sin \phi \int ydx$.

Since $y=F(x)$, it follows from [1], [2], that the limit of $\Sigma F(x)\Delta x$ is $\int F(x)dx$...[1],

where $\Sigma F(x)\Delta x$ stands for $\{F(x)+F(x+\Delta x)+F(x+2\Delta x)+\ldots+F(x+n\Delta x)\}\Delta x$.

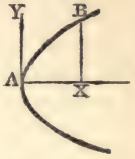
This circumstance sufficiently explains why the long *s* is placed before a differential to indicate the integral of it:—the prefix is the initial letter of the word "sum," the integral being the sum of the *elements* or successive increments

$$F(x)\Delta x,\ F(x+\Delta x)\Delta x,\ \ldots\ldots,\ F(x+n\Delta x)\Delta x,$$

which make up that integral, when these elements, by the continuous diminution of Δx, become individually smaller and smaller, and at length vanish. It is thus said that "the integral is the sum of an infinite number of infinitely small quantities:" but as an infinitely small quantity, or an "infinitesimal," cannot have *any* magnitude, the statement is tantamount to saying that the integral is the sum of an infinite number of *nothings*. In either form the declaration is repugnant to the understanding; and conveys a meaning that is not in reality intended, and which is open to the same objection as the statement that nothing divided by nothing produces *something*. In the differential calculus nothings are *not* divided, as we have already shown (522), nor in the integral calculus are nothings added. In both cases general algebraic quantities are operated upon, and general results obtained; these, in virtue of their generality, are open to any special interpretation it may suit us to give to them: such an interpretation may be given that if the same interpretation had been given to the general symbols under the sign of operation at the outset, the form $\frac{0}{0}$ or $0\times\infty$ might have presented itself; but so far from any use having been made of either of these forms in arriving at the result, it is rather the obtained result that has suggested them. In the matter before us, that particular case of a general result which in this way leads to $0\times\infty$, is shown above to be $\int F(x)dx$ between assigned limits of x.

542. In the preceding diagram the curvilinear area considered is supposed to be bounded by the fixed ordinates aA, bB, corresponding to the abscissæ Oa, Ob, which we may call a, b. It is within these limits, therefore, that the integral $\int ydx$, representing the area indefinitely, must be taken: the *definite integral*, as it is then called, is expressed by the notation $\int_a^b ydx$, or $\int_a^b (Fx)dx$, which implies that the general, or indefinite integral, being found, we are to put for x the given values a and b successively, and then subtract the former result from the latter. Each of the two results will of course involve the same supplementary constant, which will therefore disappear in the remainder.

(1) Required the area of the parabola $y^2=ax$, or $y=(ax)^{\frac{1}{2}}$. Here, between the limits, $x=0$, and $x=x$, Area$=\int_0^x y\,dx=a^{\frac{1}{2}}\int_0^x x^{\frac{1}{2}}\,dx=a^{\frac{1}{2}}\frac{x^{\frac{3}{2}}}{\frac{3}{2}}=\frac{2}{3}\,x(ax)^{\frac{1}{2}}$,

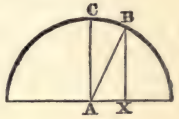


that is, Area $ABX=\frac{2}{3}\,xy$, since for $x=0$, the integral vanishes.

The area here taken lies between the limits AY, XB, the corresponding values of x being $x=0$, and $x=x$, that is, *any* value AX.

(2) Required the area of the circle $y^2=r^2-x^2$, or $y=(r^2-x^2)^{\frac{1}{2}}$,

$\therefore \int y\,dx=\int(r^2-x^2)^{\frac{1}{2}}\,dx$, or representing the radius by unity,



$$\int y\,dx=\int(1-x^2)^{\frac{1}{2}}\,dx=\int(1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}-\ldots)dx,$$

$$\therefore \text{Area}=x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{112}-\frac{5x^9}{1152}-\ldots+C.$$

When $x=0$, the area$=0$, $\therefore C=0$, $\therefore \int_0^x=x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{112}-\frac{5x^9}{1152}-\ldots$

Let the arc $CB=30°$, $\therefore x=AX=\frac{1}{2}$, $\therefore$ Area $ACBX=$

$\cdot5-\cdot0208333-\cdot0007812-\cdot0000698-\cdot0000085-\cdot0000012-\ldots=\cdot4783055.$

Also since $x=\frac{1}{2}$, $y=\sqrt{\frac{3}{4}}=\frac{1}{2}\sqrt{3}$, $\therefore$ area of triangle $ABX=$

$\frac{1}{4}\times\frac{1}{2}\sqrt{3}=\frac{1}{8}\sqrt{3}=\cdot2165063$, which, subtracted from $\cdot4783055$ leaves $\cdot2617992=$ Area sector ACB, and this multiplied by 12 gives 3·14159... for the area of the whole circle.

(3) Required the area of the ellipse $y=\frac{b}{a}\sqrt{(a^2-x^2)}$. $\int y\,dx=\frac{b}{a}\int\sqrt{(a^2-x^2)}\,dx$:

but $\int_0^a\sqrt{(a^2-x^2)}\,dx=$area of quadrant of a circle of radius a, by last example,

and $\frac{b}{a}$ times this is the area of the corresponding quadrant of the ellipse,

$\therefore$ the whole area of the ellipse$=\frac{b}{a}$ the area of the circle on $2a$ as diameter.

(4) Required the area between the curve and the asymptotes of an hyperbola.

Let the asymptotes be inclined at any angle φ, then the equation of the curve in reference to these as axes is

$$xy=\frac{a^2+b^2}{4},\ \text{and (541) Area}=\sin\varphi\int y\,dx=\frac{a^2+b^2}{4}\sin\varphi\int\frac{dx}{x}=$$

$$\frac{a^2+b^2}{4}\sin\varphi\log x+C.$$

We must, of course, fix the limits between which this area is to be taken, or at least the ordinate from which it is to commence. Let it

commence from the ordinate corresponding to $x=1$: then since $\log 1=0$, and that the area is 0 when $x=1$, $\therefore C=0$,

$$\therefore \text{Area}=\int_1^x=\frac{a^2+b^2}{4}\sin\varphi\log x\ldots\ldots[1].$$

Suppose, for ex., in the diagram at p. 339, $On=1$, then any hyperbolic space nP_1, nP_2, &c., is given by the above expression. As the log in this expression is Napierian, it follows that the areas nP_1, nP_2, &c., are the logs, to modulus $\frac{a^2+b^2}{4}\sin\varphi$, of the numbers whose linear representatives are nT' nt', &c., the scale being $On=1$. If the hyperbola be equilateral,

$$a=b, \text{ and } \sin\varphi=1, \text{ and if also } On=\sqrt{\frac{a^2}{2}},$$

the area is simply $\log_e x$, the modulus being then the square mn, which is unity. From this property of the equilateral hyperbola, Napier's logs were formerly called hyperbolic logs; but logs to any modulus have equal claims to this designation, as appears from the above.

If $x=\infty$ in the expression [1] above, the area must be ∞, since $\log\infty=\infty$. Hence the entire space between the curve and its asymptotes is infinite in magnitude.

These examples must suffice for the present in showing the application of the integral calculus to the *quadrature of curves*, as the determination of their areas is called. It was anticipated that the invention of this calculus would lead to a finite expression for the quadrature of the circle, but it was soon found that the necessary integration could not be effected in any finite terms which did not themselves involve a circular arc.

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543. **Volumes of Revolution.**—The expression for the volume generated by the revolution of a plane curve about an axis is easily obtained from what is done in the preceding article: thus, if the curvilinear area $AabB$ (page 510) revolve round ab as a fixed axis, every ordinate y will generate a circle of area πy^2. If the x of this ordinate take the increment Δx, the corresponding circle generated will be $\pi(y+\Delta y)^2$, and the increment of the solid, that is, the slice lying between these circles, will always be intermediate between the two cylinders generated, the one by the revolution of the rectangle Bb', and the other by the revolution of the rectangle Cb, supposing Δx to represent bb'. The ratio of these two cylinders is

$$\frac{\pi y^2\Delta x}{\pi(y+\Delta y)^2\Delta x}=\frac{y^2}{(y+\Delta y)^2}, \text{ which, when } \Delta x=0, \text{ and } \therefore \Delta y=0, \text{ is } \frac{y^2}{y^2}=1.$$

Consequently, Δvolume being always intermediate between $\pi y^2\Delta x$, and $\pi(y+\Delta y)^2\Delta x$, the ratio of Δvolume to either of these cylinders must, in the limit, be 1; that is,

$$\text{in the limit, } \frac{\Delta\text{volume}}{\pi y^2\Delta x}=1, \text{ or } \frac{dV}{\pi y^2dx}=1, \therefore V=\pi\int y^2dx,$$

the integral being taken between the assigned limits, or values of x.

(1) Required the volume of a right cone, the radius of whose base is r, and whose altitude is a.

Let x represent any variable distance from the vertex of the cone along a towards the base, and y the radius of the circular section at that distance: then, by similar triangles,

$$\frac{y}{x}=\frac{r}{a},\ \therefore y=\frac{r}{a}x,\ \therefore \pi\int y^2dx=\pi\int\frac{r^2}{a^2}x^2dx=\frac{\pi r^2}{a^2}\int x^2dx=\frac{\pi r^2}{3a^2}x^3+C,$$

$$\therefore\ V=\pi\int_0^a y^2dx=\pi r^2\frac{a}{3}=\text{area of base}\times\tfrac{1}{3}\text{ altitude.}$$

(2) Volume of a Sphere.—Let r be the radius of the generating circle, and put x for any variable distance from the extremity of the fixed diameter or axis: then the corresponding y^2 will be $y^2=2rx-x^2$,

$$\therefore\ \pi\int y^2dx=\pi\int(2rx-x^2)dx=2\pi r\int xdx-\pi\int x^2dx=\pi rx^2-\frac{\pi}{3}x^3+C,$$

$$\therefore\ V=\pi\int_0^x y^2dx=\pi rx^2-\frac{\pi}{3}x^3=\frac{\pi}{6}(6r-2x)x^2.$$

This expresses the volume of a spherical segment of height x, and agrees with the expression [1] at p. 273, where D is put for $2r$.

For the whole sphere, we have $V=\pi\int_0^{2r}y^2dx=4\pi r^3-\frac{\pi}{3}8r^3=\frac{4}{3}\pi r^3$,

which is two-thirds the volume of the circumscribing cylinder (p. 272).

(3) Volume of a Paraboloid.—This is the solid generated by the revolution of a semi-parabola about its axis. The equation of the generating curve being $y^2=4ax$,

$$\therefore\ \pi\int y^2dx=4a\pi\int xdx=2a\pi x^2+C,\ \therefore\ V=\pi\int_0^x y^2dx=2a\pi x^2=\frac{1}{2}\pi y^2x.$$

Hence the volume of a paraboloid of any height is one-half the volume of the circumscribing cylinder.

(4) Volume of a Spheroid.—This solid is generated by the revolution of a semi-ellipse about either the major or minor diameter: if the major be the fixed axis the solid is called a *prolate* spheroid; if the minor be the fixed axis it is called an *oblate* spheroid. In the former case,

the equation of the generating curve being $y^2=\frac{b^2}{a^2}(a^2-x^2)$, (p. 311),

$$\therefore\ \pi\int y^2dx=\frac{\pi b^2}{a^2}\int(a^2-x^2)dx=\frac{\pi b^2}{a^2}\left(a^2x-\frac{x^3}{3}\right)+C,$$

$$\therefore\ V=\pi\int_0^a y^2dx=\frac{\pi b^2}{a^2}\left(a^3-\frac{a^3}{3}\right)=\frac{2}{3}\pi ab^2=\text{vol. of prolate semi-spheroid.}$$

If the minor be the fixed axis, or the axis of x, the equation of the generating curve is

$$y^2=\frac{a^2}{b^2}(b^2-x^2),\ \therefore\ V'=\pi\int_0^b y^2dx=\frac{\pi a^2}{b^2}\left(b^3-\frac{b^3}{3}\right)=\frac{2}{3}\pi ba^2,\ \therefore\ \frac{2V}{2V'}=\frac{b}{a},$$

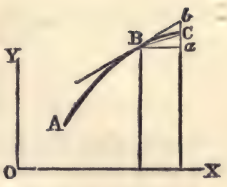
so that the volumes are as the rotating axes.

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544. **Lengths of Curve Lines.**—Let $AB=s$ be an arc of any curve, and BC any increment of it Δs: call the chord BC, c. Then the point B being (x, y), the point C will be $(x+\Delta x,\ y+\Delta y)$, and for the length of the chord BC we shall have $c^2=(\Delta x)^2+(\Delta y)^2$,

$$\therefore \frac{\Delta s}{\Delta x}=\frac{\Delta s}{c}\cdot\frac{c}{\Delta x}=\frac{\Delta s}{c}\sqrt{\left\{1+\left(\frac{\Delta y}{\Delta x}\right)^2\right\}}.$$

Now the ratio of an arc Δs to its chord c is ultimately a ratio of equality, for chords drawn from one extremity of an arc to another, approach nearer to coincidence with the arc as they increase in number and individually diminish in length: hence, passing to the limits,



$$ds=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}},\ \therefore s=\int\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx.$$

It may be remarked in reference to this conclusion, that since $\frac{dy}{dx}$ is the tangent of the inclination of the line, touching the curve at (x, y), to the axis of x, therefore $\frac{ds}{dx}$ must be the secant of that angle, so that the sides Ba, ab, Bb, of the right-angled triangle whose hypotenuse is Bb, any portion of the tangent at B, and of which the base and perp. are parallel to the rectangular axes of co-ordinates, are proportional to the differentials dx, dy, ds: such a triangle is, therefore, sometimes called the differential triangle.

(1) To find the length of an arc of the parabola $y^2=4ax$, or $y=2(ax)^{\frac{1}{2}}$.

Here $\frac{dy}{dx}=a^{\frac{1}{2}}x^{-\frac{1}{2}}$, $\therefore s=\int\sqrt{\left\{1+\frac{a}{x}\right\}}dx$. In order to effect the integration in the easiest way, put $1+\frac{a}{x}=z^2$, $\therefore x=\frac{a}{z^2-1}$, and $s=\int z\,dx$.

Integrating by parts (539) we have $s=\int z\,dx=zx-\int x\,dz=zx-\int\frac{a\,dz}{z^2-1}$,

that is (538), $s=zx-\frac{a}{2}\log\frac{z-1}{z+1}+C$. This, by replacing the value of z, and making the arc to commence at the vertex of the curve, is

$$s=\int_0^z=\sqrt{(x^2+ax)}+\frac{a}{2}\log\frac{a+2x+2\sqrt{(x^2+ax)}}{a}.$$

Suppose, for example, it were required to find the length of the parabolic path of a shot, of which the horizontal range is 4800 feet, and its greatest height (x) 1600 feet.

Here $x=1600$, $y=2400$, and $a=\frac{y^2}{4x}=900$,

$$\therefore s=\sqrt{(1600^2+900\times1600)}+450\log\frac{900+3200+2\sqrt{(1600^2+900\times1600)}}{900}$$

$$=2000+450\log 9=2000+450\times2\cdot197225=3735\cdot9 \text{ feet.}$$

(2) To find the length of an arc of a circle. According as the rectangular axes originate at the centre or at the circumference, we shall have

$$y^2=r^2-x^2, \text{ or } y^2=2rx-x^2,\ \therefore s=r\int\frac{dx}{\sqrt{(r^2-x^2)}}, \text{ or } s=r\int\frac{dx}{\sqrt{(2rx-x^2)}},$$

and the integration in each case will itself involve a circular arc (537):

the length cannot, therefore, be found in finite terms. By (531) the differential of an arc whose tangent is t and radius 1, is

$$d\tan^{-1}t=\frac{dt}{1+t^2},\ \therefore\ \tan^{-1}t=\int\frac{dt}{1+t^2}=\int(1-t^2+t^4-t^6+\ldots)dt,$$

$$\therefore\ \tan^{-1}t=t-\frac{1}{3}t^3+\frac{1}{5}t^5-\frac{1}{7}t^7+\ldots$$

Since, when the arc is 0, t is 0, the correction C is 0, and the series is the same as that otherwise determined at p. 217.

If, however, we develope $(1-x^2)^{-\frac{1}{2}}$, the coef. of dx in the preceding differential, when $r=1$, we shall have

$$s=\sin^{-1}x=x+\frac{1}{3}\cdot\frac{1}{2}x^3+\frac{1}{5}\cdot\frac{1.3}{2.4}x^5+\frac{1}{7}\cdot\frac{1.3.5}{2.4.6}x^7+\ldots$$

which, when $x=\frac{1}{2}$, or the arc$=30°$, gives

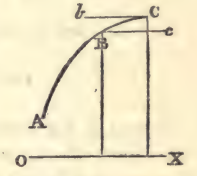
$$\text{arc of } 30°=\frac{1}{2}+\frac{1}{2.3.2^3}+\frac{1.3}{2.4.5.2^5}+\frac{1.3.5}{2.4.6.7.2^7}+\ldots=\cdot5235987\ldots,$$

$$\therefore\ 30°\times6=\pi=3\cdot141592\ldots$$

In like manner, the expression for an arc of an ellipse involves a differential that can be integrated only in a series.

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545. **Surfaces of Solids of Revolution.**—Let S be the surface generated by the revolution of the curve $s=AB$ about the fixed axis of x, OX, and let BC be any increment Δs of s, ΔS being the corresponding increment of the surface. Draw Bc, Cb, parallel to the fixed axis, and each equal to BC or Δs: then, since every point in BC, with the exception of the point B, is farther from the fixed axis than the corresponding points of Bc, that is, than the points having the same abscissa, the surface generated by BC must exceed the cylindrical surface generated by Bc; and since every point in Cb (except C) is farther from the fixed axis than the point having the same abscissa, of CB, the cylindrical surface generated by Cb must exceed the surface generated by CB; hence, ΔS always lies between $2\pi y\Delta s$, and $2\pi(y+\Delta y)\Delta s$, provided, at least, that BC be so small that the curve has no sinuosities or bends in that interval, a condition that may always be satisfied by taking BC sufficiently small.



Now the ratio $\frac{2\pi y\Delta s}{2\pi(y+\Delta y)\Delta s}=\frac{y}{y+\Delta y}=1$ in the limit, or when $\Delta y=0$.

Hence the ratio of ΔS to either of these cylindrical surfaces is 1 in the limit,

$$\therefore\ \frac{\Delta S}{2\pi y\Delta s}=1 \text{ in the limit, that is, } \frac{dS}{2\pi y ds}=1,$$

$$\therefore\ dS=2\pi y ds, \text{ but (544) } ds=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}dx,$$

$$\therefore dS=2\pi y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}dx,$$

$\therefore S=2\pi\int yds$, or $S=2\pi\int y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}dx$, between the assigned limits.

(1) Required the surface of a sphere of radius r.

The equation of the generating semicircle is $x^2+y^2=r^2$, $\therefore 2xdx+2ydy=0$,

$$\therefore \frac{dy}{dx}=-\frac{x}{y},\ \therefore \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}=\frac{\sqrt{(x^2+y^2)}}{y}=\frac{r}{y},\ \therefore ds=\frac{r}{y}dx,$$

$$\therefore 2\pi\int yds=2\pi\int rdx=2\pi rx+C,\ \therefore S=2\pi\int_0^r=2\pi r^2.$$

This is the surface of the hemisphere: hence, for the whole sphere, the surface is $S=4\pi r^2$, that is, four times the area of one of its great circles.

(2) Required the curve surface of a paraboloid.

The equation of the generating curve being $y^2=4ax$, or $y=2(ax)^{\frac{1}{2}}$,

we have, as at (544), $ds=\sqrt{\left\{1+\frac{a}{x}\right\}}dx$, $\therefore yds=2a^{\frac{1}{2}}(x+a)^{\frac{1}{2}}dx$,

$$\therefore 2\pi\int yds=4\pi a^{\frac{1}{2}}\int(x+a)^{\frac{1}{2}}dx=\frac{8}{3}\pi a^{\frac{1}{2}}(x+a)^{\frac{3}{2}}+C,$$

$$\therefore S=4\pi a^{\frac{1}{2}}\int_0^x=\frac{8}{3}\pi a^{\frac{1}{2}}\left\{(x+a)^{\frac{3}{2}}-a^{\frac{3}{2}}\right\},$$

the expression for the curve surface of a paraboloid of any altitude.

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546. From the last few articles it must be sufficiently apparent that the length of any curve line between assigned limits, as also the area of any plane surface, can always be found, provided that a certain differential expression, deducible by known rules from the equation of the curve, can be integrated: and the same may be said in reference to any surface or solid of revolution.

The integral in each case being of the form $\int_a^b Xdx$, where X is a function of x, it is the *limit* of the sum of a series of which the first term is what $X\Delta x$ becomes when $x=a$, and the last term what $X\Delta x$ becomes when $x=b$:—the intermediate terms arise from making x successively equal to $x+\Delta x$, $x+2\Delta x$, $x+3\Delta x$, and so on, till the entire interval between $x=a$, and $x=b+\Delta x$, is filled up (541). The limit of the sum of the terms, following one after another according to this law of succession, is what that sum actually amounts to only when, by continuously diminishing the arbitrary interval Δx between every consecutive pair of terms, each interval becomes reduced to zero, and consequently the number of them becomes infinite. It is the special office of the integral calculus to furnish us, in all cases, with this limiting value, to the exclusion of all the other values, in the continuous series of values which this terminates. The differential calculus and the integral calculus are both occupied exclusively with the limiting values of continuously varying quantities: the former calculus furnishes us with the exact interpretation, as to value, of what, in the absence of such interpretation, would

be merely the vague symbol $\frac{0}{0}$: the latter supplies the exact interpretation of what would otherwise be merely the equally vague symbol $0 \times \infty$.

547. From thus regarding an integral as the limit of a summation, or what the series becomes when Δx becomes dx, the foregoing general expressions for areas, volumes, &c., may be deduced with great facility: thus, since (see fig. p. 510),

$$1.\ \text{Area} = \text{limit of } \Sigma_a^{b+\Delta x}(y\Delta x),\ \therefore\ \text{Area} = \int_a^b y dx.$$

The parallelogram $y\Delta x$, or rather this in the limit, namely, ydx, is called an *element* of the surface; the surface being made up of an infinite number of these elements.

2. A solid of revolution may, in like manner, be regarded as made up of circular slices, each slice of thickness Δx, and all perp. to the fixed axis. If, however, we consider first—not the solid itself generated by the curved area above (fig. p. 510)—but that generated by the assemblage of parallelograms, the result will be a series of cylinders; and y representing the radius of the base of any one indifferently, the sum of all will be expressed by $\Sigma_a^{b+\Delta x}(\pi y^2 \Delta x)$, which sum, in the limit, or when the thickness Δx of each cylinder is reduced to zero, accurately gives the volume of the proposed solid, that is,

$$\text{Volume} = \text{limit of } \Sigma_a^{b+\Delta x}(\pi y^2 \Delta x),\ \therefore\ \text{Volume} = \pi \int_a^b y^2 dx.$$

The cylinder $\pi y^2 \Delta x$, in the limit, that is, $\pi y^2 dx$, is the element of the solid.

3. In like manner, the element of a surface of revolution is

$$2\pi y \sqrt{\{(\Delta x)^2 + (\Delta y)^2\}} \text{ in the limit;}$$

the chord of the arc Δs being in a ratio of equality with that arc at the limit,

$$\therefore\ \text{Surface} = \text{limit of } \Sigma_a^{b+\Delta x} 2\pi y \sqrt{\{(\Delta x)^2 + (\Delta y)^2\}} =$$

$$\text{limit of } \Sigma_a^{b+\Delta x} 2\pi y \sqrt{\left\{1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right\}} \Delta x,$$

$$\text{that is, Surface} = 2\pi \int_a^b y \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} dx.$$

The student will do well to consider attentively the train of ideas which lead to these several results. In the case of a plane area included between two bounding ordinates corresponding to $x = a$, and $x = b$, and between the portions of curve and axis of x intercepted by these ordinates, we mentally proceed thus. Commencing with the first or initial ordinate, we conceive it to move parallel to itself, preserving its original length, towards the final ordinate, till a parallelogram of breadth Δx is generated, Δx being entirely under our control as to degree of smallness: the moving ordinate then stops, and increases, or diminishes, as the case may be, till it again reaches the curve: thus modified, its motion recommences, and another parallelogram, of the same uniform breadth Δx, is generated; and in this way a succession of parallelograms, with one angle of each on

the curve, is formed by intermittent motion, the last parallelogram extending beyond the final ordinate.

We then imagine Δx to continuously diminish, and thus the number of parallelograms, each getting narrower and narrower, to increase, the intermittent motion above alluded to approaching nearer and nearer to a *continuous* motion, which, when actually attained, the continuously increasing or decreasing ordinate generates, by its uniform motion—not a series of inscribed parallelograms—but the curve surface itself, which is the *limit* to the sum of that series when it consists of an infinite number of terms, and of which limit we have seen that the definite integral $\int_a^b y dx$ expresses the exact value.

In like manner in reference to a solid of revolution, bounded by two circular sections, through the centres of which the fixed axis passes. Commencing with the first bounding circle, we conceive it to move parallel to its own plane, preserving its original magnitude, towards the final circle, till a cylinder of breadth Δx is generated: the circle then expands or contracts till it again becomes a section of the solid; and then a second cylinder of breadth Δx is generated; and so on. The limit of the sum of this series of cylinders, which limit is approached to closer and closer as Δx diminishes, and which is ultimately reached only when $\Delta x = 0$, is the proposed solid, and we have seen that its value is expressed by the definite integral $\pi \int_a^b y^2 dx$. And a similar process applies to the curve line, and to the curve surface.

548. Whatever function of the single variable x, X may represent, it is plain that $y = X$ will be the equation of some plane curve; so that we may regard every definite integral $\int_a^c X dx$ as a case of *quadrature*, or as a plane area taken between the limits $x = a$, $x = c$. If b be an intermediate value of x, between a and c, the area will consist of the portion between the limits $x = a$, $x = b$, and of the portion between the limits $x = b$, $x = c$, as is obvious: hence

$$\int_a^c X dx = \int_a^b X dx + \int_b^c X dx,$$

and in this way may any definite integral be cut up into component definite integrals, as many as we please.

Since the notation $\int_a^c$ denotes the general integral when $x = c$, diminished by the general integral when $x = a$, it follows that $\int_a^c X dx = -\int_c^a X dx$, and from this we may conclude that $\int_0^a (Fx) dx$ is always the same as $\int_0^a F(a-x) dx$. For, in the former, put $a - z = x$, then $dx = -dz$, and the integral is the same as $-\int_a^0 F(a-z) dz$, because, when $z = 0$, $x = a$, and when $z = a$, $x = 0$: but it was before seen that

$$-\int_a^0 F(a-z) dz = \int_0^a F(a-z) dz, \quad \therefore \int_0^a F(x) dx = \int_0^a F(a-x) dx,$$

the last definite integral being, of course, the same, whatever letter whether z or x, the constants 0 and a replace. We shall now return the differential calculus.

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549. **Successive Differentiation.**—Since the differential coefficient, derived from any function of a variable, may itself also contain that variable, or be a new function of it, this new function may, in like manner, be differentiated, and thus a *second* differential coefficient be obtained. If the variable appear also in this, we may obtain a *third* differential coefficient, and so on, till we arrive at a differential coefficient

into which the variable does not enter, and at which, therefore, this process of *successive differentiation* must stop. For example, taking the function $y=ax^4+bx^2+c$, we have

First differential coefficient $\frac{dy}{dx}=4ax^3+2bx=f_1(x)$.

Second ,, $\frac{df_1(x)}{dx}=12ax^2+2b=f_2(x)$.

Third ,, $\frac{df_2(x)}{dx}=24ax=f_3(x)$.

Fourth ,, $\frac{df_3(x)}{dx}=24a=f_4(x)=$constant

But the usual notation for successive differential coefficients is this, namely,

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \frac{d^4y}{dx^4},\ \ldots\ldots\frac{d^ny}{dx^n},$$

where the small figures over the d denote not powers, but repetitions of the operation of differentiating; and where the exponents over the x apply to dx, that is, dx^n is merely a shorter way of writing $(dx)^n$. Whenever, hereafter, a differential of the power x^n is to be indicated, a point will be interposed, thus, $d.x^n$. We shall give an example or two of successive differentiation of explicit functions.

(1) $y=x^n$,

$\therefore \frac{dy}{dx}=nx^{n-1}$,

$\frac{d^2y}{dx^2}=n(n-1)x^{n-2}$,

$\frac{d^3y}{dx^3}=n(n-1)(n-2)x^{n-3}$,

$\vdots \qquad \vdots$

$\frac{d^ry}{dx^r}=n(n-1)\ldots[n-(r-1)]x^{n-r}$.

(2) $y=\log x$,

$\frac{dy}{dx}=x^{-1}$,

$\frac{d^2y}{dx^2}=(-1)x^{-2}$,

$\frac{d^3y}{dx^3}=(-1)^2 2x^{-3}$,

$\vdots$

$\frac{d^ry}{dx^r}=(-1)^{r-1}2.3.4\ldots(r-1)x^{-r}$.

(3) $u=yz$, where y and z are both functions of the independent variable x,

$$\frac{du}{dx}=y\frac{dz}{dx}+z\frac{dy}{dx},$$

$$\frac{d^2u}{dx^2}=y\frac{d^2z}{dx^2}+\frac{dy}{dx}\cdot\frac{dz}{dx}+z\frac{d^2y}{dx^2}+\frac{dz}{dx}\cdot\frac{dy}{dx}=y\frac{d^2z}{dx^2}+2\frac{dy}{dx}\cdot\frac{dz}{dx}+z\frac{d^2y}{dx^2},$$

$$\frac{d^3u}{dx^3}=y\frac{d^3z}{dx^3}+3\frac{dy}{dx}\cdot\frac{d^2z}{dx^2}+3\frac{d^2y}{dx^2}\cdot\frac{dz}{dx}+z\frac{d^3y}{dx^3},\ \&c.,\ \&c.$$

Thus far the numerical coefficients are those of the expanded binomial, and the law may be shown to be general, as follows:—

550. **The Theorem of Leibnitz.**—If $u=yz$, the rth diff. coef. of u, with respect to x, will be

$$\frac{d^ru}{dx^r}=y\frac{d^rz}{dx^r}+r\frac{dy}{dx}\cdot\frac{d^{r-1}z}{dx^{r-1}}+\frac{r(r-1)}{2}\frac{d^2y}{dx^2}\cdot\frac{d^{r-2}z}{dx^{r-2}}+$$

$$\frac{r(r-1)(r-2)}{2.3}\frac{d^3y}{dx^3}\cdot\frac{d^{-3}z}{dx^{r-3}}+\ldots+\frac{d^ry}{dx^r}z\ldots[1].$$

If this be true, we should have, by differentiating,

$$\frac{d^{r+1}u}{dx^{r+1}}=\left(y\frac{d^{r+1}z}{dx^{r+1}}+\frac{dy}{dx}\cdot\frac{d^rz}{dx^r}\right)+r\left(\frac{dy}{dx}\cdot\frac{d^rz}{dx^r}+\frac{d^2y}{dx^2}\cdot\frac{d^{r-1}z}{dx^{r-1}}\right)+$$

$$\frac{r(r-1)}{2}\left(\frac{d^2y}{dx^2}\cdot\frac{d^{r-1}z}{dx^{r-1}}+\frac{d^3y}{dx^3}\cdot\frac{d^{r-2}z}{dx^{r-2}}\right)+$$

$$\frac{r(r-1)(r-2)}{2.3}\left(\frac{d^3y}{dx^3}\cdot\frac{d^{r-2}z}{dx^{r-2}}+\frac{d^4y}{dx^4}\cdot\frac{d^{r-3}z}{dx^{r-3}}\right)+\ldots+\left(\frac{d^ry}{dx^r}\cdot\frac{dz}{dx}+\frac{d^{r+1}y}{dx^{r+1}}z\right),$$

that is, arranging the terms according to the differentials of z,

$$\frac{d^{r+1}u}{dx^{r+1}}=y\frac{d^{r+1}z}{dx^{r+1}}+(r+1)\frac{dy}{dx}\cdot\frac{d^rz}{dx^r}+\frac{(r+1)r}{2}\frac{d^2y}{dx^2}\cdot\frac{d^{r-1}z}{dx^{r-1}}+$$

$$\frac{(r+1)r(r-1)}{2.3}\frac{d^3y}{dx^3}\cdot\frac{d^{r-2}z}{dx^{r-2}}+\ldots+\frac{d^{r+1}y}{dx^{r+1}}z\ldots[2].$$

Now this series is what [1] becomes when r is changed into $r+1$; so that if [1] be true for any value of r, it is necessarily true for the next value $r+1$, and consequently for the next $r+2$, and so on. But, as shown in last page, it is true for $r=3$, hence it is true for $r=4$, for $r=5$, and generally for $r=$ any positive integer whatever.

Examples for Exercise.

To prove the following results, the first three by successive differentiation, and the fourth by the theorem of Leibnitz.

(1) $y=\log x$, $\therefore \frac{d^6y}{dx^6}=-\frac{2.3.4.5}{x^6}$.

(2) $y=(x^2+a^2)\tan^{-1}\frac{x}{a}$,

$$\therefore \frac{d^3y}{dx^3}=\frac{4a^3}{(x^2+a^2)^2}.$$

(3) $y=a^x$,

$$\therefore \frac{d^ry}{dx^r}\div y=\text{constant}=(\log a)^r.$$

(4) $u=e^{ax}y$, $\therefore \frac{d^nu}{dx^n}=$

$$e^{ax}\left\{a^ny+na^{n-1}\frac{dy}{dx}+\frac{n(n-1)}{2}a^{n-2}\frac{d^2y}{dx^2}+\ldots+\frac{d^ny}{dx^n}\right\}.$$

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551. **Development of Functions: Maclaurin's Theorem.**—If y be a function of x, which it is possible to develope in a series of ascending powers of that variable, then will the development be

$$y=[y]+\left[\frac{dy}{dx}\right]x+\frac{1}{2}\left[\frac{d^2y}{dx^2}\right]x^2+\frac{1}{2.3}\left[\frac{d^3y}{dx^3}\right]x^3+\ldots\ldots[1],$$

where the brackets are intended to imply that, *after* the differential coefficients which they enclose are obtained in a general form, x in each is to be put $=0$. Or, if instead of representing the function by y, we write it $f(x)$, and as at (549), put $f_1(x)$, $f_2(x)$, &c., for the successive derived functions, or differential coefficients, the development may be written thus:

$$f(x)=f(0)+f_1(0)x+\frac{1}{2}f_2(0)x^2+\frac{1}{2.3}f_3(0)x^3+\ldots\ldots[2].$$

Since, by hypothesis, the function *is* developable in a series of ascending powers of x—of course with coefficients finite and independent of x—we may assume

$$y=A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\ldots,\ \therefore\ [y]=A_0,$$

$$\therefore \frac{dy}{dx}= A_1+2A_2x+3A_3x^2+4A_4x^3+\ldots,\ \therefore \left[\frac{dy}{dx}\right]=A_1,$$

$$\frac{d^2y}{dx^2}= 2A_2+2.3A_3x+3.4A_4x^2+\ldots,\ \therefore \left[\frac{d^2y}{dx^2}\right]=2A_2,$$

$$\frac{d^3y}{dx^3}= 2.3A_3+2.3.4A_4x+\ldots,\ \therefore \left[\frac{d^3y}{dx^3}\right]=2.3A_3,$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{d^ny}{dx^n}= 2.3.4\ldots nA_n+\ldots,\ \therefore \left[\frac{d^ny}{dx^n}\right]=2.3.4\ldots nA_n.$$

The values of the assumed coefficients are, therefore,

$$A_0=[y],\ A_1=\left[\frac{dy}{dx}\right],\ A_2=\frac{1}{2}\left[\frac{d^2y}{dx^2}\right],\ A_3=\frac{1}{2.3}\left[\frac{d^3y}{dx^3}\right],\ \&c.,$$

thus establishing the truth of [1], which is called *Maclaurin's Theorem*, although it appears to have been first given by *Stirling*.

552. It is of importance that the student be not betrayed into attributing to this theorem an extent of generality not justified by the hypothesis upon which the reasoning that leads to it is founded. He must take notice that the original assumption is not only that

$$y=A_0+A_1x+A_2x^2+A_3x^3+\&c.,$$

but also that the quantity comprehended in the "&c.," whatever it be, must, equally with the terms following the first term of the prefixed series, vanish when $x=0$, and moreover, that all the successive differential coefficients derived from this quantity must vanish when $x=0$. In all those cases in which the series [1] *terminates*, there is, of course, no supplementary correction; the "&c." is then superfluous, and there is complete harmony between the premises and the conclusion.

It is only when such strict equality exists that the second member of [1] can ever be regarded as the complete development of the first, and we have accordingly suppressed the "&c.," which most writers append, and which is only an unintelligible symbol, covering a meaning of which, in most cases, we know nothing.

It will be remembered that when we equate a function with its development, we express merely an *identity*: the two expressions equated are different in form, but always the same in value. If the second member of [1] is the development of the first, the equation is true, whatever particular value be given to x: such a value may be given to it as that the more of the terms of the series we sum up the closer do we approach to the exact value of the undeveloped function, and $\therefore$ the more insignificant do the terms neglected become: the portion of the whole thus neglected approaching continuously to zero, as the terms retained become more numerous. In every such case the "&c.," which, in different circumstances, implies a correction, is, of course, superfluous: it is to such cases alone that the theorem rigidly applies.

Maclaurin's theorem, then, always gives the development of a function in the declared form, whenever, by successive differentiation of that function, we arrive at a diff. coef. which is constant, that coef. being the last of the differential coefficients in [1]: the development is then general and *finite*.

The theorem always gives the development for a particular value of the variable x, even when the series is interminable—from a *constant* differential coefficient never occurring—provided that for the particular value of x assigned, that series is convergent, or that there is no supplementary " &c.," and provided, moreover, that for the proposed value of x, none of the differential coefficients become infinite. In this latter case, we should, of course, conclude that the development in the prescribed form with *finite* coefficients is impossible. We say " of course," because, if finite coefficients were possible, the above process would supply them. [The student may re-peruse the observations on general developments at pp. 77, 83, and 167.]

553. In certain applications of Maclaurin's theorem to particular functions, the successive differentiations may become laborious: but when the first differential coefficient $\frac{dy}{dx}$ is obtained, we may frequently deduce all the others by simply developing $\frac{dy}{dx}$ by common algebra. Thus, by the theorem, since $\frac{dy}{dx}$ is a function of x, we have

$$\frac{dy}{dx}=\left[\frac{dy}{dx}\right]+\left[\frac{d^2y}{dx^2}\right]x+\frac{1}{2}\left[\frac{d^3y}{dx^3}\right]x^2+\frac{1}{2.3}\left[\frac{d^4y}{dx^4}\right]x^3+\ldots[3],$$

and comparing this with the development of y, as exhibited in the theorem [1], we see that [1] may be deduced from [3] by adding to $[y]$ the several terms of [3], when they are respectively multiplied by x, $\frac{1}{2}x$, $\frac{1}{3}x$, &c. Hence, we have this rule to find the terms after $[y]$.

Develope $\frac{dy}{dx}$ in a series of ascending powers of x, whenever this development is easy by common algebra. Increase the exponent of x in each term by unity, and divide the term by the exponent thus increased.

An example of the facilities furnished by this rule is given in the development of $\tan^{-1}x$ in next page.

554. **Applications of Maclaurin's Theorem.**—(1) To develope $(a+x)^n$.

Putting $y=(a+x)^n$, therefore $[y]=a^n$,

$$\therefore \frac{dy}{dx}=n(a+x)^{n-1}, \quad ,, \quad \left[\frac{dy}{dx}\right]=na^{n-1},$$

$$\frac{d^2y}{dx^2}=n(n-1)(a+x)^{n-2}, \quad ,, \quad \left[\frac{d^2y}{dx^2}\right]=n(n-1)a^{n-2},$$

$$\frac{d^3y}{dx^3}=n(n-1)(n-2)(a+x)^{n-3} \quad ,, \quad \left[\frac{d^3y}{dx^3}\right]=n(n-1)(n-2)a^{n-3},$$

&c. &c.

Substituting these values in the general theorem [1], we have

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{2.3}a^{n-3}x^3+\ldots,$$

which is the well-known binomial theorem.

(2) To develope $\log(a+x)$.

Putting $y=\log(a+x)$, $\therefore [y]=\log a$, and

$$\frac{dy}{dx}=(a+x)^{-1}, \quad \text{therefore} \quad \left[\frac{dy}{dx}\right]=a^{-1},$$

$$\frac{d^2y}{dx^2}=-(a+x)^{-2}, \quad \text{,,} \quad \left[\frac{d^2y}{dx^2}\right]=-a^{-2},$$

$$\frac{d^3y}{dx^3}=2(a+x)^{-3}, \quad \text{,,} \quad \left[\frac{d^3y}{dx^3}\right]=2a^{-3},$$

$$\frac{d^4y}{dx^4}=-2.3(a+x)^{-4}, \quad \text{,,} \quad \left[\frac{d^4y}{dx^4}\right]=-2.3a^{-4},$$

&c., &c.,

$$\therefore \log(a+x)=\log a+\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\frac{x^4}{4a^4}+\ldots$$

If $a=0$, or the function be $\log x$ instead of $\log(a+x)$, then $[y]$, $\left[\frac{dy}{dx}\right]$, &c.,

would all be infinite: we infer, therefore, that the development of $\log x$ in the proposed form, with finite coefficients, is impossible. But we know that

$$\log x=(x-1)-\tfrac{1}{2}(x-1)^2+\tfrac{1}{3}(x-1)^3-\ldots :$$

we may infer, therefore, that $-(1+\frac{1}{2}+\frac{1}{3}+\ldots)$ is infinite.

(3) To develope $\sin x$.

Putting $y=\sin x$, $\therefore [y]=0$,

$$\therefore \frac{dy}{dx}=\cos x, \ \therefore \left[\frac{dy}{dx}\right]=1, \ \frac{d^2y}{dx^2}=-\sin x, \ \therefore \left[\frac{d^2y}{dx^2}\right]=0,$$

$$\frac{d^3y}{dx^3}=-\cos x, \ \therefore \left[\frac{d^3y}{dx^3}\right]=-1, \ \frac{d^4y}{dx^4}=\sin x, \ \therefore \left[\frac{d^4y}{dx^4}\right]=0.$$

The preceding values now recur, $\therefore \sin x=x-\frac{x^3}{2.3}+\frac{x^5}{2.3.4.5}-\ldots\ldots$

the arcual development of the sine; and in a similar way may the development of $\cos x$ be obtained: but, by differentiating the above, we get

$$\cos x=1-\frac{x^2}{2}+\frac{x^4}{2.3.4}-\ldots\ldots$$

(4) To develope $\tan^{-1}x$.

Putting $y=\tan^{-1}x$, we have $[y]=\pm n\pi$, where n is any integer.

$$\text{Also } \frac{dy}{dx}=\frac{1}{1+x^2}, \ \therefore \left[\frac{dy}{dx}\right]=1, \ \frac{d^2y}{dx^2}=-\frac{2x}{(1+x^2)^2}, \ \therefore \left[\frac{d^2y}{dx^2}\right]=0.$$

Now it is easily seen that the continuation of this process would soon become laborious; we shall therefore avail ourselves of the rule given in last page. Thus, developing the value of $\frac{dy}{dx}$ by common division, we have

$\frac{1}{1+x^2}=1-x^2+x^4-x^6+x^8-x^9+$&c., and consequently, by the rule referred to,*

$$\tan^{-1}x=\pm n\pi+x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\frac{1}{9}x^9-\ldots,$$ as at p. 220.

Examples for Exercise.

(1) To develope a^x.
(2) ,, $\sin^{-1}x$.
(3) ,, $\cos^{-1}x$.
(4) To develope $(\tan x)^4$.
(5) ,, $(\cos x+\sqrt{-1}\sin x)^n$.
(6) ,, $(a+bx+cx^2+\ldots)^n$.

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555. **Development of Functions: Taylor's Theorem.**—We have seen in the preceding articles that whenever a function $f(x)$ is developable in a series, proceeding according to the ascending powers of x, Maclaurin's theorem will always enable us to determine the coefficients of those powers. We have now to show that whenever a function of the form $f(x+h)$ is developable in a series, proceeding according to the ascending powers of h, another theorem, that of Taylor, will, in like manner, always enable us to determine the coefficients of the powers. But before entering upon the investigation of Taylor's theorem, it will be necessary to establish the following principle, which is all but self-evident.

Lemma.—If, in any function of $p+q$, it be known that one of the quantities p, q, is constant, and that the other is variable, we may derive the correct differential coefficients from the function $f(p+q)$, without inquiring which of the two is the variable, and which the constant; for the several differential coefficients will be the same, whether we differentiate $f(p+q)$ on the presumption that p is the variable, or on the presumption that q is the variable: that is, we shall have

$$\frac{df(p+q)}{dp}=\frac{df(p+q)}{dq},$$ and $\therefore$ the following coefficients also equal.

For since the function contains but one variable, we may put $p+q=f(x)$; and whichever of the parts p, q, of x, takes the increment k, the result $f(x+k)$ is necessarily the same,

and $\therefore \frac{f(x+k)-f(x)}{k}$ the same (whatever be k), whether k be regarded as Δp, or Δq:

hence, passing to the limit, $\frac{df(x)}{dp}$ is the same as $\frac{df(x)}{dq}$, that is,

$\frac{df(p+q)}{dp}$ is the same as $\frac{df(p+q)}{dq}$, $\therefore \frac{d^nf(p+q)}{dp}=\frac{d^nf(p+q)}{dq}$.

For example: let $y=3(p+q)^2-6(p+q)$. Differentiating with respect to p,

$\frac{dy}{dp}=6(p+q)^2-6$; and diff. with respect to q, $\frac{dy}{dq}=6(p+q)^2-6$.

Again: let $y=\log(p+q)+\sin(p+q)$, $\therefore \frac{dy}{dp}=\frac{1}{\log(p+q)}+\cos(p+q)=\frac{dy}{dq}$, &c.

This principle being admitted, let $f(x+h)$ be developable in a series of the form

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* This simple rule, which is to be found in most recent works on the calculus, the author believes to have been first given by himself in 1831, in his "Integral Calculus," p. 82.

$$f(x+h)=A_0+A_1h+A_2h^2+A_3h^3+A_4h^4+\ldots\ldots[1],$$

$$\therefore \frac{df(x+h)}{dx}=\frac{dA_0}{dx}+\frac{dA_1}{dx}h+\frac{dA_2}{dx}h^2+\frac{dA_3}{dx}h^3+\ldots\ldots[2],$$

$$\text{and } \frac{df(x+h)}{dh}=A_1+2A_2h+3A_3h^2+4A_4h^3+\ldots\ldots[3].$$

By the lemma, the series [2], [3], are equal for all values of h,

$$\therefore A_1=\frac{dA_0}{dx},\ A_2=\frac{1}{2}\frac{dA_1}{dx}=\frac{1}{2}\frac{d^2A_0}{dx^2},$$

$$A_3=\frac{1}{3}\frac{dA_2}{dx}=\frac{1}{2.3}\frac{d^3A_0}{dx^3},\ A_4=\frac{1}{4}\frac{dA_3}{dx}=\frac{1}{2.3.4}\frac{d^4A_0}{dx^4},\ \&c.$$

Now, for $h=0$, in [1], the first member is $f(x)$, and the second A_0,

$$\therefore A_0=f(x),\ \therefore A_1=f_1(x),\ A_2=\frac{1}{2}f_2(x),\ A_3=\frac{1}{2.3}f_3(x),\ A_4=\frac{1}{2.3.4}f_4(x),\ \&c.,$$

$$\therefore f(x+h)=f(x)+f_1(x)h+f_2(x)\frac{h^2}{2}+f_3(x)\frac{h^3}{2.3}+f_4(x)\frac{h^4}{2.3.4}+\cdots$$

Or, putting y for $f(x)$,

$$f(x+h)=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{2}+\frac{d^3y}{dx^3}\frac{h^3}{2.3}+\frac{d^4y}{dx^4}\frac{h^4}{2.3.4}+\ldots[1],$$

which development is *Taylor's theorem*, h being either positive or negative. From this theorem of Taylor that of Maclaurin may be easily deduced thus:—

After the general differential coefficients have been obtained, and the above general identity exhibited, let x take the particular value $x=0$, then

$$f(h)=[y]+\left[\frac{dy}{dx}\right]h+\left[\frac{d^2y}{dx^2}\right]\frac{h^2}{2}+\left[\frac{d^3y}{dx^3}\right]\frac{h^3}{2.3}+\left[\frac{d^4y}{dx^4}\right]\frac{h^4}{2.3.4}+\cdots$$

Now each of the bracketed coefficients contains constants only; they are, therefore, all independent of the value of h, which value may $\therefore$ be any whatever, without affecting these coefficients: we may, therefore, replace it by x, or by any other symbol, so that

$$f(x)=y=[y]+\left[\frac{dy}{dx}\right]x+\frac{1}{2}\left[\frac{d^2y}{dx^2}\right]x^2+\frac{1}{2.3}\left[\frac{d^3y}{dx^3}\right]x^3+\ldots\ldots$$

which is the theorem before established.

We shall now proceed to apply Taylor's theorem to one or two developments in different cases of the function $f(x+h)$: but the principal uses of the theorem will be shown hereafter. Its importance in effecting algebraic developments is small in comparison with its value in suggesting geometrical and analytical theorems by a discussion of its coefficients.

(1) To develope $\sin(x+h)$ in a series of powers of the arcual measure of the angle h.

$$\text{Put } y=\sin x,\ \therefore \frac{dy}{dx}=\cos x,\ \frac{d^2y}{dx^2}=-\sin x,\ \frac{d^3y}{dx^3}=-\cos x,\ \&c.,$$

$$\therefore \sin(x+h)=\sin x+\cos x\,.\,h-\sin x\frac{h^2}{2}-\cos x\frac{h^3}{2.3}+\sin x\frac{h^4}{2.3.4}-\cdots$$

$$=\sin x\left(1-\frac{h^2}{2}+\frac{h^4}{2.3.4}-\ldots\right)+\cos x\left(h-\frac{h^3}{2.3}+\frac{h^5}{2.3.4.5}-\ldots\right),$$

which merely brings us back to the known property

$$\sin(x+h)=\sin x\cos h+\cos x\sin h.$$

(2) Develope $\tan^{-1}(x+h)$ by Taylor's theorem. Put $y=\tan^{-1}x$,

$$\therefore \frac{dy}{dx}=\frac{1}{\sec^2 y}=\cos^2 y,\ \frac{d^2y}{dx^2}=-2\sin y\cos y\frac{dy}{dx}=-\sin 2y\cos^2 y,$$

$$\frac{d^3y}{dx^3}=-2(\cos 2y\cos^2 y-\sin 2y\sin y\cos y)\frac{dy}{dx}=-2\cos(y+2y)\cos y\frac{dy}{dx}$$

$$=-2\cos 3y\cos^3 y,\ \frac{d^4y}{dx^4}=2.3(\sin 3y\cos^3 y+\cos 3y\sin y\cos^2 y)\frac{dy}{dx}$$

$$=2.3\sin 4y\cos^2 y\frac{dy}{dx}=2.3\sin 4y\cos^4 y,\ \&c.,$$

$$\therefore \tan^{-1}(x+h)=y+\cos^2 y.h-\frac{\sin 2y\cos^2 y}{2}h^2-\frac{\cos 3y\cos^3 y}{3}h^3+\frac{\sin 4y\cos^4 y}{4}h^4+\ldots$$

For exercises in Taylor's theorem the student may develope the functions $\cos(x+h)$, $\log(x+h)$, and $\tan(x+h)$.

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556. **Limits of Taylor's Theorem.**—From attentively examining the reasoning by which the theorem of Taylor has been established, it will be seen that there is strict algebraic equivalence between $f(x+h)$ and the series which that theorem affirms to be its development, only on one or other of two conditions.

1. The series must spontaneously terminate, in consequence of a *constant* differential coefficient being arrived at.

2. But if no such constant differential coefficient can ever be reached, and the series be therefore interminable, there can be strict equivalence between the infinite series and its involopment $f(x+h)$, only for such values of h as cause that series to be *convergent*, either from the beginning, or after a finite number of terms.

As to the cases in which, for particular values of x, any of the differential coefficients become *infinite*, they are, of course, excluded from the theorem; the development in the proposed form, with *finite* coefficients, being then impossible; as in the analogous cases of Maclaurin's theorem. Such are improperly called the *failing cases* of Taylor's theorem: they will be examined into hereafter; our business, at present, is with *Taylor's theorem*, and not with cases that do not belong to it, and which are expressly excluded from it by the hypothetical conditions on which it is founded.

The purport of the present article is to show how the *Remainder*, or supplementary correction of the series [1], p. 526, may be estimated, when we stop the summation of it at any particular term, thus neglecting all that follow, inclusive of what may be implied in the "&c.," when the series is not convergent.

Let $f(x)$ be a function of x, such that it and its several differential coefficients are finite and continuous between the limits $x=a$, $x=a+h$. And in the expression

$$f(a+x)-f(a)-f_1(a)x-f_2(a)\frac{x^2}{2}-f_3(a)\frac{x^3}{2.3}-\ldots-f_{n-1}(a)\frac{x^{n-1}}{2.3\ldots(n-1)}-R\frac{x^n}{2.3\ldots n}\ldots[1],$$

let R be such a finite quantity, not involving x, that when $x=h$, the ex-

pression [1] may $=0$: then, since the expression is also 0 when $x=0$, it follows that the function [1] becomes zero for two different values of the variable x. Consequently it cannot continually increase, nor continually diminish, as x passes through its intermediate values, from $x=0$, up to $x=h$:—it must *somewhere*, in the interval, change from an increasing state to a diminishing state, or vice versâ; that is, calling the function [1] Y, $\frac{\Delta Y}{\Delta x}$, and consequently $\frac{dY}{dx}$, must change from $+$ to $-$, or from $-$ to $+$, at least *once* in the interval. But, for a continuously varying finite quantity to thus change its sign, it must first pass through the value zero, $\therefore$ there is some value x_1 of x, between 0 and h, for which the differential coefficient of [1] is zero, that is, for which

$$f_1(a+x)-f_1(a)-f_2(a)x-f_3(a)\frac{x^2}{2}-\ldots-R\frac{x^{n-1}}{2.3\ldots(n-1)}\ldots[2],$$

is zero. But [2] is also zero, for $x=0$: hence, as before, there must be some value x_2 of x, between 0 and h, for which the differential coefficient of [2] is zero.

Continuing this reasoning up to n differentiations of [1], we have finally $f_n(a+x)-R=0$, for some value x_n of x, between 0 and h. Let this value be $x=\theta h$, where θ is some proper fraction; then $R=f_n(a+\theta h)$. Substituting this value of R in [1], and remembering that [1]$=0$ when $x=h$, we have

$$f(a+h)=f(a)+f_1(a)h+f_2(a)\frac{h^2}{2}+f_3(a)\frac{h^3}{2.3}+\ldots+f_{n-1}(a)\frac{h^{n-1}}{2.3\ldots(n-1)}+$$
$$f_n(a+\theta h)\frac{h^n}{2.3\ldots n}\ldots[1],$$

which is Lagrange's theorem on the Limits of Taylor's theorem, to which it gives the necessary completion; for it is plain that the constant functions $f(a)$, $f_1(a)$, &c., are no other than what $f(x)$, $\frac{df(x)}{dx}$, &c., become when x takes *any* value a: the letter a may therefore be replaced by x.* The symbol θ denotes some fraction, about which all we know is, that it must be less than unity, that is, it must be between 0 and 1, so that when Taylor's theorem, in any particular case, is interminable, and we take n terms of the series for the development of the function $f(a+h)$, we know that the *Remainder*, or supplementary correction necessary to complete that development, will be a quantity lying between the limits

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* The foregoing investigation of Lagrange's theorem is due to Mr. Homersham Cox, who published it originally in the Cambridge and Dublin Mathematical Journal, and afterwards in an appendix to his rudimentary work on the Integral Calculus, in Mr. Weale's well-known series. It is this latter version of Mr. Cox's method of proof which, with slight modifications, we have adopted above. For a different way of establishing the theorem, the student is referred to the large work on the Calculus, by Professor De Morgan, a work which every student of the Higher Analysis ought to possess, as it not only comprehends a more extensive amount of information on the Differential and Integral Calculus, than any other English publication, but being the work of one who has evidently examined every point of difficulty for himself, and has exercised an independent judgment upon all, it is replete with matter for profitable reflection, even to the advanced analyst.

$f_n(a)\,\frac{h^n}{2.3...n}$, and $f_n(a+h)\,\frac{h^n}{2.3...n}$. If h be so small that $\frac{h^n}{2.3...n}$ tends continuously to zero as n increases, becoming confounded with it when $n=\infty$, in other words, if the series be convergent, this correction vanishes; and the series is complete in the form in which it was established at (555). Let us apply Lagrange's correction of Taylor's theorem to an example or two: and first for the development of $(a+h)^n$. Here we have

$$(a+h)^n=a^n+n(a+\theta h)^{n-1}h,$$

$$\text{or}\quad =a^n+na^{n-1}h+n\frac{n-1}{2}(a+\theta h)^{n-2}h^2,$$

$$\text{or}\quad =a^n+na^{n-1}h+n\frac{n-1}{2}a^{n-2}h^2+n\frac{n-1}{2}\cdot\frac{n-2}{3}(a+\theta h)^{n-3}h^3,$$

and so on, where it must be observed that although θ is less than unity in every one of these expressions, it is not the same in all.

Again, for the complete development of $\log(x+h)$, we have

$$\log(x+h)=\log x+\frac{h}{x}-\frac{1}{2}\frac{h^2}{x^2}+\frac{1}{3}\frac{h^3}{x^3}-\ldots\frac{(-1)^{n-2}}{n-1}\frac{h^{n-1}}{x^{n-1}}+\frac{(-1)^{n-1}}{n}\frac{h^n}{(x+\theta h)^n}.$$

557. **Limits of Maclaurin's Theorem.**—Putting $a=0$ in [1], and replacing h, which may be anything, by x, we have

$$f(x)=f(0)+f_1(0)x+f_2(0)\frac{x^2}{2}+f_3(0)\frac{x^3}{2.3}+\ldots+f_{n-1}(0)\frac{x^{n-1}}{2.3...(n-1)}+$$

$$f_n(\theta x)\frac{x^n}{2.3...n}\ldots\ldots[2].$$

For example: let $f(x)=\sin x$, $\therefore f(0)=0$, $f_1(0)=1$, $f_2(0)=0$, $f_3(0)=-1$, &c., as at p. 524,

$$\therefore \sin x=0+\cos\theta x\,.\,x=x-\sin\theta x\,.\,\frac{x^2}{2}=x-\cos\theta x\,.\,\frac{x^3}{2.3}$$

$$=x-\frac{x^3}{2.3}+\sin\theta x\,.\,\frac{x^4}{2.3.4}=x-\frac{x^3}{2.3}+\cos\theta x\,.\,\frac{x^5}{2.3.4.5}$$

$$=x-\frac{x^3}{2.3}+\frac{x^5}{2.3.4.5}-\sin\theta x\,.\,\frac{x^7}{2.3.4.5.6.7},\text{ and so on.}$$

And, in like manner, for the complete development of e^x we have

$$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{2.3}+\ldots+\frac{x^{n-1}}{2.3...(n-1)}+e^{\theta x}\frac{x^n}{2.3...n}.$$

Since θ lies between 0 and 1, e^θ must lie between 1 and e, so that the remainder after n terms lies between

$$\frac{x^n}{2.3...n}\text{ and }e^x\frac{x^n}{2.3...n}.$$

It may be remarked in reference to the development of an exponential function, that the series gives but one value—the arithmetical value as it is usually called. If x were $=\frac{1}{2}$, the series would not supply the negative square root of e, though $\sqrt{e^\theta}$ would enter it, thus seeming to provide for the double value; for this negative root is minus the *entire series*. It must be remembered that differentiation is applicable only when the variable to

which it is applied varies *continuously:* if x vary from $x=0$ to $x=1$, the interval $e^0 \ldots\ldots e^1$ must be capable of being filled up by a continuous series of values, uniting these two extremes: isolated values of e^x, occurring within the range $x=0$, $x=1$, and which fall out of the continuous series, are unrecognized by the calculus; and we must not look to its results to account for them. It is, no doubt, common for functions of x, during the continuous progress of that variable from one value to another, to change from positive to negative; but the change is from a continuous series of positive values, to a continuous series of negative values; and each series may be regarded as distinct, and the interval be accordingly cut up into partial intervals, as instanced at p. 519. But in neither series can a value be accompanied by what may be called its congeneric value, of opposite sign, as is plain; for such a value would break the continuity. If the multiple values of e^x are required to be symbolized in its development, we must connect with that development the factor 1^x.

In the remaining portion of the present work we shall have but very little occasion to employ Lagrange's correction of Taylor's development, which correction, as we have seen, is necessary only when that development is a diverging infinite series, and such series, being useless for the purposes either of practical calculation, or of theoretical research, will always be rejected (see NOTE, p. 83).

Whenever the development is a converging infinite series, that series alone, taken, of course, in all its infinity of terms, is the complete development, no correction of it or supplementary addition to it being requisite. It is true that even in *such* series, Lagrange's correction will exhibit the limits of error committed in summing up only a specified number of terms, and rejecting all the rest, and so far it may be useful in numerically evaluating functions approximatively, in particular cases; but the doing this will frequently involve the necessity of actually computing the involvement itself, the labour of which it is usually the object of the development to avoid, even when that labour is practicable. For instance, in calculating sin x in terms of the arcual value of x (p. 529), if we wish to know the numerical limits of error committed in stopping at the third term of the series, we must actually know the value of sin x, which is supposed at first to be unknown. Nevertheless, it is of interest to learn that whatever the correct value of sin x may be, the sum of the three terms mentioned differs from that correct value by less than the $\frac{x^7}{2.3.4.5.6.7}$ part of the whole. And, in like manner, by taking n terms of the series for e^x, the sum of them will differ from the correct value by less than the $\frac{x^n}{2.3 \ldots n}$ part of the whole; but to find the actual numerical value of this part, for a proposed value of x, would require that e^x be independently computed.

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558. **Differentiation of Compound Functions.** — Let $u=F(y)$, where $y=f(x)$; and let it be required to determine $\frac{du}{dx}$. Here we have

$$\Delta y = f(x+\Delta x) - f(x), \text{ and } \Delta u = F(y+\Delta y) - F(y).$$

By Taylor's theorem,

$$\Delta y=\frac{dy}{dx}\Delta x+\frac{d^2y}{dx^2}\frac{(\Delta x)^2}{2}+\frac{d^3y}{dx^3}\frac{(\Delta x)^3}{2.3}+\dots$$

Also, by the same theorem,

$$\Delta u=\frac{du}{dy}\Delta y+\frac{d^2u}{dy^2}\frac{(\Delta y)^2}{2}+\frac{d^3u}{dy^3}\frac{(\Delta y)^3}{2.3}+\dots$$

$$=\frac{du}{dy}\left\{\frac{dy}{dx}\Delta x+\frac{d^2y}{dx^2}\frac{(\Delta x)^2}{2}+\dots\right\}+\frac{1}{2}\frac{d^2y}{dy^2}\left\{\frac{dy}{dx}\Delta x+\frac{d^2y}{dx^2}\frac{(\Delta x)^2}{2}+\dots\right\}^2+\dots$$

$$\therefore\ \frac{u}{\Delta x}=\frac{du}{dy}.\frac{dy}{dx}+P\Delta x+Q(\Delta x)^2+\dots,\ \therefore\ \frac{du}{dx}=\frac{du}{dy}.\frac{dy}{dx}\dots[1].$$

Hence the differential coefficient $\frac{du}{dx}$ is found by differentiating u, on the hypothesis that y is the independent variable, and then multiplying the coefficient thus obtained by that derived from y, considered as a function of x. Whenever y varies uniformly with x,

$\frac{dy}{dx}$ must be constant, for then $\frac{\Delta y}{\Delta x}$ is always the same:

hence, in this case, $\frac{d^2y}{dx^2}=0$, $\therefore\ \frac{\Delta u}{\Delta x}=\frac{du}{dy}.\frac{\Delta y}{\Delta x}\dots[2]$.

The following examples will illustrate the application of [1]:—

(1) $u=a^y$, where $y=b^x$,

$$\frac{du}{dy}=a^y\log a,\ \frac{dy}{dx}=b^x\log b,$$

$$\therefore\ \frac{du}{dx}=\frac{du}{dy}.\frac{dy}{dx}=a^{b^x}.b^x\log a.\log b.$$

(2) $u=\log y$, where $y=\log x$,

$$\frac{du}{dy}=\frac{1}{y},\ \frac{dy}{dx}=\frac{1}{x},$$

$$\therefore\ \frac{du}{dx}=\frac{du}{dy}.\frac{dy}{dx}=\frac{1}{x\log x}.$$

(3) $u=\cot a^y$, where $y=\log\frac{x}{\sqrt{(a^2+x^2)}}$

$$\frac{du}{dy}=-\operatorname{cosec}^2a^y.a^y\log a,\ \frac{dy}{dx}=\frac{a^2}{x(a^2+x^2)},$$

$$\therefore\ \frac{du}{dx}=-\operatorname{cosec}^2a^y.a^y\log a.\frac{a^2}{x(a^2+x^2)}.$$

(4) $u=\sin\left(\frac{x}{a+x}\right)^2$. Put $\left(\frac{x}{a+x}\right)^2=y$,

$$\therefore\ \frac{du}{dy}=\frac{d\sin y}{dy}=\cos y,\ \frac{dy}{dx}=\frac{2ax}{(a+x)^3},$$

$$\therefore\ \frac{du}{dx}=\cos\left(\frac{x}{a+x}\right)^2.\frac{2ax}{(a+x)^3}.$$

559. Suppose now that $u=F(y,z)$, where y and z are both functions of x, and that it be required to determine $\frac{du}{dx}$.

Then $\Delta u=F\{(y+\Delta y),\ (z+\Delta z)\}-F(y,\ z)$,

which difference of u, by adding and subtracting the same quantity, as below, may be split into the two differences,

$$\Delta u=[F\{y,\ (z+\Delta z)\}-F(y,\ z)]+[F\{(y+\Delta y),\ (z+\Delta z)\}-F\{y,\ (z+\Delta z)\}].$$

Now the first of these two differences is got by changing the z in u into $z+\Delta z$, and leaving the y untouched, as if it were a constant: the second of the differences is got by taking the changed function, namely, $F\{y,(z+\Delta z)\}$, as an original function, and changing y into $y+\Delta y$, leaving the z untouched as if it were a constant, and diminishing the result, as

before, by that original function. Hence, so far as the first partial difference is concerned,

$$\frac{\Delta u}{\Delta x}=\frac{\Delta F(y,\ z),\text{ when } z \text{ alone takes an inc. } \Delta z}{\Delta z}\cdot\frac{\Delta z}{\Delta x}.$$ And so far as the second is concerned,

$$\frac{\Delta u}{\Delta x}=\frac{\Delta F\{y,\ (z+\Delta z)\},\text{ when } y \text{ alone takes an inc. } \Delta y}{\Delta y}\cdot\frac{\Delta y}{\Delta x}.$$

Hence, in the limit, when Δx, and consequently Δy, Δz, each become zero, we have for the complete differential coefficient $\frac{du}{dx}$,

$$\frac{du}{dx}=\frac{du}{dz}\frac{dz}{dx}+\frac{du}{dy}\frac{dy}{dx}\text{.....[3]},$$

that is, we are to differentiate the compound function u with respect to each of the component functions y, z, as if the other were a constant, and are then to add the two results together, each result being of the form [1] given in last article. If z is simply the independent variable x, the expression [3] becomes

$$\left\{\frac{du}{dx}\right\}=\frac{du}{dx}+\frac{du}{dy}\frac{dy}{dx}\text{.....[4]},$$

where the *total* differential coefficient is enclosed in braces to distinguish it from the *partial* diff. coef. on the right, which is the diff. coef. derived from u by treating y as if it were not a variable, but a constant. We shall give an application or two of these forms, y being always a function of x.

(1) $u=\cot x^y$. By [4],

$$\left\{\frac{du}{dx}\right\}=\frac{du}{dx}+\frac{du}{dy}\frac{dy}{dx}.$$

$$\text{Now } \frac{du}{dx}=-\operatorname{cosec}^2 x^y\,\frac{dx^y}{dx}$$

$$=-\operatorname{cosec}^2 x^y\,.\,yx^{y-1},$$

$$\frac{du}{dy}=-\operatorname{cosec}^2 x^y\,\frac{dx^y}{dy}=-\operatorname{cosec}^2 x^y\,.\,x^y\log x,$$

$$\therefore\ \left\{\frac{du}{dx}\right\}=-x^y\operatorname{cosec}^2 x^y\left(\frac{y}{x}+\frac{dy}{dx}\log x\right),$$

where the operation $\frac{dy}{dx}$ must remain unexecuted till the function y is assigned.

(2) $u=\log\tan\frac{x}{y}$,

$$\therefore\ \frac{du}{dx}=\frac{d\tan\frac{x}{y}}{\tan\frac{x}{y}\,dx}=\frac{\sec^2\frac{x}{y}\,.\,\frac{1}{y}}{\tan\frac{x}{y}},$$

$$\frac{du}{dy}=\frac{d\tan\frac{x}{y}}{\tan\frac{x}{y}\,dy}=-\frac{\sec^2\frac{x}{y}\,.\,\frac{x}{y^2}}{\tan\frac{x}{y}},$$

$$\therefore\ \left\{\frac{du}{dx}\right\}=\frac{\sec^2\frac{x}{y}}{y^2\tan\frac{x}{y}}\left(y-x\frac{dy}{dx}\right)$$

$$=\frac{y-x\frac{dy}{dx}}{y^2\sin\frac{x}{y}\cos\frac{x}{y}}=\frac{y-x\frac{dy}{dx}}{\frac{1}{2}y^2\sin 2\frac{x}{y}}.$$

(3) $u=\tan^{-1}(xy)$, where $y=e^x$,

$$\frac{du}{dx}=\frac{y}{1+(xy)^2},\ \frac{du}{dy}=\frac{x}{1+(xy)^2}.$$

$$\text{Also } \frac{dy}{dx}=e^x,$$

$$\therefore\ \left\{\frac{du}{dx}\right\}=\frac{y+xe^x}{1+(xy)^2}=\frac{(1+x)e^x}{1+x^2e^{2x}}.$$

(4) $u=z^2+y^3+zy$, where $z=\sin x$, and $y=e^x$,

$$\frac{du}{dz}=2z+y,\ \frac{du}{dy}=3y^2+z.$$

$$\text{Also } \frac{dz}{dx}=\cos x,\text{ and } \frac{dy}{dx}=e^x,$$

$$\therefore\text{ [3], }\left\{\frac{du}{dx}\right\}=(2z+y)\cos x+(3y^2+z)e$$

$$=(2\sin x+e^x)\cos x+(3e^{2x}+\sin x)e^x$$

$$=\sin 2x+e^x(\sin x+\cos x)+3e^{3x}.$$

NOTE.—It is sometimes easier to replace y and z by the functions of x which they represent, and then to differentiate u directly: thus, in the above example,

$$u=\sin^2 x+e^{3x}+e^x \sin x, \text{ which differentiated, gives}$$

$$\frac{du}{dx}=2 \sin x \cos x+3e^{3x}+e^x \cos x+\sin x e^x=\sin 2x+e^x(\sin x+\cos x)+3e^{3x},$$

the same, of course, as before.

EXAMPLES FOR EXERCISE.

(1) $u=x^3+3axy+y^3$.
(2) $u=x \log y$.
(3) $u=\log\{x-a+\sqrt{(x^2-2ax)}\}$.
(4) $u=(\cos x)^y$, where $y=\sin x$.
(5) $u=y+z$, where $y=\sqrt{(1-x^2)}$, and $z=\sin^{-1}x$.

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560. **Differentiation of Implicit Functions.**—If the variables x, y, are connected together in an equation $F(x, y)=0$, then the function that y is of x is not explicitly given: it is only implied in this expression of the relation between the two variables. But to obtain the differential coefficient $\frac{dy}{dx}$ there is no necessity to solve the equation for y, and thus to determine what function of x, y really denotes; the formula [4] suffices for this purpose: for putting

$$u=F(x, y)=0, \therefore \left\{\frac{du}{dx}\right\}=\frac{du}{dx}+\frac{du}{dy}\frac{dy}{dx}=0, \therefore \frac{dy}{dx}=-\frac{du}{dx}\div\frac{du}{dy};$$

we have only, therefore, to differentiate the several terms of the expression, first with respect to x, as if y were constant, then with respect to y, as if x were constant; to divide the former result by the latter, and to prefix the minus sign. Or, without any special rule, it is enough to differentiate the several terms of u in the usual way—remembering the form [1], since y is a function of x—and then, by equating the result to 0, to determine $\frac{dy}{dx}$.

(1) $u=x^3+3axy+y^3=0$,

$$\therefore \frac{du}{dx}=3x^2+3ay, \frac{du}{dy}=3ax+3y^2, \therefore \frac{dy}{dx}=-\frac{x^2+}{ax+y^2}.$$

Or, differentiating in the usual manner,

$$3x^2+3ay+3ax\frac{dy}{dx}+3y^2\frac{dy}{dx}=0,$$

$$\therefore (ax+y^2)\frac{dy}{dx}=-(x^2+ay), \therefore \frac{dy}{dx}=-\frac{x^2+ay}{ax+y^2}.$$

If the second differential coefficient be required, we have

$$\frac{d^2y}{dx^2}=-\frac{(ax+y^2)\left(2x+a\frac{dy}{dx}\right)-(x^2+ay)\left(a+2y\frac{dy}{dx}\right)}{(ax+y^2)^2}. \left(\text{Or substituting for } \frac{dy}{dx}\right),$$

$$=-\frac{(ax+y^2)(2ax^2+2xy^2-ax^2-a^2y)-(x^2+ay)(a^2x+ay^2-2x^2y-2ay^2)}{(ax+y^2)^3}$$

$$=-\frac{2xy^4+6ax^2y^2+2x^4y-2a^3xy}{(ax+y^2)^3}=-\frac{2xy(y^3+3axy+x^3-a^3)}{(ax+y^2)^3}.$$

But $x^3+3axy+y^3=0$, $\therefore \frac{d^2y}{dx^2}=\frac{2a^3xy}{(ax+y^2)^3}$.

(2) Given $y^3-3y+x=0$: to develope y in ascending powers of x.

$$3y^2\frac{dy}{dx}-3\frac{dy}{dx}+1=0, \ \therefore \frac{dy}{dx}=\frac{1}{3(1-y^2)}, \ \frac{d^2y}{dx^2}=\frac{6y\frac{dy}{dx}}{9(1-y^2)^2}=\frac{2y}{3^2(1-y^2)^3},$$

$$\frac{d^3y}{dx^3}=\frac{3^2(1-y^2)^3.2\frac{dy}{dx}+2y.3^3(1-y^2)^2 2y\frac{dy}{dx}}{3^4(1-y^2)^6}=\frac{2(1+5y^2)\frac{dy}{dx}}{3^2(1-y^2)^4}, \text{ \&c.}$$

Now when $x=0$, $y=0$, or $y=\sqrt{3}$; taking only the first value, we have

$$[y]=0, \ \left[\frac{dy}{dx}\right]=\frac{1}{3}, \ \left[\frac{d^2y}{dx^2}\right]=0, \ \left[\frac{d^3y}{dx^3}\right]=\frac{2}{3^3}, \text{ \&c.}$$

Hence, by Maclaurin's theorem, we have, as far as the first three terms, the second being 0,

$$y=\frac{x}{3}+\frac{x^3}{3^4}+\text{\&c.}=\frac{1}{3}\left\{x+\left(\frac{x}{3}\right)^3+\text{\&c.}\right\}\ldots\ldots[1].$$

It has not been thought worth while to carry on the differentiation further: the example is given merely for the purpose of showing how Maclaurin's theorem may be applied to the development of an implicit function of an independent variable. The process here exemplified has been affirmed to be applicable to the solution of cubic and higher equations, by means of infinite series; but its efficiency in this way is so exceedingly limited, that it is all but valueless for the purpose; since, to be of any use, the resulting series must be convergent. In the present example, for instance, if any given number a be put for x in [1], a root of the equation $y^3-3y+a=0$ might seem to be given by the series, whatever that number a may be; but if it be such as to render the series divergent—which is more likely than otherwise—the expression is useless, on account of the unknown value concealed under the "&c."

Examples for Exercise.

(1) $y^3+2xy+x^2=a^2$, to find $\frac{dy}{dx}$.

(2) $x^3+y^3=a^3$, to find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$.

(3) $y^3-2xy+a^2=0$, to find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$.

(4) $\frac{y^2}{(x-a)^2}=2y\frac{\sqrt{(x-b)}}{x-a}-x$, to find $\frac{dy}{dx}$.

(5) $\text{Log}\frac{a+\sqrt{(a^2-y^2)}}{y}=\frac{x+\sqrt{(a^2-y^2)}}{a}$, to find $\frac{dy}{dx}$.

(6) $y^3-xy=1$, to develope y in powers of x, as far as x^3.

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561. **Vanishing Fractions.**—Expressions which have this designation have already been discussed in the Algebra (p. 144); and the

student will have perceived by this time that the process at (190), for finding the value of any such fraction, is virtually a process of differentiation, though by aid of the simple theorem given at (134), common algebra sufficed to suggest and carry on the operations there necessary.

The determination of the true value which $\frac{F(x)}{f(x)}$ represents, when for a particular value a of x the undeveloped fraction assumes the form $\frac{0}{0}$, can always be effected whenever $F(x+h)$ and $f(x+h)$ are developable in a series, with finite coefficients, proceeding according to the positive powers of h, whether integral or fractional. This it is our present purpose to show, explaining first how the evaluation of the vanishing fraction is to be determined when the developments spoken of are given by Taylor's theorem, and then how the value is to be found when that theorem does not apply, in consequence of the occurrence of a differential coefficient, which becomes infinite in the proposed hypothesis of $x=a$; such occurrence being a sure indication that, for the assigned value of x, the development is *not* according to the positive integral powers of h (556). The following lemma must be premised:—

Lemma.—In the general development

$$F(x+h)=F(x)+F_1(x)h+F_2(x)\frac{h^2}{2}+\ldots,$$

it is impossible that any particular value given to x can cause $F(x)$, and at the same time *all* the differential coefficients $F_1(x)$, $F_2(x)$, &c., to vanish. [It is, of course, implied, from the absence of the "&c." after the series, that h is always sufficiently small to render the series convergent for the proposed value of x.]

For if such could be the case, then, for that particular value a of x, we should have $F(a+h)=0$; but by hypothesis $F(a)=0$; hence, whatever values put for a satisfy this equation, the same values put for $a+h$ must satisfy $F(a+h)=0$. In other words, $a=a+h$, which is impossible, however small h may be short of $h=0$.

This truth being established, in the proposed fraction $\frac{F(x)}{f(x)}$, which takes the form $\frac{0}{0}$ when $x=a$, let x be changed into $x+h$, then, by Taylor's theorem,

$$\left.\frac{F(x+h)}{f(x+h)}=\frac{F(x)+F_1(x)h+\frac{1}{2}F_2(x)h^2+\frac{1}{2.3}F_3(x)h^3+\ldots\ldots}{f(x)+f_1(x)h+\frac{1}{2}f_2(x)h^2+\frac{1}{2.3}f_3(x)h^3+\ldots\ldots}\right\}\ldots[1].$$

In this identity put a for x, then, since by hypothesis $F(a)=0$, and $f(a)=0$,

$$\left.\frac{F(a+h)}{f(a+h)}=\frac{F_1(a)+\frac{1}{2}F_2(a)h+\frac{1}{2.3}F_3(a)h^2+\ldots\ldots}{f_1(a)+\frac{1}{2}f_2(a)h+\frac{1}{2.3}f_3(a)h^2+\ldots\ldots}\right\}\ldots[2].$$

And this, when $h=0$, becomes $\frac{F(a)}{f(a)}=\frac{0}{0}=\frac{F_1(a)}{f_1(a)}$,

and, therefore, if $F_1(a)$, $f_1(a)$, do not themselves both vanish, the value of the fraction, in the proposed hypothesis, is determined.

But if $F_1(a)=0$, and $f_1(a)=0$, then [2] becomes simply

$$\frac{F(a+h)}{f(a+h)}=\frac{F_2(a)+\frac{1}{3}F_3(a)h+\ldots\ldots}{f_2(a)+\frac{1}{3}f_3(a)h+\ldots\ldots}, \quad \therefore \frac{F(a)}{f(a)}=\frac{F_2(a)}{f_2(a)}.$$

In like manner, if this equivalent of the proposed vanishing fraction should itself also be a vanishing fraction, we should have to equate the former to $\frac{F_3(a)}{f_3(a)}$; and so on, till, by successive differentiation of $F(x), f(x)$, we at length arrive at results which do not both vanish for $x=a$:—results which must inevitably be attainable in virtue of the preceding lemma. All this is in complete consistency with what is established at art. 190, where the process here described has been sufficiently exemplified in the case of a common algebraic fraction: we shall here, therefore, give a few examples of its application to *transcendental* functions only, that is, to functions involving exponential, logarithmic, or trigonometrical expressions.

(1) Required the value of

$$\frac{a^x-b^x}{x}, \text{ when } x=0.$$

$$\text{Here } \frac{F_1(x)}{f_1(x)}=\log a.a^x-\log b.b^x,$$

$$\therefore \frac{F_1(0)}{f_1(0)}=\log a-\log b=\log\frac{a}{b},$$

the value required.

(2) Required the value of

$$\frac{e^x-e^{-x}}{\sin x}, \text{ when } x=0.$$

$$\text{Here } \frac{F_1(x)}{f_1(x)}=\frac{e^x+e^{-x}}{\cos x},$$

$$\therefore \frac{F_1(0)}{f_1(0)}=\frac{1+1}{1}=2,$$

the value required.

(3) Required the value of

$$\frac{1-\cos x}{x^2}, \text{ when } x=0.$$

$$\text{Here } \frac{F_1(x)}{f_1(x)}=\frac{\sin x}{2x}=\frac{0}{0}, \text{ when } x=0,$$

$$\frac{F_2(x)}{f_2(x)}=\frac{\cos x}{2}, \quad \therefore \frac{F_2(0)}{f_2(0)}=\frac{1}{2},$$

the value required.

(4) Required the value of

$$\frac{2x\sin x-\pi}{\cos x}, \text{ when } x=\frac{\pi}{2}.$$

$$\text{Here } \frac{F_1(x)}{f_1(x)}=\frac{2\sin x+2x\cos x}{-\sin x},$$

$$\therefore \frac{F_1\left(\frac{\pi}{2}\right)}{f_1\left(\frac{\pi}{2}\right)}=-2, \text{ the value required.}$$

(5) Required the value of

$$\frac{(x-a)^n}{e^x-e^a}, \text{ when } x=a.$$

$$\text{Here } \frac{F_1(x)}{f_1(x)}=\frac{n(x-a)^{n-1}}{e^x}.$$

Hence, according as $n>1$, <1, or $=1$, we have

$$\frac{F_1(a)}{f_1(a)}=0, \text{ or } \infty, \text{ or } e \ .$$

If, in seeking to determine the value of a vanishing fraction by the above process, we should arrive at a differential coefficient which, for the assigned value of the variable, becomes infinite, we shall be apprized that, for that particular value of the variable, the development of the function, according to the positive integral powers of x, is impossible. But a method analogous to the foregoing is still to be applied; thus: Let $\frac{F(x)}{f(x)}$

become $\frac{0}{0}$ when $x=a$: substitute $a+h$ for x, and let the terms of the fraction thus changed be developed either by involution, the extraction of roots, or any other algebraic process, so that we may have

$$\frac{F(a+h)}{f(a+h)}=\frac{Ah^{\alpha}+Bh^{\beta}+\ldots}{A'h^{\alpha'}+B'h^{\beta'}+\ldots},$$

the terms of each series being arranged so that α and α' are the smallest exponents in the respective series, β and β' the next in magnitude, and so on: one or other of these three cases will present itself; either

1. $\alpha>\alpha'$, or 2. $\alpha=\alpha'$, or 3. $\alpha<\alpha'$.

In the first case, by dividing numerator and denominator by $h^{\alpha'}$, and then putting $h=0$,

the result will be $\frac{F(a)}{f(a)}=\frac{0}{A'}=0$.

In the second case, the result of the same operation will be $\frac{F(a)}{f(a)}=\frac{A}{A'}$.

In the third case, by dividing the num. and den. by h^{α},

the result for $h=0$ will be $\frac{F(a)}{f(a)}=\frac{A}{0}=\infty$.

It appears, therefore, that the development of numerator and denominator of the fraction, after $a+h$ is put for x, need not be extended beyond the first significant term, or that containing the lowest exponent of h: and that, according as this leading exponent in the numerator is greater than, equal to, or less than, that in the denominator, will the true value of the fraction be 0, finite, or infinite. And the finding these leading exponents by common algebra will sometimes be an easier way of arriving at the value of the fraction than by the process of differentiation. The examples at p. 146 will illustrate this method, to which the two following are added.

(1) Since when $x=a$, $\frac{\sqrt{x}-\sqrt{a}+\sqrt{(x-a)}}{\sqrt{(x^2-a^2)}}=\frac{0}{0}$, put $a+h$ for x, then

$$\frac{F(a+h)}{f(a+h)}=\frac{(a+h)^{\frac{1}{2}}-a^{\frac{1}{2}}+h^{\frac{1}{2}}}{h^{\frac{1}{2}}(2a+h)^{\frac{1}{2}}}=\frac{h^{\frac{1}{2}}+\ldots}{(2ah)^{\frac{1}{2}}+\ldots}, \therefore \frac{F(a)}{f(a)}=\frac{1}{(2a)^{\frac{1}{2}}}.$$

(2) Since when $x=a$, $\frac{(x^2-a^2)^{\frac{3}{2}}+x-a}{(1+x-a)^3-1}=\frac{0}{0}$, put $a+h$ for x, then

$$\frac{F(a+h)}{f(a+h)}=\frac{h^{\frac{3}{2}}(2a+h)^{\frac{3}{2}}+h}{(1+h)^3-1}=\frac{h+\ldots}{3h+\ldots}, \therefore \frac{F(a)}{f(a)}=\frac{1}{3}.$$

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562. **Fractions of the Form $\frac{\infty}{\infty}$.**—When this is the form which the general fraction assumes for a particular value of the variable, it may easily be converted into the form $\frac{0}{0}$ just discussed. For if

$$\frac{F(a)}{f(a)}=\frac{\infty}{\infty}, \text{ then also } \frac{F'(a)}{f(a)}=\frac{\frac{1}{f(a)}}{\frac{1}{F(a)}}=\frac{0}{0}\ldots[1].$$

Now $d\left(\frac{1}{f(x)}\right)=\frac{-f_1(x)}{\{f(x)\}^2}$, $\therefore$ [1], $\frac{F(a)}{f(a)}=\left\{\frac{F(a)}{f(a)}\right\}^2\cdot\frac{f_1(a)}{F_1(a)}$, $\therefore$ $1=\frac{F(a)}{f(a)}\cdot\frac{f_1(a)}{F_1(a)}$,

and $d\left(\frac{1}{F(x)}\right)=\frac{-F_1(x)}{\{F(x)\}^2}$ $\therefore$ $\frac{F(a)}{f(a)}=\frac{F_1(a)}{f_1(a)}$, as in the other form.

so that the methods already explained equally apply, whichever of the two forms the fraction assumes: thus:—

(1) Required the value of

$$\frac{F(x)}{f(x)}=\frac{\log\sin x}{\log\sin 2x}, \text{ when } x=0.$$

Here $\frac{F(0)}{f(0)}=\frac{\infty}{\infty}$. And

$$\frac{F_1(x)}{f_1(x)}=\frac{\cos x\sin 2x}{\sin x\cos 2x}$$

$$=\frac{2\sin x\cos^2 x}{\sin x\cos 2x}=\frac{2\cos^2 x}{\cos 2x},$$

$\therefore$ $\frac{F(0)}{f_1(0)}=2$, the value required.

(2) Required the value of $\frac{x^n}{e^x}$ when $x=\infty$, and n a finite positive integer.

$$\frac{F_1(x)}{f_1(x)}=\frac{nx^{n-1}}{e^x},\ \therefore\ \frac{F_1(\infty)}{f_1(\infty)}=\frac{\infty}{\infty},$$

$$\frac{F_2(x)}{f_2(x)}=\frac{n(n-1)x^{n-2}}{e^x},\ \therefore\ \frac{F_2(\infty)}{f_2(\infty)}=\frac{\infty}{\infty},$$

$$\vdots\qquad\qquad\vdots$$

$$\frac{F_n(x)}{f_n(x)}=\frac{n(n-1)(n-2)\ldots 2.1}{e^x},$$

$\therefore$ $\frac{F_n(\infty)}{f_n(\infty)}=0$, since, n being a finite number, the numerator is finite, and the denominator infinite.

(3) Required the value of

$$\frac{e^{-\frac{1}{x^2}}}{x^{2n}},\ n \text{ being a positive integer,}$$

when $x=0$.

Put $\frac{1}{z}$ for x^2, then the fraction is

$$\frac{F(z)}{f(z)}=\frac{e^{-\frac{1}{z}}}{\frac{1}{z^n}}=\frac{z^n}{e^z},$$ which has the form

$\frac{\infty}{\infty}$ when $z=\infty$, that is, when $x=0$.

Also $\frac{F_1(z)}{f_1(z)}=\frac{nz^{n-1}}{e^z}$, $\therefore$ $\frac{F_1(\infty)}{f_1(\infty)}=\frac{\infty}{\infty}$,

$$\vdots\qquad\qquad\vdots$$

$$\frac{F_n(z)}{f_n(z)}=\frac{n(n-1)\ldots 2.1}{e^z},\ \therefore\ \frac{F_n(\infty)}{f_n(\infty)}=0.$$

[NOTE.—It has been remarked of the function $e^{-\frac{1}{x^2}}$ that "although it vanishes when $x=0$, yet x is not a factor of it," and in reference to this circumstance, a recent writer of great ability makes the following observation:—"The statement made in most of the ordinary text-books on the Theory of Algebraical Expressions, that if $F(x)=0$, when $x=0$, x is a factor of $F(x)$, or (what is equivalent) that if $F(x)=0$, when $x=a$, $x-a$ is a factor of $F(x)$, is only one of too many *universal* propositions which are unduly assumed." Those who make this statement, if any there really be,* need not go to transcendental functions to discover their error. Common algebra will supply functions, as many as we please, each of which, like

* De Morgan, in alluding to the function here commented upon, has more accurately represented "the statement made in the ordinary text-books." He says:—"Until very lately, all analysts considered functions which vanish when $x=a$ as necessarily divisible by some positive power of $x-a$" [not necessarily by some positive *integral* power of $x-a$]. "This is only one of a great many too general assumptions which are disappearing one by one from the science."

It may be proper to add here that the author of the present work has not had an opportunity of consulting Sir W. R. Hamilton's paper on this function in the Transactions of the Royal Irish Academy: the assertion above, that x is *not* a factor of it, must therefore be regarded as a quotation, at second hand, from that eminent analyst.

that here noticed, vanishes when $x=0$, and yet is not divisible by x: for instance, $\sqrt{x}$ vanishes for $x=0$, but is evidently not divisible by x. In the theory of equations it is affirmed, and *proved*, that if the polynomial which forms the first member vanish for $x=a$, the expression is necessarily divisible by $x-a$: but then it is always premised that the polynomial is rational and integral (see also art. 190).]

563. **Forms $0\times\infty$, and $\infty-\infty$.** 1. Let $F(a)=0$, and $f(a)=\infty$, then

$$F(a)\times f(a)=\frac{F(a)}{\frac{1}{f(a)}}=\frac{0}{0}=\frac{F(a)}{f'(a)},$$ where $f'(a)$ stands for $\frac{1}{f(a)}$.

2. Let $F(a)=\infty$, and $f(a)=\infty$, then

$$F(a)-f(a)=\left(\frac{1}{f(a)}-\frac{1}{F(a)}\right)\div\frac{1}{F(a)\times f(a)}=\frac{0}{0}.$$

(1) Required the value of

$$(1-x)\tan\frac{\pi x}{2}$$ when $x=1$.

Here the first factor becomes 0, and the second ∞; also $\tan\frac{\pi x}{2}=\frac{1}{\cot\frac{\pi x}{2}}$,

$$\therefore \text{the expression is } \frac{1-x}{\cot\frac{\pi x}{2}}$$

which is $\frac{0}{0}$ when $x=1$,

$$\therefore \frac{F_1(x)}{f'_1(x)}=\frac{1}{\frac{\pi}{2}\operatorname{cosec}^2\frac{\pi x}{2}},$$

$$\therefore \frac{F_1(1)}{f'_1(1)}=\frac{2}{\pi},$$ the value required.

(2) Required the value of

$$x\tan x-\frac{\pi}{2}\sec x,$$ when $x=90°$.

Here the two terms are both ∞ in the proposed hypothesis,

$$\frac{1}{f(x)}-\frac{1}{F(x)}=\frac{1}{\frac{1}{2}\pi\sec x}-\frac{1}{x\tan x},$$

$$\frac{1}{F(x)\times f(x)}=\frac{1}{x\tan x\times\frac{1}{2}\pi\sec x}$$

Dividing the first by the second,

the expression is $\frac{x\sin x-\frac{1}{2}\pi}{\cos x}$, and

differentiating num. and denom., which each become 0 when $x=90°$, we have

$$F(x)-f(x)=\frac{x\cos x+\sin x}{-\sin x},$$

$$\therefore F(90°)-f(90°)=\frac{0+1}{-1}=-1.$$

Otherwise. Since $\tan x=\frac{\sin x}{\cos x}$,

and $\sec x=\frac{1}{\cos x}$,

$$\therefore x\tan x-\frac{\pi}{2}\sec x=\frac{x\sin x-\frac{1}{2}\pi}{\cos x},$$

which is of the form $\frac{0}{0}$ when $x=90°$: and to this the form $\infty-\infty$ may often, in like manner, be reduced without recurring t the general formula.

Examples for Exercise.

(1) $\frac{x^3-ax^2-a^2x+a^3}{x^2-a^2}$, when $x=a$.

(2) $\frac{x^{\frac{3}{2}}+(x-1)^{\frac{3}{2}}-1}{(x^2-1)^{\frac{3}{2}}-(x-1)}$, when $x=1$.

(3) $\frac{\log x}{\cot x}$, when $x=0$.

(4) $\frac{1-x+\log x}{1-(2x-x^2)^{\frac{1}{2}}}$, when $x=1$.

(5) $\frac{\tan x-\sin x}{\sin x^3}$, when $x=0$.

(6) $x^n\log x$, when $x=0$.

(7) $\sec x-\tan x$, when $x=90°$.

(8) x^x, when $x=0$.

(9) $\frac{x}{x-1}-\frac{1}{\log x}$, when $x=1$.

(10) $\frac{1}{\log x}-\frac{x}{\log x}$, when $x=1$.

(11) $\frac{e^x-e^{\sin x}}{x-\sin x}$, when $x=0$.

(12) $(1+ax)^{\frac{b}{x}}$, when $x=\infty$.

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564. **To find $\frac{y}{x}=\frac{0}{0}$, when y is an Implicit Function of x.**—Let $F(x, y)=0$: it is required to determine $\frac{y}{x}=\frac{dy}{dx}$, when $x=0, y=0$. This may be effected without differentiation, as in the following examples:—

(1) Given $ay^2-x^3-bx^2=0$ to find $\frac{y}{x}$ or $\frac{dy}{dx}$, when $x=0$, and $y=0$.

Divide by x^2: then $a\left(\frac{y}{x}\right)^2-x-b=0$,

$\therefore$ when $x=0$, and $y=0$,

$$\left(\frac{y}{x}\right)=\sqrt{\frac{b}{a}}=\left(\frac{dy}{dx}\right),$$

the brackets implying that x and y are each $=0$.

(2) $x^4+ay^3-2axy^2-3ax^2y=0$.

Divide by x^3, then

$$x+a\left(\frac{y}{x}\right)^3-2a\left(\frac{y}{x}\right)^2-3a\left(\frac{y}{x}\right)=0\ldots[1],$$

which, since $x=0$, is the same as

$$\left(\frac{y}{x}\right)^2-2\left(\frac{y}{x}\right)-3=0\ldots[2].$$

And solving this quadratic, we have

$$\left(\frac{y}{x}\right)=\frac{2\pm4}{2}=3 \text{ or } -1,$$

and since [1] was divided by $\left(\frac{y}{x}\right)$ to get [2], $\therefore$ another value is $\left(\frac{y}{x}\right)=0$: hence $\left(\frac{dy}{dx}\right)=0$, 3, or -1.

(3) $y^4-96a^2y^2+100a^2x^2-x^4=0$.

Divide by x^2, then

$$y^2\left(\frac{y}{x}\right)^2-96a^2\left(\frac{y}{x}\right)^2+100a^2=0\ldots[1],$$

$$\therefore \left(\frac{y}{x}\right)=\sqrt{\frac{x^2-100a^2}{y^2-96a^2}}=\sqrt{\frac{25}{24}}=\frac{5}{12}\sqrt{6}=\left(\frac{dy}{dx}\right).$$

The last term of [1], $-x^2$, was omitted because $x=0$; and if we could have foreseen that no infinite value of $\left(\frac{y}{x}\right)$ could satisfy the equation, the first term might have been omitted, because $y=0$. The next example shows that such an omission is not generally allowable.

(4) $y^3-3axy+x^3=0$.

Divide by x^2, then, since $x=0$,

$$y\left(\frac{y}{x}\right)^2-3a\left(\frac{y}{x}\right)=0,$$

$$\therefore \left(\frac{y}{x}\right)=0, \text{ and } \left(\frac{y}{x}\right)=\frac{3a}{y}=\infty.$$

The following is the rule by which these solutions have been obtained.

RULE 1. Divide the equation by the highest power of x, that will not introduce that symbol into the denominator of any coefficient of $\left(\frac{y}{x}\right)$, $\left(\frac{y}{x}\right)^2$, &c., nor yet into the denominator of any term independent of $\left(\frac{y}{x}\right)$. 2. Expunge such of the latter terms as vanish for $x=0$, or for

$y=0$, and then treat the equation as one of common algebra, in which $\left(\frac{y}{x}\right)$ is the unknown quantity.

If the values of $\frac{dy}{dx}$ are required for $x=a$, $y=b$, instead of for $x=0$, $y=0$, then let $F(x, y)=0$ be transformed into $f(x', y')=0$, by putting $x'+a$ for x, and $y'+b$ for y, and from this equation determine $\left(\frac{y'}{x'}\right)$ as above: the values of this will evidently be those of $\frac{dy}{dx}$ in the proposed hypothesis $x=a$, $y=b$.*

EXAMPLES FOR EXERCISE.

(1) Required $\frac{dy}{dx}$, when $x=0$, $y=0$, in $ay^3-x^3y-bx^3=0$.

(2) Required $\frac{dy}{dx}$, when $x=0$, $y=0$, in $x^4+2ax^2y-ay^3=0$.

(3) Required $\frac{dy}{dx}$, when $x=0$, $y=0$, in $x^4-ayx^2+by^3=0$.

(4) Required $\frac{dy}{dx}$, when $x=0$, $y=0$, in $y^3-7x^2y-6x^3+x^4=0$.

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565. **Maxima and Minima: Explicit Functions.**—This subject, like that of vanishing fractions, has been partially discussed in the ALGEBRA, p. 147: but its complete theory requires the Differential Calculus.

An explanation of the terms maximum and minimum has been given at the page referred to: it is equivalent to the following:—In any function $y=F(x)$, let the independent variable x take a particular value, as $x=a$, as also a preceding and succeeding value, $x=a-h$, and $x=a+h$; then the corresponding values of the function being

$$F(a-h),\ F(a),\ F(a+h),$$

if a be such a value that for any finite value of h, however small, and for all intermediate values between 0 and this, the middle value $F(a)$ of the three functions *exceeds* that on each side, the value $x=a$ is said to render the proposed function a maximum; but if the middle value continue less than that on each side, between the same limits of h, the value $x=a$ is said to render the function a minimum. In other words, if x, starting from any value, by continuous increase, causes $F(x)$ to *increase* till x arrives at a certain value a, and then causes it to *decrease*, $F(a)$ is a *maximum* value of the function. But if x, starting from any value, by continuous increase, causes $F(x)$ to *decrease*, till x arrives at a certain value a, and then causes it to *increase*, $F(a)$ is a *minimum* value of the function. In order to discover what the values of x are that render the function $y=F(x)$ a maximum or a minimum, let x be changed into $x\pm h$; then, by Taylor's theorem, h being so small as to render it convergent,

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* The above method, which supersedes the more operose process by differentiation, is due to Mr. Woolhouse (Differential Calculus in Weale's Series, p. 95). The rule in which we have embodied it may perhaps be acceptable to the student

$$F(x\pm h)=F(x)\pm\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}\pm\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\ldots$$

Now, if $x=a$ make the proposed function a *maximum*, there must exist for h some finite value, such that for all intermediate values between 0 and this, we must have $F(a)>F(a\pm h)$; and consequently we must have the condition

$$\pm\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}\pm\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\ldots \textit{ negative } \ldots[1],$$

when a is put for x in the coefficients. But if the value $x=a$ make the function a *minimum*, then the condition will be

$$F(a)<F(a\pm h), \text{ so that } \pm\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}\pm\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\ldots \text{ will be } \textit{positive} \ldots[2].$$

We know, however, that a value so small may be given to h as to cause the first term in each of the series [1], [2], to exceed the sum of all the other terms (194), and that this first term will continue to do so for all other values of h between this small value and 0: hence, for each of these values of h, the sign belonging to the series, as a whole, is the same as the sign of the first term. It is impossible, therefore, that either of the conditions [1], [2], can exist, for *both* $+\frac{dy}{dx}h$ and $-\frac{dy}{dx}h$, when a is put for x, unless a be such as to satisfy the condition $\frac{dy}{dx}=0$: it follows, therefore, that, of the values of x for which Taylor's theorem does not fail (556), those only can cause the function to be a maximum or a minimum which satisfy the condition $\frac{dy}{dx}=0$. Expunging, therefore, the first term from each of the series [1], [2], it follows that if $x=a$ cause the function to be a maximum, it must also be such as to render

$$\frac{d^2y}{dx^2}\frac{h^2}{1.2}\pm\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\ldots \textit{ negative } \ldots[3],$$

and if $x=a$ make the function a minimum, a must render

$$\frac{d^2y}{dx^2}\frac{h^2}{1.2}\pm\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\ldots \textit{ positive } \ldots[4].$$

But by the principle (194) already referred to, the condition [3] cannot exist for values of h from $h=0$, up to $h=$a small value, unless $\frac{d^2y}{dx^2}$ be negative, nor can [4] exist unless $\frac{d^2y}{dx^2}$ be positive—supposing, that is, that this coefficient do not vanish when a is put for x: the inference, therefore, is this, namely:—

Of the values of x which satisfy the condition $\frac{dy}{dx}=0$, those only which also satisfy the condition $\frac{d^2y}{dx^2}=a$ *negative quantity*, belong to maximum values of the function: while those which, on the contrary, satisfy the

condition $\frac{d^2y}{dx^2}=a$ *positive quantity*, belong to minimum values of the function.

It may happen, however, that some of the values given by the equation $\frac{dy}{dx}=0$, when substituted for x in $\frac{d^2y}{dx^2}$, may cause this coefficient also to vanish, in which case the conditions [1], [2], become

$$\pm\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\frac{d^4y}{dx^4}\frac{h^4}{1.2.3.4}\pm\ldots \textit{ negative},$$

$$\text{and } \pm\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\frac{d^4y}{dx^4}\frac{h^4}{1.2.3.4}\pm\ldots \textit{ positive},$$

which, for reasons similar to those assigned above, are both impossible conditions, unless $\frac{d^3y}{dx^3}=0$, and unless also $\frac{d^4y}{dx^4}$ is *negative*, in the case of a *maximum*, and *positive*, in the case of a *minimum*.

566. It thus appears that to ascertain for what values of x the function $y=F(x)$ becomes a maximum or a minimum, we must proceed as follows:—

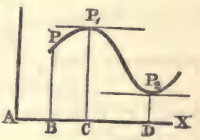
Rule.—Find the real roots of the equation $\frac{dy}{dx}=0$, and substitute them, one by one, in the coefficients $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, &c., stopping at the first of these, which does not vanish. If this be of an *odd* order, the root that we have employed is *not* one of those values of x that cause the function to be either a maximum or a minimum. But if it be of an *even* order, then, according as it is negative or positive, will the root employed correspond to a maximum or a minimum.

Should any of the real roots of the equation $\frac{dy}{dx}=0$ cause the first of the following coefficients, which does not vanish, to become infinite, we shall be apprized that the development of $F(x\pm h)$, for that particular value of x, cannot be exhibited in a series, proceeding according to the positive integral powers of h, and therefore that the foregoing rule does not apply. In a case of this kind, we must find, by actual involution, extraction, &c., the true term that ought to supply the place of that rendered infinite in Taylor's series for $x=a$. If this term be such as to change its sign, with a change of sign of h, then $x=a$ does not render the function either a maximum or a minimum; but if the sign of the term do not change with that of h, the value $x=a$ renders the function a maximum or a minimum, according as the sign of that term is negative or positive.

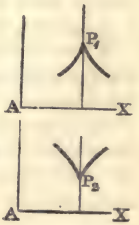
There is one more case which it is necessary to consider, in order that the general theory may be complete. We have hitherto regarded Taylor's development to have place as far, at least, as the first power of h; but there is nothing to authorize us in affirming that, among those values of x for which the coefficient of this first power becomes *infinite*, there may not be some which cause the function to satisfy the conditions of maxima or minima. Before, therefore, we can conclude in any case that the values of x, deduced from the condition $\frac{dy}{dx}=0$, comprise among them *all* those

which render the function a maximum or a minimum, we must examine the values fulfilling the condition $\frac{dy}{dx}=\infty$, by substituting each of these $\pm h$ for x in the proposed function, and observing which of the results agree with the conditions of maxima and minima, as defined at (565). The foregoing examination might have been abridged; but it will probably be acceptable to the student in the more detailed form here given to it. It may be noticed, in addition to what is shown above, that if $\frac{dy}{dx}$ be *constant*, the function, which must then be of the form $ax+b$, cannot have either a maximum or a minimum value.

567. The foregoing analytical examination may be satisfactorily illustrated geometrically. Let $y=F(x)$ be the equation of the annexed curve, conceived to be generated by the motion of the ordinate BP moving parallel to itself towards X. From P to P_1, where the curve bends, the ordinate continuously increases: from P_1 to P_2 it continuously diminishes, till reaching the next bend P_2, it again increases, and so on. At the points P_1, P_2—the points which, in the portion of curve exhibited, have the greatest and least ordinates—the tangents are parallel to the axis of x: at these points, therefore, $\frac{dy}{dx}=0$ (528). Two ordinates in the immediate vicinity of P_1, one to the right, and the other to the left of it, that is, the two ordinates $F(x+h)$, $F(x-h)$, will each be *less* than the ordinate $F(x)$ at the point, for a small value of h; that is, $F(x+h)-F(x)$ and $F(x-h)-F(x)$ will both be *negative;* and in the development of these differences by Taylor's theorem, h may be taken so small, that the first term that does not vanish—say the term $\frac{d^ny}{dx^n}\frac{h^n}{1.2...n}$, may have the sign which belongs to the entire series; that is, the sign of the whole difference: but, as just seen, the sign of the whole difference is *negative*. Hence, at the point P_1, where the ordinate $F(x)$ is a maximum, we must have $\frac{dy}{dx}=0$, and the first differential coefficient which does not vanish must be of an even order (or belong to an even power of $\pm h$), and be *negative*.



Applying similar reasoning to the point P_2, we conclude, in like manner, that at that point, where the ordinate $F(x)$ is a minimum, we must have $\frac{dy}{dx}=0$, and the first diff. coef. which does not vanish, must be of an even order, and *positive*. It might, however, happen that the points P_1, P_2, are on such curves as in the annexed diagram, the prolongation of the ordinate being a tangent to each of two uniting branches. In such a case $\frac{dy}{dx}$ will not be 0 but ∞, and this illustrates the case last considered in the above analytical investigation.



568. **Maxima and Minima: Implicit Functions.**—Let now the function that y is of x—which function is to be rendered a maximum or minimum—be only implicitly given, that is, let

$$u=F(x, y)=0, \text{ then (560), } \frac{dy}{dx}=-\frac{du}{dx}\div\frac{du}{dy}\ldots[1].$$

Hence, when $\frac{dy}{dx}=0$, we must have $\frac{du}{dx}=0$, and conversely; so that the values corresponding to maxima and minima, as determinable from the condition $\frac{dy}{dx}=0$ of the preceding article, are now to be determined from the conditions

$$u=0, \frac{du}{dx}=0\ldots\ldots[2].$$

The values of x, and the accompanying values of y, being found from these equations, we must substitute them in succession for x, and y in $\frac{d^2y}{dx^2}$, when those values of y will be maxima values, that render this coefficient negative, and those will be minima that render it positive: and so on, as in last article.

In order to derive the second diff. coef., put for brevity $\frac{dy}{dx}=-\frac{M}{N}$ for [1] above, then

$$\frac{d^2y}{dx^2}=-\frac{N\left(\frac{dM}{dx}+\frac{dM}{dy}\frac{dy}{dx}\right)-M\left(\frac{dN}{dx}+\frac{dN}{dy}\frac{dy}{dx}\right)}{N^2},$$

and, therefore, dividing num. and den. by N, for those particular values of x which satisfy the condition

$$\frac{M}{N}, \text{ or } \frac{dy}{dx}=0, \text{ we must have } \frac{d^2y}{dx^2}=-\frac{d^2u}{dx^2}\div\frac{du}{dy}\ldots[3].$$

NOTE.—For examples the student may return to those at pp. 150 and 280–4; and in solving the additional examples given below, he must keep in remembrance the precepts 5, 6, at p. 280.

(1) $y=a+\sqrt[3]{(a^3-2a^2x+ax^2)}$.

Here the additive constant a may be suppressed, the remaining expression may be cubed, and then the constant factor a removed: hence y is a max. or min., if $a^2-2ax+x^2$ be so, or suppressing the a^2, if x^2-2ax be so: putting, therefore,

$$u=x^2-2ax, \text{ we have } \frac{du}{dx}=2x-2a=0.$$

$$\text{Also } \frac{d^2u}{dx^2}=2, \text{ a } \textit{positive} \text{ quantity;}$$

$\therefore$ y is a *minimum* when $x=a$,

and the expression has no maximum.

(2) $y=b+\sqrt[3]{(x-a)^5}$. Omitting b and the radical, and then taking the 5th root, we have $u=x-a$, $\therefore \frac{du}{dx}=1$, which being constant, the function has neither a maximum nor a minimum.

(3) $y=x^x$, $\therefore \frac{dy}{dx}=x^x(1+\log x)=0$,

$$\frac{d^2y}{dx^2}=x^x\left\{\frac{1}{x}+(1+\log x)^2\right\}.$$

The factor x^x can never become 0,

$\therefore 1+\log x=0$, $\therefore \log x=-1$, $\therefore x=e^{-1}$,

and $\therefore \frac{d^2y}{dx^2}=\left(\frac{1}{e}\right)^{\frac{1}{e}}e$, a positive quantity,

$$\therefore \text{ when } x=\frac{1}{e},\ x^x \text{ is a minimum,}$$

but it has no maximum.

(4) $y=x-\sqrt{(a^2-x^2)}$,

$$\therefore \frac{dy}{dx}=1+(a^2-x^2)^{-\frac{1}{2}}x=0,$$

$$\therefore x+(a^2-x^2)^{\frac{1}{2}}=0 \ldots\ldots [1],$$

$$\therefore x^2=a^2-x^2 \ldots\ldots [2],$$

$$\therefore x=\frac{a\sqrt{2}}{2}.$$

For this value of x the second differential coefficient will be found to be *plus*, so that we might be inclined to infer that for this value of x the proposed function is a *minimum*, which, however, would be a mistake. In fact, the value of x which satisfies [2] does not satisfy [1]: the former equation is equivalent to

$$\{x+(a^2-x^2)^{\frac{1}{2}}\}\{x-(a^2-x^2)^{\frac{1}{2}}\}=0,$$

and it is the *second* of these factors which becomes zero for $x=\frac{a\sqrt{2}}{2}$, and not the first, whether we take $\sqrt{2}$ plus or minus: hence the proposed function does not admit of either a maximum or of a minimum value, for it is plain that, for the first member of [1] to become *infinite*, x must be ∞ (see the remarks at p. 285).

But it is easy to show, from other considerations, that the proposed function cannot be either a maximum or a minimum: for let $a=1$; then we may replace the function by $\sin x'-\cos x'$, where from the above determination $\sin x'=\frac{1}{2}\sqrt{2}$,

$\therefore x'=45°$. Now, however small h may be,

$$\sin(45°+h)-\cos(45°+h)=+, \text{ and}$$

$$\sin(45°-h)-\cos(45°-h)=-;$$

results inconsistent with either a maximum or a minimum.

(5) Divide a given number a into so many equal parts, that the product of them may be a maximum.

Let x be the number of equal parts;

$$\text{then } \left(\frac{a}{x}\right)^x = \text{a maximum,}$$

$$\therefore x\log\frac{a}{x}=\text{max.},\ \therefore \log\frac{a}{x}-1=0,$$

$$\therefore \frac{a}{x}=\log^{-1}1=e,\ \therefore x=\frac{a}{e}.$$

Hence each of the equal parts must be $e=2{\cdot}171828\ldots$, and the number of them $\frac{a}{e}$, in order that their product, which will be $(e)^{\frac{a}{e}}$, may be a maximum.

(6) $u=x^3-3axy+y^3=0$,

$$\therefore \frac{du}{dx}=3x^2-3ay,\ \frac{du}{dy}=3y^2-3ax,$$

$$\therefore [2],\ \left.\begin{matrix} x^3-3axy+y^3=0 \\ 3x^2-3ay=0 \end{matrix}\right\},$$

$$\therefore y=\frac{x^2}{a},\ \therefore x^6-2a^3x^3=0,$$

$$\therefore x=0,\ x=a\sqrt[3]{2},$$

$$\therefore [3],\ \frac{d^2y}{dx^2}=-\frac{d^2u}{dx^2}\div\frac{du}{dy}=$$

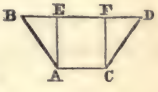
$$-6x\div 3\left(\frac{x^4}{a^2}-ax\right)=-\frac{2a^2}{x^3-a^3}$$

$$=\frac{2}{a},\text{ or } -\frac{2}{a}.$$

Hence, when $x=0$, $y=0$, a min.

,, $x=a\sqrt[3]{2}$, $y=a\sqrt[3]{4}$, a max.

(7) Given the length of a man's foot, and the distance between his heels, to find the position of his feet when he stands the firmest.



Let AB, CD, represent his feet, and A, C, his heels: then he will stand the firmest when the area $ABDC$ is the greatest (419). Draw AE perp. to BD, and put $AC=a$, $AB=b$, $BE=x$, $\therefore AE=\sqrt{(b^2-x^2)}$, and the area

$$ABDC=(a+x)\sqrt{(b^2-x^2)}=\text{max.},$$

$$\therefore x\sqrt{(b^2-x^2)}-\frac{(a+x)x}{\sqrt{(b^2-x^2)}}=0,$$

$\therefore b^2-x^2-ax-x^2=0$, or $2x^2+ax-b^2=0$,

$$\therefore x=\frac{-a+\sqrt{(8b^2+a^2)}}{4}.$$

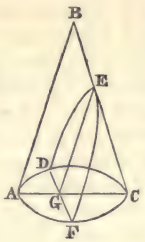
If $a=0$, that is, if the heels touch,

$$x=b\cdot\frac{1}{2}\sqrt{2},\text{ but } \frac{1}{2}\sqrt{2}=\sin 45°,$$

$\therefore BE=AB\sin 45°$, $\therefore$ the angles BAE, DCF, are each 45°, so that if the heels touch, the position will be the firmest when the feet are at right angles to each other.

(8) To cut the greatest parabola from a given cone BAC.

Let AGC be the diameter of the base, perp. to DGF. Put $AC=a$, $AB=b$, $CG=x$, then $AG=a-x$, and since EG is parallel to BA (403), we have by the property of the circle,



$DG=\sqrt{(ax-x^2)}$,

$\therefore DF=2\sqrt{(ax-x^2)}$;

and by sim. triangles,

$$a : b :: x : GE=\frac{bx}{a}.$$

Hence (p. 512), the area of the parabola is

$$\frac{4}{3}\frac{bx}{a}\sqrt{(ax-x^2)}=\text{max.},$$

$$\therefore x^2(ax-x^2)=\text{max.},$$

$$\therefore 3ax^2-4x^3=0\ldots\ldots[1],$$

$$\therefore x=0,\ x=\frac{3}{4}a.$$

The second value substituted for x in the differential of [1] gives a negative result: hence, for this value of CG, the section is a maximum: for the other value, the section is merely a point: it corresponds to neither a max. nor a min., since, for $x=0$, the second diff. coef. vanishes, but the third does not (566).

(9) To inscribe the greatest rectangle in an ellipse.

Let (x, y) denote one of the vertices of the inscribed rectangle: then $4xy$, or xy, and $\therefore x^2y^2$ is to be a maximum; that is,

since $y^2=\frac{b^2}{a^2}(a^2-x^2)$, $x^2(a^2-x^2)=\text{max.}$,

$$\therefore 2a^2x-4x^3=0,\ \therefore 2a^2-4x^2=0,$$

$$\therefore x^2=\frac{2a^2}{4},\ \therefore x=\frac{1}{2}a\sqrt{2},$$

$$\therefore y=\frac{b}{a}\sqrt{\left(a^2-\frac{a^2}{2}\right)}=\frac{1}{2}b\sqrt{2}$$: hence the area, or $4xy=2ab=$one half the circumscribing rectangle, or of any circumscribing parallelogram, whose sides are parallel to conjugate diameters (p. 319).

Examples for Exercise.

(1) The greatest rectangle that can be inscribed in a parabola is that whose height is $\frac{2}{3}$ the height of the parabola: required the proof.

(2) The greatest parabola that can be inscribed in another parabola, the vertex of the former being at the middle of the base of the latter, is that whose height is $\frac{2}{3}$ the height of the given parabola: required the proof.

(3) Prove that the greatest parabola that can be inscribed in a given isosceles triangle is that whose altitude is $\frac{2}{3}$ the alt. of the triangle.

(4) Given the base b of an inclined plane, to find its altitude a, so that the *horizontal* velocity which a body would acquire by descending down it may be the greatest possible.

(5) Determine a number x such that the xth root of it may be the greatest possible.

(6) Determine those conjugate diameters of an ellipse which form the greatest angle.

(7) Given the volume of a cone: what must its base and altitude be, in order that its whole surface may be a minimum?

(8) What radius must a circle have, in order that an arc of it, of given length a, may belong to a segment of the greatest area?

(9) A conical ale-glass is 6 inches in depth, and 5 inches in diameter at top: what must be the diameter of a heavy sphere which, dropped into the vessel when full, shall expel the greatest quantity of the liquid?

(10) Required the angle of inclination of a pair of gates for a lock on a canal, so that the resistance to the fluid pressure may be the greatest possible.

(11) Find for what angle, less than 180°, of arcual measure θ, the product $\theta \sin\theta$ is a maximum.*

(12) Given $y^2+2x^2y+32x-48=0$: required the values of x for which y is a maximum or a minimum.

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* This will lead to the equation $\tan x+1=0$, an equation which may be solved by the method of Double Position, as explained at (181).

569. **Functions of Two Independent Variables.**—Every function hitherto considered has been regarded as passing through its various changes, solely in consequence of the variation—the *uniform* variation—of but one of the quantities composing that function, all the others remaining constant: in other words, we have treated, as yet, only of functions of a single independent variable. We shall now briefly examine functions of two independent variables, or those whose changes take place in consequence of independent changes in two of the quantities which compose the function. As an example of a simple case of such a function, let V be the volume of a parallelopiped of length x, breadth y, and therefore of depth $\frac{V^3}{xy}$. The surface S of this solid will evidently be

$$S=2\left(xy+\frac{V^3}{x}+\frac{V^3}{y}\right),$$

S being a function of the two *independent* variables x and y: a change in x, the length, does not affect y, the breadth; nor does a change in y affect x, though a change in either, or in both, affects S; there is evidently, therefore, no necessity that these independent changes should take place simultaneously.

Let $u=F(x, y)$, and let x become $x+h$, y remaining unchanged; then, by Taylor's theorem, observing that y undergoes no change,

$$F(x+h, y)=u+\frac{du}{dx}h+\frac{d^2u}{dx^2}\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\frac{h^3}{1.2.3}+\ldots[1].$$

But if y also change, becoming $y+k$, then, in [1]

u will become $u+\frac{du}{dy}k+\frac{d^2u}{dy^2}\frac{k^2}{1.2}+\frac{d^3u}{dy^3}\frac{k^3}{1.2.3}+\ldots$,

$$\therefore \frac{du}{dx} \quad \text{,,} \quad \frac{du}{dx}+\frac{d\frac{du}{dy}}{dx}k+\frac{d\frac{d^2u}{dy^2}}{dx}\frac{k^2}{1.2}+\frac{d\frac{d^3u}{dy^3}}{dx}\frac{k^3}{1.2.3}+\ldots,$$

$$\therefore \frac{d^2u}{dx^2} \quad \text{,,} \quad \frac{d^2u}{dx^2}+\frac{d^2\frac{du}{dy}}{dx^2}k+\frac{d^2\frac{d^2u}{dy^2}}{dx^2}\frac{k^2}{1.2}+\frac{d^2\frac{d^3u}{dy^3}}{dx^2}\frac{k^3}{1.2.3}+\ldots,$$

$$\therefore \frac{d^3u}{dx^3} \quad \text{,,} \quad \frac{d^3u}{dx^3}+\frac{d^3\frac{du}{dy}}{dx^3}k+\frac{d^3\frac{d^2u}{dy^2}}{dx^3}\frac{k^2}{1.2}+\frac{d^3\frac{d^3u}{dy^3}}{dx^3}\frac{k^3}{1.2.3}+\ldots,$$

and so on. But before we actually make these substitutions, we shall write, for abridgment,

$$\frac{d^2u}{dydx} \text{ for } \frac{d\frac{du}{dy}}{dx}; \quad \frac{d^3u}{dy^2dx} \text{ for } \frac{d\frac{d^2u}{dy^2}}{dx}; \text{ and generally,}$$

$$\frac{d^{p+q}u}{dy^pdx^q} \text{ for } \frac{d^q\frac{d^pu}{dy^p}}{dx^q}. \text{ Hence}$$

$$F(x+h,\ y+k)=u+\begin{vmatrix} \frac{du}{dx}h \\ \frac{du}{dy}k \end{vmatrix}+\begin{vmatrix} \frac{d^2u}{dx^2}\frac{h^2}{1.2} \\ \frac{d^2u}{dydx}kh \\ \frac{d^2u}{dy^2}\frac{k^2}{1.2} \end{vmatrix}+\begin{vmatrix} \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} \\ \frac{d^3u}{dydx^2}\frac{kh^2}{1.2} \\ \frac{d^3u}{dy^2dx}\frac{k^2h}{1.2} \\ \frac{d^3u}{dy^3}\frac{k^3}{1.2.3} \end{vmatrix}+\ldots[2].$$

In this development of the changed function, we have supposed, first the variable x to change, and then the variable y. The result would, of course, have been the same if y had changed first, and then x: but in this case we should evidently have got

$$F(x+h,\ y+k)=u+\begin{vmatrix} \frac{du}{dx}h \\ \frac{du}{dy}k \end{vmatrix}+\begin{vmatrix} \frac{d^2u}{dx^2}\frac{h^2}{1.2} \\ \frac{d^2u}{dxdy}hk \\ \frac{d^2u}{dy^2}\frac{k^2}{1.2} \end{vmatrix}+\begin{vmatrix} \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} \\ \frac{d^3u}{dx^2dy}\frac{h^2k}{1.2} \\ \frac{d^3u}{dxdy^2}\frac{hk^2}{1.2} \\ \frac{d^3u}{dy^3}\frac{k^3}{1.2.3} \end{vmatrix}+\ldots[3].$$

As the developments [2], [3], are identical, we have, by equating the coefficients of like powers of h and k,

$$\frac{d^2u}{dydx}=\frac{d^2u}{dxdy},\ \frac{d^3u}{dydx^2}=\frac{d^3u}{dx^2dy},\ \text{and generally,}\ \frac{d^{p+q}u}{dy^pdx^q}=\frac{d^{q+p}u}{dx^qdy^p},$$

so that whether we determine the pth differential coefficient relatively to y, and then the qth diff. coef. of this relatively to x, the final result is the same as if we first determine the qth diff. coef. relatively to x, and then the pth of this relatively to y; that is, the final result is the same in whichever order the differentiations be performed.

570. The preceding development may be put in the following form, which is analogous to the development of $F(x+h)$ by Taylor's theorem, and becomes identical to it when y is constant, or $k=0$, namely,

$$F(x+h,\ y+k)=u+\left(\frac{du}{dx}h+\frac{du}{dy}k\right)+\frac{1}{2}\left(\frac{d^2u}{dx^2}h^2+2\frac{d^2u}{dxdy}hk+\frac{d^2u}{dy^2}k^2\right)$$
$$+\frac{1}{2.3}\left(\frac{d^3u}{dx^3}h^3+3\frac{d^3u}{dx^2dy}h^2k+3\frac{d^3u}{dxdy^2}hk^2+\frac{d^3u}{dy^3}k^3\right)+\ldots[1],$$

where the *partial* differential coefficients in each term are the differentials of those in the immediately preceding term: thus,

$$du=\frac{du}{dx}dx+\frac{du}{dy}dy\ldots\ldots[2],$$

in which du on the left is the total differential made up of the differential of u with respect to x, and of the differential of u with respect to y. In

like manner, differentiating again, remembering that $\frac{du}{dx}$, $\frac{du}{dy}$, are also functions of the independent variables x and y, we have

$$d\frac{du}{dx}=\frac{d^2u}{dx^2}dx+\frac{d^2u}{dxdy}dy, \text{ and } d\frac{du}{dy}=\frac{d^2u}{dydx}+\frac{d^2u}{dy^2}dy,$$

$$\therefore d^2u=\frac{d^2u}{dx^2}dx^2+2\frac{d^2u}{dxdy}dxdy+\frac{d^2u}{dy^2}dy^2\ldots\ldots[3].$$

In like manner, these coefficients being functions of x and y, we shall find, by differentiating each separately, and then, as before, uniting the several differentials, that

$$d^3u=\frac{d^3u}{dx^3}dx^3+3\frac{d^3u}{dx^2dy}dx^2dy+3\frac{d^3u}{dxdy^2}dxdy^2+\frac{d^3u}{dy^3}dy^3\ldots\ldots[4],$$

and so on; the numerical coefficients agreeing with those in the corre sponding powers of the developed binomial.

The development [1] may be regarded as the extension of Taylor's theorem to two independent variables, each of which receives increments; and we may easily deduce from it a similar extension of Maclaurin's theorem: thus, supposing x and y each to be 0, [1] will be the development of $F(h, k)$; and writing x, y, for the general symbols h, k, bracketing the coefficients as at (551), to imply that, when they are computed, x and y are to be made zero, we have

$$u=F(x,y)=[u]+\left(\left[\frac{du}{dx}\right]x+\left[\frac{du}{dy}\right]y\right)+\frac{1}{2}\left(\left[\frac{d^2u}{dx^2}\right]x^2+2\left[\frac{d^2u}{dxdy}\right]xy+\left[\frac{d^2u}{dy^2}\right]y^2\right)+\ldots$$

571. We may now extend the theory of Maxima and Minima to functions of two independent variables, the condition being that if the function $u=F(x, y)$ become a maximum or a minimum for particular values of x and y, then h and k must be finite increments, sufficiently small, that between 0 and those small values, we must always have, for the aforesaid particular values of x and y,

$F(x, y)>F(x\pm h, y\pm k)$ for a max., and $F(x, y)<F(x\pm h, y\pm k)$ for a min.,

and consequently [1] last article,

$$\left(\pm\frac{du}{dx}h\pm\frac{du}{dy}k\right)+\frac{1}{2}\left(\frac{d^2u}{dx^2}h^2\pm2\frac{d^2u}{dxdy}hk+\frac{d^2u}{dy^2}k^2\right)+\ldots[1],$$

must be negative in the former case, and positive in the latter. But since h and k may be taken so small that the sign of [1] may be the sign of its first significant term, and that the term which now appears first changes its sign with changes of sign of h and k, this first term must be 0; that is, h and k being independent, we must have $\frac{du}{dx}=0$, and $\frac{du}{dy}=0$, for those values of x and y which make the proposed function a maximum or a minimum. And it will be the former, or the latter, according as, for the substituted values of x and y, the next term, or its double,

$$\frac{d^2u}{dx^2}h^2\pm2\frac{d^2u}{dxdy}hk+\frac{d^2u}{dy^2}k^2,$$

is negative or positive. For brevity, write this, after dividing by k^2, thus:

$$A\left(\frac{h}{k}\right)^2 \pm 2B\left(\frac{h}{k}\right)+C\text{.....[2]:}$$

we have then to ascertain the conditions which A, B, C, must satisfy, in order that this expression may be always of the same sign, whatever be the ratio $\frac{h}{k}$; and it is plain that it will be of the same sign, whichever be the sign of the middle term, only when one, at least, of the quantities A, C, is of that sign. But (68) we know that it will be of the same sign, whatever be $\frac{h}{k}$, only when $AC-B^2$ is *positive*, and that when such is not the case, the expression will be positive or negative, according to the value of $\frac{h}{k}$; so that then there can be neither a maximum nor a minimum. Hence, for there to be a maximum or a minimum, we must have $AC-B^2$ *positive*, and consequently A, C, must be *both of the same sign*. If this sign be $+$, then [2] will be wholly $+$, whatever be $\frac{k}{h}$: if the sign common to A, C, be $-$, [2] will be wholly $-$. Hence, for a max. or min. to exist, we must have

$$\frac{du}{dx}=0, \frac{du}{dy}=0, \text{ and also } \frac{d^2u}{dx^2}\cdot\frac{d^2u}{dy^2}-\left(\frac{d^2u}{dxdy}\right)^2 \textit{ positive } \text{.....[3],}$$

and these conditions being satisfied, the function $u=F(x, y)$, is a maximum for such values of the variables as make $\frac{d^2u}{dx^2}$, $\frac{d^2u}{dy^2}$, both *negative;* and a minimum for such as make them both *positive*. The condition [3] having been first shown to be necessary by Lagrange, is usually called Lagrange's condition. If it fail, there cannot be either a maximum or a minimum.

Ex. 1. Among all rectangular parallelopipeds to determine that which, having a given volume a^3, shall have the least possible surface. Let x, y, z, represent three contiguous edges, then

$$u=2xy+2xz+2yz=\text{minimum : but since } xyz=a^3, \therefore z=\frac{a^3}{xy},$$

$$\therefore u=2xy+2\frac{a^3}{y}+\frac{2a^3}{x}=\text{minimum},$$

$$\therefore \frac{du}{dx}=2y-\frac{2a^3}{x^2}=0, \frac{du}{dy}=2x-\frac{2a^3}{y^2}=0, \therefore x=y=a, \therefore z=a.$$

Now, for these values, $\frac{d^2u}{dx^2}$, and $\frac{d^2u}{dy^2}$, are evidently both *positive;* and Lagrange's condition holds: hence the surface is a minimum when the figure is a *cube*.

(2) For what values of x and y is the expression x^2y+xy^2-axy a maximum or a minimum? Here, calling the expression u,

$$\frac{du}{dx}=2xy+y^2-ay=0, \text{ and } \frac{du}{dy}=x^2+2xy-ax=0 : \text{ these are satisfied by}$$

$$x=0,\ y=0\ ;\ x=a,\ y=0\ ;\ x=0,\ y=a\ ;\ x=\frac{a}{3},\ y=\frac{a}{3}\ ;$$

the last of which is the only pair of values which satisfy Lagrange's condition:

$$\text{these make } \frac{d^2u}{dx^2} \text{ and } \frac{d^2u}{dy^2}, \textit{ positive}, \text{ and therefore, for}$$

$$x=y=\frac{a}{3}, \text{ the expression is a minimum, namely } -\left(\frac{a}{3}\right)^3.$$

(3) Prove that a rectangular cistern, open at top, in order to contain a given quantity of water under the least internal surface, must have a square base, and its depth equal to half a side of the base, that is, the cistern must be half a cube.

NOTE.—If the values which satisfy $\frac{du}{dx}=0$, $\frac{du}{dy}=0$, be any of them such as to cause the next following partial differential coefficients to vanish, the character of such values can be ascertained only by an appeal to the coefficients still more advanced, as in the analogous case at (566): the general tests would, however, be very complicated.

We may notice here that, although we have designated the differential coefficients derived from u as *partial* differential coefficients, yet there is no *total* differential coefficient furnished by a function of two or more independent variables: there is a total *differential*, but as there are two or more *independent* increments, there can be no *total* coefficient to either.*

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572. **Change of the Independent Variable.**—Every independent variable is always considered to vary uniformly; but if we wish to introduce the hypothesis that this variation is not uniform, we may assume the variable to be itself dependent upon a new variable, which does change uniformly: this change of the independent variable may be effected as follows:—

Let $y=F(x)$, and let $x=f(t)$, the variation of x depending upon the uniform variation of t, then (558),

$$\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}, \therefore \frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}\ ;\ \frac{d^2y}{dt^2}=\frac{d^2y}{dx^2}\frac{dx^2}{dt^2}+\frac{dy}{dx}\frac{d^2x}{dt^2}.$$

Hence, putting for brevity, $\frac{(dy)}{(dx)}$ for $\frac{dy}{dt}\div\frac{dx}{dt}$, we have

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* In connection with the preceding article, the student may consult, with advantage, the work of Ramchundra, referred to at p. 280 (Chap. IV.). On the same subject, see also Jephson's "Fluxional Calculus," Garnier's "Calcul Différentiel," and Puissant's "Problèmes de Géométrie."

$$\frac{dy}{dx}=\frac{(dy)}{(dx)},\ \frac{d^2y}{dx^2}=\frac{(d^2y)(dx)-(d^2x)(dy)}{(dx)^3},\ \&c.\ldots\ldots[1].$$

If we make $t=y$, the hypothesis is equivalent to making y the independent, and x the dependent variable. In this case

$$(dy)=\frac{dy}{dy}=1,\ (d^2y)=0,\ (dx)=\frac{dx}{dy},\ (d^2x)=\frac{d^2x}{dy^2},\ \therefore [1],$$

$$\frac{dy}{dx}=1\div\frac{dx}{dy},\ \frac{d^2y}{dx^2}=-\frac{(d^2x)}{(dx)^3}=-\frac{d^2x}{dy^2}\div\left(\frac{dx}{dy}\right)^3\ldots\ldots[2].$$

In certain investigations it is sometimes found necessary thus to alter the hypothesis respecting the variable originally regarded as independent: in such a case the above substitutions may be made at any stage of the inquiry.

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573. **Failure of Taylor's Theorem.**—This, although the authorized phrase, is a very improper designation of the cases we are now about to examine:—the cases, namely, to which Taylor's development is not applicable, simply because they do not conform to the conditions for which that theorem stipulates:—it is not the *theorem* that fails, but the *function;* which, in the cases mentioned, is deficient in the requisites demanded by the theorem. The theorem fails in the same sense that the 47th of the first book of Euclid fails, when the triangle submitted to it has no right angle.

The indispensable condition that Taylor's theorem shall give the development of $F(x+h)$, whatever particular value be substituted for x in that development, is that such particular value never causes any of the coefficients to become infinite: it is only for such values of x as violate this condition that the theorem is said to fail, that is to say, it fails for such values of x as are roots either of the equation $F(x)=\infty$, or $\frac{1}{F(x)}=0$, or else roots of the equation $\frac{1}{F_n(x)}=0$, where $F_n(x)$ stands for any one of the differential coefficients. Take, for example, the simple case $F(x)=(x-a)^{-m}$: then by Taylor's theorem, the first term of the development of $F(x+h)=(x-a+h)^{-m}$, namely $(x-a)^{-m}$, would be ∞ for $x=a$: the inference is that h^{-m} cannot be developed in the form that theorem assigns.

Again: suppose we have a function of the form

$$F(x)=X+(x-a)^{m+\frac{p}{q}}f(x),\ \text{in which neither } X \text{ nor } f(x) \text{ involve the factor } x-a,$$

m being a positive whole number, and $\frac{p}{q}$ a proper fraction: then, as soon as by successive differentiation we have exhausted the integer m, the differential coefficients which follow will each have a fractional power of $x-a$ entering its denominator as a factor; and $\therefore$ for $x=a$, all the coefficients, after the coefficient $F_m(x)$, will be infinite, so that, from that term, the development fails. Up to the point of failure, however, the development, as furnished by Taylor's theorem, is always true; for we know, regardless of these so-called failing cases, that

$$F(x+h)=F(x)+F_1(x)h+F_2(x)\frac{h^2}{1.2}+\ldots+F_n(x+\theta h)\frac{h^n}{1.2\ldots n}$$

is always true (556), provided the failure do not occur previously to the introduction of Lagrange's correction.

574. The usual way to obtain the true development of the function, for those particular values of the variable for which Taylor's theorem fails, is to have recourse to the ordinary processes of common algebra. For example, suppose the function were $F(x)=2ax-x^2+a\sqrt{(x^2-a^2)}$, and that we required the development of $F(x+h)$ for $x=a$. Differentiating, we have

$$\frac{dy}{dx}=2(a-x)+\frac{ax}{\sqrt{(x^2-a^2)}}, \text{ which, for } x=a, \text{ is } \infty.$$

As the theorem thus fails to be applicable after the first term, the true development, when the terms are arranged according to the ascending powers of h, must have a fractional power of h in the second term. Developing $(2a+h)^{\frac{1}{2}}$ by the binomial theorem, we have

$$\begin{aligned}F(a+h)&=a^2-h^2+a(2ah+h^2)^{\frac{1}{2}}=a^2-h^2+ah^{\frac{1}{2}}(2a+h)^{\frac{1}{2}}\\&=a^2-h^2+ah^{\frac{1}{2}}\left\{(2a)^{\frac{1}{2}}+\frac{h}{2(2a)^{\frac{1}{2}}}-\frac{h^2}{2.4(2a)^{\frac{3}{2}}}+\ldots\right\}\\&=a^2+a\left\{(2a)^{\frac{1}{2}}h^{\frac{1}{2}}+\frac{h^{\frac{3}{2}}}{2(2a)^{\frac{1}{2}}}-\frac{h^2}{a}-\frac{h^{\frac{5}{2}}}{2.4(2a)^{\frac{3}{2}}}+\ldots\right\}\end{aligned}$$

Again: let the function be $F(x)=\sqrt{x}+(x-a)^2\log(x-a)$, and let it be required to develope $F(a+h)$. Since here $F(a)=\sqrt{a}-0\times\infty$, it becomes necessary to ascertain whether the expression which assumes this form is really finite or infinite: applying then the method explained at (563), we find that when $x=a$, the true value of

$$(x-a)^2\log(x-a), \text{ or of } \frac{\log(x-a)}{\frac{1}{(x-a)^2}}, \text{ is } 0.$$

Hence the first term of the development is $\sqrt{a}$. Differentiating then $F(x)$, we have

$$\frac{dF(x)}{dx}=\frac{1}{2\sqrt{x}}+2(x-a)\log(x-a)+x-a, \text{ which, for } x=a, \text{ is } \frac{1}{2\sqrt{a}}.$$

Consequently the true development of $F(x+h)$ agrees with Taylor's form, as far, at least, as the first two terms. Differentiating again, we have

$$\frac{d^2F(x)}{dx^2}=-\frac{1}{4x\sqrt{x}}+2\log(x-a)+3, \text{ which, for } x=a, \text{ is } -\infty.$$

Hence the true development departs from Taylor's form at the third term: the first three terms are as follows:—

$$(a+h)^{\frac{1}{2}}+h^2\log h=a^{\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}h+\left(\log h-\frac{1}{2.4}a^{-\frac{3}{2}}\right)h^2+\ldots,$$

and we know that $\log h$, which enters the third term, does not admit of development, according to the positive integral powers of h (p. 524).

575. **Failure of Maclaurin's Theorem.**—The failing cases of Maclaurin's theorem are not at all analogous to the failing cases of Taylor's. The latter theorem becomes inapplicable in any case only for a particular value of the variable: the former, when inapplicable at all, is inapplicable generally for the function, whatever value be put for the variable: for in Maclaurin's theorem, the coefficients, for the same function, are fixed in value, and are unaffected, therefore, by changes of value in the variable: these coefficients are determined by putting 0 for the variable in the general values of them. If this substitution of 0 for x cause $F(x)$, or any of the coefficients derived from it, to become infinite, the theorem is inapplicable, no matter what finite value be put for x. It is inapplicable, moreover, when this substitution causes $F(x)$ and all its derived functions, to vanish, as in $F(x)=e^{-\frac{1}{x^2}}$.

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576. **Asymptotes to Curves.**—A rectilinear asymptote to a curve is a straight line which the curve approaches continually nearer and nearer to contact with, but which it actually touches only at an infinite distance. It may thus be regarded as the extreme or ultimate position of the tangent; so that the determination of the asymptote, whenever such line exists, is the determination of the tangent, on the hypothesis that the co-ordinates of the point of contact are one or both infinite, while at the same time the intercepts of the axes, between the origin and the line thus determined, are one or both finite: if this latter condition have not place, it is plain that the entire line itself, as well as the point of contact, must be infinitely distant, and therefore no part of it can have existence within finite bounds. The equation of the tangent at any point (x', y') of a plane curve is (528)

$$y-y'=\frac{dy'}{dx'}(x-x'),$$

and, therefore, by putting in this $y=0$, and $x=0$, successively, we have for the intercepts of the axes of x, and y, between the origin and the tangent, the expressions

$$x'-\left(y'\div\frac{dy'}{dx'}\right), \text{ and } y'-x'\frac{dy'}{dx'} \quad \ldots\ldots[1].$$

If for $x'=\infty$, both these are finite, they will determine two points, one on each axis, through which an asymptote passes. If for $x'=\infty$, the first expression is finite, and the second infinite, the first will determine a point on the axis of x, and the second will show that a parallel to the axis of y, through this point, will be an asymptote to the curve. But if, on the contrary, the second expression is finite, and the first infinite, the asymptote will pass through the point in the axis of y determined by the finite value, and will be parallel to the axis of x.

In like manner, if for $y'=\infty$, one or both of the expressions [1] are finite, the curve will have an asymptote determinable in a similar way. Should both expressions be infinite, the curve cannot have an asymptote: —no line can be parallel to both axes. When the expressions [1] are both 0 for $x'=\infty$, the asymptote will pass through the origin: it is, of course, unnecessary to say that if one of the intercepts [1] be 0, the other must be 0.

Ex. 1. Let the curve be the common parabola, its equation being $y=2(mx)^{\frac{1}{2}}$.

Here $\frac{dy}{dx}=\left(\frac{m}{x}\right)^{\frac{1}{2}}$, $\therefore\ y-x\frac{dy}{dx}=2(mx)^{\frac{1}{2}}-(mx)^{\frac{1}{2}}=(mx)^{\frac{1}{2}}$, which is ∞ for $x=\infty$.

Also since $\frac{dy}{dx}$ is 0 for $x=\infty$, the first of [1] is also ∞:

hence the parabola cannot have an asymptote.

(2) Let the curve be the hyperbola, its equation being $y=\frac{b}{a}(x^2-a^2)^{\frac{1}{2}}$.

Here $\frac{dy}{dx}=\frac{b}{a}x(x^2-a^2)^{-\frac{1}{2}}$, $\therefore\ x-\left(y\div\frac{dy}{dx}\right)=x-\frac{x^2-a^2}{x}=\frac{a^2}{x}$,

which is 0 when $x=\infty$: hence the hyperbola has an asymptote, and it passes through the origin. To find its equation, which must be of the form

$y=\frac{dy'}{dx'}x$, we have only to put $x=\infty$ in the foregoing expression for $\frac{dy}{dx}$, that is,

in $\frac{b}{a}x(x^2-a^2)^{-\frac{1}{2}}=\frac{b}{a}\frac{1}{\left(1-\frac{a^2}{x}\right)^{\frac{1}{2}}}$, and we find the equation to be

$y=\pm\frac{b}{a}$; so that there are *two* asymptotes.

577. In many cases asymptotes to curves may be more readily determined, without the aid of the differential calculus, from the consideration that to the same abscissa the difference between the ordinate to the curve, and the ordinate to the asymptote, becomes less and less as x increases, and vanishes when $x=\infty$. Or else the difference of the abscissas to the same ordinate becomes less and less as y increases, and vanishes when $y=\infty$. If neither of these results have place, there is no asymptote; if both have place, there are at least two asymptotes: an example will illustrate this method.

Let the equation of the curve be $xy-ay-bx=0$: then $y=\frac{bx}{x-a}$.

Performing the division, $y=b+\frac{ab}{x}+\frac{a^2b}{x^2}+\ldots\ldots$

Now as x increases, the difference between this and b becomes less and less, and vanishes altogether when $x=\infty$: hence the curve has an asymptote $y=b$.

Again: from the equation of the curve, $x=\frac{ay}{y-b}=a+\frac{ab}{y}+\frac{ab^2}{y^2}+\ldots$,

therefore the curve has another asymptote $x=a$.

We thus see that the object is to develope one of the variables according to the *descending* powers of the other, and then to reject all the terms involving the variable in a denominator; those retained, equated with the other variable, if of only the first degree, will represent an asymptote: if they are of a higher degree, there will be no rectilinear asymptote.

When, as in the example above, the equation to the asymptote is simply $y=A$, or $x=B$, showing that that asymptote is parallel to an axis, then there may be an asymptote parallel to the other axis, as above, so that it will be necessary to ascertain whether or not such be the case. The development spoken of is to be effected either by common division, or extraction, the binomial theorem, Maclaurin's theorem, after putting $\frac{1}{z}$ for x, or for y, or else by indeterminate coefficients, as in the example following.

(2) Let the equation of the curve be $y^3+x^3-axy=0$.

Assume $y=Ax+B+Cx^{-1}+\ldots$, $\therefore\ y^3=A^3x^3+3A^2Bx^2+\ldots$,

and $axy=Aax^2+Bax+\ldots$,

$\therefore\ y^3+x^3-axy=(A^3+1)x^3+(3A^2B-Aa)x^2+\ldots=0$,

$\therefore\ A^3+1=0,\ 3A^2B-Aa=0$,

$$\therefore\ A=-1,\ B=-\frac{a}{3}.$$

Hence the line $y=-x-\frac{a}{3}$ is an asymptote.

Applying this method to the two examples previously given, we have, in the case of the parabola,

$$y^2=4mx,\text{ or } x=\frac{1}{4m}y^2+0y+0+\frac{0}{y}+\ldots\ldots,$$

so that the curve has no asymptote. In the case of the hyperbola,

$$y=\pm\frac{b}{a}(x^2-a^2)^{\frac{1}{2}}=\pm\left(\frac{b}{a}x-\frac{1}{2}\frac{a^2}{x}+\ldots\right),\ \therefore\ y=\pm\frac{b}{a}x \text{ denote the two asymptotes.}$$

NOTE.—Whenever asymptotes exist which are parallel to either of the co-ordinate axes, as in the case of the first of the preceding examples, they may be readily ascertained to exist by noticing that for some finite value of one of the variables the other will become infinite: thus, taking the example referred to, namely, $(x-a)y-bx=0$, we see that for $x=a$, we have $0\times y-ab=0$, which can be possible only when $y=\infty$: the equation $x=a$ is therefore that of an asymptote. The proposed equation is also $(y-b)x-ay=0$, which for $y=b$, is $0\times x-ab=0$, and this requires that $x=\infty$: the equation $y=b$ is therefore that of an asymptote.

EXAMPLES FOR EXERCISE.

(1) Prove that the curve $y^3=ax^2-x^3$ has the asymptote $y=-x+\frac{a}{3}$.

(2) Prove that the axis of x is an asymptote to the curve $y(a^2+x^2)-a^2(a-x)=0$.

(3) Show that the axis of x is an asymptote to the logarithmic curve $y=a^x$.

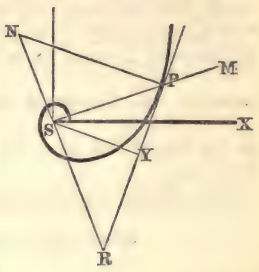
(4) Prove that the Cissoid of Diocles, namely, the curve $y^2=\frac{x^3}{2a-x}$, has no asymptote.

(5) Prove that the curve $y^2(x-2a)-x^3+a^3=0$ has the asymptotes $x=2a$, $y=\pm(x+a)$.

(6) Prove that $x=a$, and $y=x+\frac{a-b}{3}$, are asymptotes to the curve

$$(y^3+by^2)(x-a)-x^4=0.$$

578. **Spiral Curves.**—If, while a straight line revolves about a fixed point S, a point P move along it, this point will trace out a curve called a *spiral*. From the mode of its generation, it is pretty clear that a spiral curve is analytically represented, in the most convenient manner, by means of polar co-ordinates: the object of the present article is to show, by employing polar co-ordinates, how the subtangents, asymptotes, &c., of spirals are to be determined.



Let the curve be any whatever; and let the point S be taken for pole, SX being the fixed axis. Put $SP=r$, and the angle $PSX=\theta$: then if PR be a tangent at any point P, and PN the normal, and if RSN be perp. to the radius vector SP, the part SR will be the *polar subtangent*, and the part SN the *polar subnormal*. If the curve were originally referred to rectangular axes, SX, and a perpendicular to SX, we should have

$$x=r\cos\theta,\ y=r\sin\theta,\text{ and } x^2+y^2=r^2;$$

and the equation of the curve in polar co-ordinates would be of the form

$$r=F(\theta),\text{ or } F(r,\ \theta)=0.$$

$$\text{Now } SR=SP\tan SPR=r\tan(\theta-\alpha)=r\,\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}.$$

$$\text{But } \tan\theta=\frac{y}{x},\text{ and } \tan\alpha=\frac{dy}{dx},$$

the axes being rectangular: hence the subtangent SR is

$$T_1=SR=r\left(\frac{y}{x}-\frac{dy}{dx}\right)\div\left(1+\frac{y}{x}\frac{dy}{dx}\right)=r\left(y-x\frac{dy}{dx}\right)\div\left(x+y\frac{dy}{dx}\right).$$

But the differential coefficient $\frac{dy}{dx}$, which implies that x is the principal variable, must be replaced by the following expression when the variable is changed for θ, namely (572),

$$\frac{dy}{dx}=\frac{dy}{d\theta}\div\frac{dx}{d\theta},\ \therefore\ SR=r\left(y\frac{dx}{d\theta}-x\frac{dy}{d\theta}\right)\div\left(x\frac{dx}{d\theta}+y\frac{dy}{d\theta}\right).$$

$$\text{And since } \frac{dx}{d\theta}=\frac{dr}{d\theta}\cos\theta-r\sin\theta,\text{ and } \frac{dy}{d\theta}=\frac{dr}{d\theta}\sin\theta+r\cos\theta,$$

we have by substitution,

$$\text{Subtangent } T_1=SR=r^2\div\frac{dr}{d\theta},\ \therefore\ \text{Subnormal } N_1=SN=\frac{SP^2}{SR}=\frac{dr}{d\theta}.$$

Also for the angle P, between the tangent and the radius vector, we have

$$\tan P=\frac{SR}{r}=r\div\frac{dr}{d\theta}.$$

When for $r=\infty$ the subtangent is finite, then, and only then, the curve has an asymptote, as is evident.

(1) *The logarithmic spiral.*—If the point P so move along the revolving line SP that the arcual measure of the angle between the radius vector

SP and the fixed axis SX, has a constant ratio to the logarithm of the radius vector, the path of P is the logarithmic spiral:

its equation, therefore, is $\frac{\log r}{\theta}=a$, or, which is the same, $r=a^{\theta}$.

This justifies the name given to the curve; for if a be the base of any system of logs, the log of the radius vector r will be the number expressing the arcual measure of the exponent θ. Differentiating this equation, we have

$$\frac{dr}{d\theta}=a^{\theta}\log a,\ \therefore\ r^2\div\frac{dr}{d\theta}=a^{2\theta}\div a^{\theta}\log a=\frac{a^{\theta}}{\log a}=\frac{r}{\log a}=SR=T_1.$$

If $a=e$, the base of the Napierian system of logs, the subtangent at any point will always be equal to the radius vector of that point.

For the angle P we have $\tan P=r\div\frac{dr}{d\theta}=\frac{1}{\log a}$,

which being constant, it follows that this spiral cuts all its radii vectores at the same angle: it is in consequence of this peculiarity that it is frequently called the *equiangular* spiral:—if $a=e$, the constant angle is 45°. When $r=\infty$, T_1 is also ∞, hence this spiral has no rectilinear asymptote.

(2) *The spiral of Archimedes.*—The equation of this spiral is $r=a\theta$,

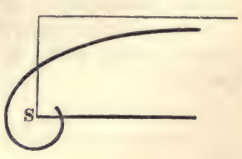
$$\therefore\ \frac{dr}{d\theta}=a,\ \therefore\ r^2\div\frac{dr}{d\theta}=(a\theta)^2\div a=r\theta=T_1,$$

so that here the subtangent is equal to the length of the circular arc to radius r, which is described while the revolving line generates the angle θ; hence when $\theta=2\pi$, the subtangent is equal to the whole circumference, so that if this spiral could be accurately constructed, we might draw a straight line accurately, equal to the circumference of a circle. Since T_1 is infinite when r is, there is no asymptote to this curve. The equation $r=a\theta^n$ comprehends an endless variety of spirals. If $n=-1$, it is the *hyperbolic*, or *reciprocal* spiral. If $n=\frac{1}{2}$, it is the *parabolic* spiral: if $n=-\frac{1}{2}$, the spiral is called the *lituus*.

(3) *The hyperbolic spiral.*—The equation being $r\theta=a$, we have

$$\frac{dr}{d\theta}=-\frac{a}{\theta^2},\ \therefore\ r^2\div\frac{dr}{d\theta}=\frac{a^2}{\theta^2}\div-\frac{a}{\theta^2}=T_1=-a,\ \text{a constant}$$

length. Since $r=\frac{a}{\theta}$, when $\theta=0$, $r=\infty$:



hence this spiral has a rectilinear asymptote parallel to the fixed axis, and at the distance $-a$ from the pole. The radius vector, in its initial or first position, is therefore infinite, the generating angle being positive.

(4) In a similar manner it may be shown that in the parabolic spiral the subtangent is

$$T_1=2\frac{r^3}{a^2}:\ \text{and that in the lituus it is } T_1=2\frac{a^2}{r}.$$

In this spiral the fixed axis is an asymptote. [NOTE.—For negative angles there is always another spiral.]

Although the formulæ in the present article have been applied exclusively to spirals, yet they are equally applicable to every curve related to polar co-ordinates: thus, take the hyperbola, the polar equation of which, when the centre is the pole, is

$$r^2=\frac{a^2(e^2-1)}{1-e^2\cos^2\theta},\ \therefore \text{ when } e^2\cos^2\theta=1,\ r=\infty:$$

hence the asymptotes are so situated as that

$$\cos^2\theta=\frac{1}{e^2}=\frac{a^2}{a^2+b^2},\ \therefore \frac{1}{\cos^2\theta}=\tan^2\theta+1=\frac{a^2+b^2}{a^2},$$

$$\therefore \tan\theta=\pm\frac{b}{a},\ \therefore \text{ the asymptotes are } y=\pm\frac{b}{a}x.$$

A curve which is referred to polar co-ordinates is usually called a *polar curve.*

579. **Circular Asymptotes.**—Hitherto we have considered only rectilinear asymptotes, but if, in any spiral curve, r should continue finite, though θ increase without limit, it will follow that the spiral must make an infinite number of convolutions about its pole before it can reach the circumference of a circle, described from that point as centre, and of which the radius is the finite value of r corresponding to $\theta=\infty$. In such a case, therefore, the spiral has a *circular asymptote.* Moreover, if the value of r for $\theta=\infty$ be greater than the value of r for every other value of θ, the spiral will be wholly included *within* its circular asymptote; but otherwise it will be wholly *without* it. For example:—

In the spiral whose equation is $(r^2-ar)\theta^2=1$, or $\theta=\frac{1}{\sqrt{(r^2-ar)}}$, $\theta=\infty$, when $r=a$;

and for a less value of r, θ is imaginary, $\therefore$ the spiral can never approach so near to the pole as $r=a$ during any finite number of revolutions: hence the asymptotic circle, whose radius is a, is *within* the spiral.

If, however, the equation had been $\theta=\frac{1}{\sqrt{(ar-r^2)}}$,

then, although as before, $\theta=\infty$ when $r=a$, yet, since for all other real values of θ, r is less than a, the spiral must be entirely enclosed within the asymptotic circle of radius a.

Perpendicular upon the tangent from the pole.—An expression for this perpendicular (SY) is often necessary, especially in astronomical inquiries.

Call it p: then $p^2=r^2\sin^2 SPR=r^2\frac{\tan^2 SPR}{1+\tan^2 SPR}$,

$$\therefore \frac{1}{p^2}=\frac{1}{r^2}\frac{\tan^2 SPR+1}{\tan^2 SPR}=\frac{1}{r^2}(1+\cot^2 SPR)\ldots\ldots[1].$$

But (578), $\tan SPR=r\div\frac{dr}{d\theta}$, $\therefore \cot^2 SPR=\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2$.

Put $u=\frac{1}{r}$, then $\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$: hence [1], $\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2$.

For other expressions see (583).

580. **Rectification of Polar Curves.**—It was shown at (544) that s being the arc of any curve related to rectangular co-ordinates,

$\left(\frac{ds}{dx}\right)^2=1+\left(\frac{dy}{dx}\right)^2$. Now (572), $\frac{ds}{d\theta}=\frac{ds}{dx}\frac{dx}{d\theta}$, and (578), $\frac{dy}{dx}=\frac{dy}{d\theta}\div\frac{dx}{d\theta}$,

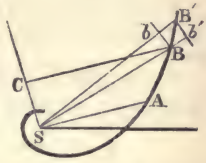
$\therefore \left(\frac{ds}{d\theta}\right)^2=\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2$. But since $x=r\cos\theta$, and $y=r\sin\theta$,

$\frac{dx}{d\theta}=\cos\theta\frac{dr}{d\theta}-r\sin\theta$, and $\frac{dy}{d\theta}=\sin\theta\frac{dr}{d\theta}+r\cos\theta$,

$\therefore \left(\frac{ds}{d\theta}\right)^2=r^2+\left(\frac{dr}{d\theta}\right)^2$, $\therefore s=\int\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}d\theta$...[1].

This integral, taken between the limits $\theta=\theta_1$ and $\theta=\theta_2$, will give the length of curve included between the two corresponding values of r, that is, between Sr_1 and Sr_2.

581. **Sectorial Area: Polar Co-ordinates.**—Let SAB be a sectorial area, and BSB' an increment of it $=\Delta$ Area. Then, if from centre S, and with radii SB, SB' arcs of circles Bb, $B'b'$ be described, terminating in the radii vectores to B, B' the increment Δ Area, if taken sufficiently small, will always lie between the circular sectors, that is,



between $\frac{1}{2}r^2\Delta\theta$, and $\frac{1}{2}(r+\Delta r)^2\Delta\theta$: hence

$\frac{\Delta\text{ Area}}{\Delta\theta}$ lies between $\frac{1}{2}r^2$, and $\frac{1}{2}(r+\Delta r)^2$, $\therefore$ in the limit, when $\Delta\theta$,

and consequently Δ Area, becomes 0, we have $\frac{d\text{ Area}}{d\theta}=\frac{1}{2}r^2$,

$\therefore$ Area$=\frac{1}{2}\int r^2d\theta$...[1], taken between assigned limits of θ.

Otherwise. Conceive the proposed sectorial area to be cut up into component sectors, by drawing radii vectores so that the angles ($\Delta\theta$) between every two consecutive pair may be equal: if the extremities of these radii be joined, we shall have a sectorial polygon, which will approach nearer and nearer to the curvilinear area as the angles spoken of diminish. The area of any one of the component plane triangles will be generally expressed by

$\frac{1}{2}rr'\sin\Delta\theta$: but when $\Delta\theta$ diminishes down to $d\theta$, and therefore r and r' become equal, the sum of the infinite number of these zero triangles, that is, the sectorial area, is

$\frac{1}{2}\int r^2d\theta$...[2], the differential of it being $\frac{1}{2}r^2d\theta$.* This same differential, that is, the zero-triangle, is also $\frac{1}{2}pds$, p being the perp. (SY) on the tangent at the extremity of s. (See fig. p. 558).

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* The following is good advice :—"We would recommend to the student in pursuing any problem in the Integral Calculus, never for one moment to lose sight of the manner in which he would do it, if a rough solution, for practical purposes only, were required." —*De Morgan* (*Elementary Illustrations*, &c.).

582. **Sectorial Area: Rectangular Co-ordinates.**—It is sometimes necessary to have an expression for a sectorial area when the co-ordinates are rectangular, and there is a very easy way of arriving at such an expression. In the above diagram SA, SC, being rectangular axes, (x, y) being any point B in the curve, we have for the area $SABC$ the expression $\int xdy$, making y the independent variable (546).

Consequently, the sectorial area SAB is $\int xdy - \frac{1}{2}xy$, because $\frac{1}{2}xy$ is the triangle SBC.

By differentiating this, we have d sectorial Area $= \frac{xdy - ydx}{2}$: hence (last art.),

$$p = r^2 \div \frac{ds}{d\theta} = \frac{xdy - ydx}{ds}.$$

From the first of these we have

$\frac{ds}{d\theta} = \frac{r^2}{p}$, $\therefore s = \int \frac{r^2}{p} d\theta$...[1], which is another form for the length of a polar curve.

We shall now apply the above formulæ to an example or two.

(1) Required the length of a circular arc subtending the angle θ at the centre.

Taking the expression [1] of the present article, observing that in the circle $p = r$, we have

$\int_0^\theta r d\theta = r\theta$. Let $\theta = \pi$, then the arc is a semicircumference $= r\pi$.

(2) Required the length of an arc of a logarithmic spiral included between two assigned radii vectores r_1, r_2. The equation is $r = a^\theta$:

$$\therefore \frac{dr}{d\theta} = a^\theta \log a = r \log a, \text{ also } d\theta = \frac{dr}{r \log a} : \text{ hence}$$

$$s = \int \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{1}{2}} d\theta = \int \left\{ r^2 + (r \log a)^2 \right\}^{\frac{1}{2}} \frac{dr}{r \log a} = \int \left\{ \frac{1 + (\log a)^2}{(\log a)^2} \right\}^{\frac{1}{2}} dr,$$

which, between the proposed limits, is $s = \frac{\sqrt{(1 + \log^2 a)}}{\log a}(r_2 - r_1)$.

This result shows that the curve is such that the arc of it intercepted by any two radii, having the same difference, is always the same in length.

(3) From an extremity of the diameter $2a$ of a circle a chord r is drawn: required the area of the sector between this chord and the diameter. Let θ be the angle of the sector, then for the chord r, we have

$$r = 2a \cos\theta, \therefore \frac{1}{2}\int r^2 d\theta = 2a \int \cos^2\theta d\theta, \therefore \text{Area} = 2a^2 \int_0^\theta \cos^2\theta d\theta.$$

This form we have not as yet integrated: we may proceed as follows:—

By integ. by parts (539) $\int \cos\theta d \sin\theta = \cos\theta \sin\theta - \int \sin\theta d \cos\theta$,

that is, $\int \cos^2\theta d\theta = \cos\theta \sin\theta + \int \sin^2\theta d\theta$.

Add now the first member to each side: then since $\sin^2\theta + \cos^2\theta = 1$,

$$2\int \cos^2\theta d\theta = \cos\theta \sin\theta + \int d\theta,$$

$$\therefore \text{Area} = 2a^2 \int_0^\theta \cos^2\theta d\theta = a^2(\cos\theta \sin\theta + \theta) = a^2\left(\frac{1}{2}\sin 2\theta + \theta\right)$$

If $\theta=\frac{1}{2}\pi$, the area will evidently be that of the whole semicircle, and since in this case $\sin 2\theta=0$, the expression gives Area$=\frac{1}{2}a^2\pi$, which we otherwise know to be true.

The integration of the foregoing expression might have been easily obtained without the aid of the formula at (539): thus

By Trigonometry (244) or (259) $\cos^2\theta=\frac{1}{2}\cos 2\theta+\frac{1}{2}$, $\therefore \int\cos^2\theta d\theta=\frac{1}{2}\int\cos 2\theta d\theta+\frac{1}{2}\theta$

$=\frac{1}{4}\sin 2\theta+\frac{1}{2}\theta$, $\therefore 2a^2\int_0^\theta\cos^2\theta d\theta=a^2\left(\frac{1}{2}\sin 2\theta+\theta\right)$, as before.

The former mode of integration is given chiefly for the purpose of exemplifying to the student one among many of the expedients it is frequently necessary to adopt (though not necessary in *this* case) in order to reduce unknown integrals to known forms. The direct rules for integration are so few, comparatively, that the operation, even in apparently very simple instances, can sometimes be performed only by analytical artifice and ingenuity. If the student desire to test the truth of the above general expression for a sectorial area he can do so very readily: if he draw a radius to the extremity of the chord the area will become divided into a central sector and an isosceles triangle: the first part of the expression, namely, $\frac{1}{2}a^2\sin 2\theta$, is the area of this triangle, and the other part $a^2\theta$, the area of the central sector.

We shall conclude this examination of polar curves by appending a table of the principal straight lines connected with them: the several equations the student can easily verify for himself by an inspection of the figure, p. 558, and by a reference to the preceding formulæ: the angle SPY will be called P.

583. *Polar tangent, subtangent, normal, subnormal, &c.*

Tangent $PR=\frac{r}{\cos P}=\frac{r^2}{\sqrt{(r^2-p^2)}}=r\frac{ds}{dr}$.

Subtangent $SR=r\tan P=\frac{rp}{\sqrt{(r^2-p^2)}}$

$=r^2\frac{d\theta}{dr}$.

Normal $PN=\frac{r}{\sin P}=\frac{r^2}{p}=\frac{ds}{d\theta}$.

Subnormal $SN=\frac{r}{\tan P}=\frac{r}{p}\sqrt{(r^2-p^2)}$

$=\frac{dr}{d\theta}$.

$SY=p=r\sin P=r^2\frac{d\theta}{ds}=\left[\frac{xdy-ydx}{ds}\right]$.

$PY=r\cos P=\sqrt{(r^2-p^2)}=r\frac{ar}{ds}$.

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584. **Contact of Curves: Osculation.**—Let $y=f(x)$, $Y=F(x)$ be the equations of two plane curves, in the former of which the constants a, b, c, are known, and therefore the curve determinate. In the latter, however, the constants A, B, C, &c., will be regarded as unknown or arbitrary, and therefore open to any conditions we may choose to subject them to. The first is therefore an individual curve completely given: the second is any one whatever of a particular class, species, or *family* of curves. The arbitrary constants which enter into the equation of a curve are usually called the *parameters*.

Let x take the increment h, and let the corresponding ordinates y', Y' be developed by Taylor's theorem; then

$$y'=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\text{.....[1].}$$

$$Y'=Y+\frac{dY}{dx}h+\frac{d^2Y}{dx^2}\frac{h^2}{1.2}+\frac{d^3Y}{dx^3}\frac{h^3}{1.2.3}+\text{.....[2].}$$

The parameters which enter [2] being arbitrary, they may be determined in terms of x (whatever particular value we may choose to give to this variable), and of the known constants a, b, c, &c., so as to satisfy as many of the conditions

$$y=Y,\ \frac{dy}{dx}=\frac{dY}{dx},\ \frac{d^2y}{dx^2}=\frac{d^2Y}{dx^2},\ \text{&c.....[3],}$$

as there are arbitrary constants, or parameters in $Y=F(x)$.

The values of A, B, C, &c., determined in this way, when substituted in [2], will cause so many of the leading terms in both series to become *identical*, whatever particular value we put for x. Now the greater the number of leading terms in the two developments which are thus rendered identical, the nearer will the developments themselves approach to identity, provided h may be taken as small as we please:—a proviso, indeed, always to be understood in reasoning from Taylor's theorem.

When the first of the conditions [3] exist, the curves have a common point, inasmuch as, for the same abscissa, they have a common ordinate: if this be the *only* condition, we infer, therefore, that the curves intersect.

When the second condition also exists, we infer (528) that the curves have a common tangent at the point: if these two be the only conditions, the curves have simple contact, or *contact of the first order*.

When the third condition also exists, the approach of the two curves in the immediate vicinity of this contact will be closer, that is, the difference between y' and Y' will be less; and they will touch with *contact of the second order*, as it is called. And so on. Hence of all curves $Y=F(x)$ of a given species, *that* will touch any fixed curve $y=f(x)$, at a proposed point, with the closest contact, whose parameters are *all* determined agreeably to the conditions [3]. No other curve implied in the equation $Y=F(x)$—whatever values we give to the constants—can approach so nearly to coincidence with the given curve $y=f(x)$, in the immediate vicinity of the point of contact, as this; so that no other curve of the same species can pass between it and the fixed curve. A touching curve thus determined is said to be the *osculating curve*, of that species, to the fixed curve, at the proposed point: its contact is of the nth order, if it have $n+1$ parameters, the values of which have all been fixed by their being made to satisfy the $n+1$ conditions [3]. If the values thus determined happen to be such that they spontaneously, as it were, satisfy also the $n+1$th of the conditions [3], the contact at the point is, of course, still more intimate.

If the contact of two curves be of an odd order, they do not intersect at the point of contact:—if of an even order, they do intersect. For in the former case the difference $y'-Y'$, of the series [1], [2], has an *even* power of h in its leading term; and in the latter case, an *odd* power of h; so that this difference $y'-Y'$, for sufficiently small values of h, has the

same sign whether h be + or −, in the one case, and *opposite* signs in the other case. This shows that in the immediate vicinity of the equal and coincident ordinates y, Y, the ordinate y' is greater or less than the ordinate Y', on *both* sides of the ordinate common to both curves, whenever the contact is of an *odd* order; but that whenever it is of an *even* order, y' is greater than Y' on one side the common point, and less on the other. In the diagram at p. 329, the linear tangent at P, which has contact with the ellipse, of an odd order (the first order) is wholly outside the curve in the vicinity of P, but the touching *circle*, in the vicinity of P, is inside the curve to the left, and outside to the right: its contact with the ellipse being of the second order, since there are three parameters, or arbitrary constants, in the general equation of the circle.

For reasons stated at p. 327, the osculating circle is that which, of all curves, is preferred to test the curvature of another curve at any proposed point of the latter, and we shall, therefore, proceed to show how the radius of the osculating circle—the radius of curvature—is to be determined, be the given curve whatever plane curve it may.

It is as well, however, just to remind the student, that if the curve be such that, for the point chosen in it, the coefficients [3] any of them become infinite, thus implying that for the assigned value of x, Taylor's theorem is no longer applicable, the preceding theory fails for that point. Should the failure commence at the second, or at a more advanced coefficient, the curve will have such a peculiar form at the point, as not to admit of orders of contact, as we shall hereafter see: but if it be the *first* differential coefficient $\frac{dy}{dx}=\frac{dY}{dx}$, that becomes infinite, we know, independently of Taylor's theorem, that simple contact, at least, will be implied, since the two curves will have a common linear tangent at the point, this tangent being parallel to the axis of y.

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585. **Radius of Curvature: Rectangular Co-ordinates.** —Let $y=f(x)$ be the equation of any given plane curve: it is required to affix such values to the arbitrary constants or parameters α, β, ρ, in the general equation $(x-\alpha)^2+(Y-\beta)^2=\rho^2$, of a circle, that the particular circle thus fixed may osculate the proposed curve at an assigned point (x, y). By [3] the parameters are to be determined from the three conditions

$$y=Y, \frac{dy}{dx}=\frac{dY}{dx}, \frac{d^2y}{dx}=\frac{d^2Y}{dx^2}.$$

Differentiating $(x-\alpha)^2+(Y-\beta)^2=\rho^2$, we have $(x-\alpha)+\frac{dY}{dx}(Y-\beta)=0$,

$$\left[\therefore \frac{dY}{dx}=-\frac{x-\alpha}{Y-\beta}, \therefore \frac{d^2Y}{dx^2}=-\frac{1}{Y-\beta}-\frac{(x-\alpha)^2}{(Y-\beta)^3}=-\frac{r^2}{(Y-\beta)^3}.\right.$$

Hence, introducing the above conditions, putting y for Y, &c., we have, for determining α, β, ρ, so soon as the point (x, y) on the curve $y=f(x)$ is fixed, the three equations

$$(x-\alpha)^2+(y-\beta)^2=\rho^2 \ldots\ldots [1],$$

$$(x-\alpha)+\frac{dy}{dx}(y-\beta)=0, \text{ or } (x-\alpha)+p'(y-\beta)=0 \ldots\ldots [2],$$

$$\frac{d^2y}{dx^2}(y-\beta)^3=-\rho^2, \text{ or } p''(y-\beta)^3=-\rho^2 \ldots\ldots [3].$$

From [2], $(x-\alpha)^2=p'^2(y-\beta)^2$, $\therefore$ [1], $(p'^2+1)(y-\beta)^2=\rho^2$, $\therefore$ [3], $y-\beta=-\frac{p'^2+1}{p''}$, which, substituted in [2], gives $x-\alpha=\frac{p'(p'^2+1)}{p''}$. And from these we get for the co-ordinates of the centre, $\alpha=x-\frac{p'(p'^2+1)}{p''}$, $\beta=y+\frac{p'^2+1}{p''}$...[4], and for the radius, $\rho=\frac{(p'^2+1)^{\frac{3}{2}}}{-p''}=-\left\{\left(\frac{ds}{dx}\right)^3\div p''\right\}=-\left(\frac{N}{y}\right)^3\div p''$ (p. 498)...[5],

so that the point (x, y) in the proposed curve being assigned, we have only to put its given co-ordinates in the expressions [4] and [5], in order to find the centre and radius of the circle which will touch the curve at that point, and be in closer contact with it there, than any other circle whatever: the symbols p', p'' stand for the first and second differential coefficients derived from the equation of the given curve.

From [2], which has place even when the contact is but of the first order, we learn that the centre (α, β) of every circle touching a curve is always on the normal at the point of contact: for [2] is the same as

$\beta-y=-\frac{1}{\frac{dy}{dx}}(\alpha-x)$, which shows (α, β) to be a point on the normal at (x, y), (p. 498).

If this normal be taken for the axis of y, the equation of the circle will be

$$x^2+(y-\beta)^2=\rho^2, \text{ the point of contact } (0, y), \text{ and } \frac{dy}{dx} \text{ or } p'=0:$$

the expression for the radius of curvature is then $\rho=-\frac{1}{p''}$.

586. The sign of p'' will always make known the direction of curvature: for if a linear tangent be drawn at the proposed point (x, y), we shall have for the ordinate of that tangent corresponding to the abscissa $x+h$, the expression $y+p'h$, and for the ordinate of the curve to the same abscissa, the expression

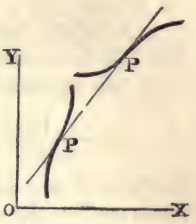
$$y+p'h+\frac{1}{2}p''h^2+\ldots\ldots,$$

and it is obvious that if p'' be *plus*, this latter ordinate, for a small value of h, must exceed the former; and thus the tangent will proceed from the point in both directions, between the curve and the axis of x, that is, the curve, at that point, will be *convex* to the axis of x. If, on the contrary, p'' is *minus*, then the ordinate of the curve will be less than that of the tangent, so that the curve will proceed in both directions from the point between the tangent and the axis of x, that is, it will be *concave* to the axis.

If p'' be 0 at the point, ρ will be ∞, whether $p'=0$ or not, so that the osculating circle degenerates into a straight line. As this straight line has contact of the second order, the parts of the curve in the immediate

vicinity of the point, in opposite directions from the point, will be on contrary sides of the tangent, as in the annexed diagram, seeing that there must then be intersection as well as contact (p. 564), provided, that is, that p''' is not also 0: but if $p'''=0$, and the next following coefficient finite, the contact will be unaccompanied by intersection:—the curve will be convex at the point, if the first finite differential coefficient, being of an even order, be +, and it will be concave if the sign be —.

A point P, at which the tangent intersects the curve, or at which the curve changes from convex to concave, is called a point of *contrary flexure*, or simply a *point of inflexion*. At this point, the curve being neither concave nor convex, the curvature is 0:—the reciprocal of the radius of curvature, which, as we have seen, is ∞; and generally, the degree of curvature of a curve at any point is always measured by $1 \div$ rad. of curvature at that point.



For exercises in finding the rad. of curvature by means of the formulæ [5], the student may take the examples worked by a different method in the Analytical Geometry (pp. 327, 350): we shall merely add here the following problem.

PROBLEM.—To determine those points in a given curve, if any such exist, at which the osculating circle shall have contact of the third order. In other words, to find for what points in the curve the values of α, β, ρ, determined as above from the first three conditions of [3] p. 564, satisfy also the fourth condition.

The differential coefficient p''', as derived from equation [3] at p. 564, is here to be equal to the corresponding coefficient derived from the equation of the given curve; so that the values which satisfy the three conditions at p. 564, are also to satisfy the fourth condition

$$(y-\beta)\,p'''+3p''p'=0,$$

and this they can do only for abscissæ, or values of x, which are real roots of the equation: if, in any particular case, the roots should be all imaginary, there will not exist any point in the curve at which the contact shall be of the third order. But when the abscissæ are real, the points indicated will be those of maximum or minimum curvature. For since

$$\rho=\frac{(p'^2+1)^{\frac{3}{2}}}{-p''},\ \therefore\ \frac{d\rho}{dx}=\frac{-3(p'^2+1)^{\frac{1}{2}}p'p''^2+(p'^2+1)^{\frac{3}{2}}p'''}{p''^2},$$

which, in the case of a maximum or a minimum value of ρ, must be 0,

$\therefore$ the numerator, or $-3p'p''^2+(p'^2+1)p'''=0$; or, dividing by p'',

and putting $y-\beta$ for its equal $-\frac{p'^2+1}{p''}$ (page 566), we have

$(y-\beta)p'''+3p''p'=0$, the equation to be satisfied above. (See also art. 594.)

NOTE.—In the figure at p. 329, the contact exhibited is of the third order; and at the principal vertices of the curves discussed in the Analytical Geometry, the contact of the osculating circle is always of the third order.

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587. In the preceding investigations x has been taken for the independent variable: but if any other quantity be chosen, the foregoing ex-

pression for ρ will require modification. To give complete generality, therefore, to the formula for the radius of curvature, we shall now suppose the independent variable to be left quite arbitrary, x and y being functions of it. Under this hypothesis p' and p'' must be replaced by the expressions at (572), that is, by

$$\frac{(dy)}{(dx)}, \text{ and } \frac{(d^2y)(dx)-(d^2x)(dy)}{(dx^3)},$$

the vincula implying that the independent variable in reference to which the differentials of x and y are taken, is arbitrary: when chosen, the differential of this independent variable, and its proper powers, are to be introduced as denominators of the above differentials. Making these substitutions, the expression for ρ becomes

$$\rho=-\frac{\{(dy)^2+(dx)^2\}^{\frac{3}{2}}}{(d^2y)(dx)-(d^2x)(dy)}=-\frac{(ds)^3}{(d^2y)(dx)-(d^2x)(dy)}, \text{ by (544).....[1],}$$

which is of the utmost generality, and will furnish the correct formula whatever be the hypothesis as to the independent variable: if x be chosen, in which case we shall have $(dx)=1$, and $\therefore (d^2x)=0$, the formula will become that at (585). If s be chosen, then we shall have $(ds)=1$,

$\therefore (ds)^2=(dx)^2+(dy)^2=1$. Differentiating this, $(d^2y)(dy)+(d^2x)(dx)=0$.....[2],

and by substituting in the denominator of [1], the expression for (d^2y), deduced from this equation, and remembering that the numerator of [1] is 1, we shall have

$$\rho=\frac{(dy)}{(d^2x)}=\frac{dy}{ds}\div\frac{d^2x}{ds^2}.$$

If the left-hand member of [2] be added to the *square* of the denom. of [1], the denom., which must remain unaltered in value, will be

$$\{(dy)^2+(dx)^2\}\{(d^2y)^2+(d^2x)^2\}=(d^2y)^2+(d^2x)^2, \therefore \rho=1\div\sqrt{\left\{\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2x}{ds^2}\right)^2\right\}}.$$

588. **Radius of Curvature: Polar Co-ordinates.**—In rectangular co-ordinates, let τ denote the angle which the linear tangent at P makes with the axis of x, then, leaving the independent variable arbitrary,

$$\tau=\tan^{-1}\frac{(dy)^*}{(dx)}, \therefore (d\tau)=d\frac{(dy)}{(dx)}\div\left(1+\frac{(dy)^2}{(dx)^2}\right) \text{ (page 501)}=\frac{(d^2y)(dx)-(d^2x)(dy)}{(dx)^2+(dy)^2},$$

$\therefore$ [1] above, $\frac{(d\tau)}{(ds)}=\frac{1}{\rho}, \therefore \rho=\frac{(ds)}{(d\tau)}$...[1]. Now the angle $\tau=\theta+SPY$ (see fig. p. 558),

that is, $\tau=\theta+\sin^{-1}\frac{p}{r}, \therefore (d\tau)=(d\theta)+\frac{r(dp)-p(dr)}{r\sqrt{(r^2-p^2)}}.$

But (583), $(d\theta)=\frac{p(dr)}{r\sqrt{(r^2-p^2)}}$, also $(ds)=\frac{r(dr)}{\sqrt{(r^2-p^2)}}$: hence

$$(d\tau)=\frac{(dp)}{\sqrt{(r^2-p^2)}}, \therefore \frac{(ds)}{(d\tau)}=\rho=r\frac{(dr)}{(dp)}.....[2].$$

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* Any first differential coefficient, in which the independent variable is assigned, may always be replaced by another, in which the independent variable is arbitrary (p. 553).

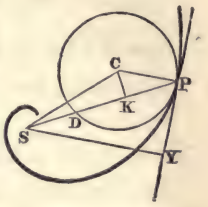
By making the substitutions for r and p at the bottom of p. 560, this expression, after some reductions, may be converted into the following, namely,

$$\rho=\frac{1}{u^3}\frac{\{u^2(d\theta)^2+(du)^2\}^{\frac{3}{2}}}{u(d\theta)^3+(d\theta)(d^2u)-(du)(d^2\theta)},\ \therefore\ \rho=\frac{1}{u}\frac{\left\{1+\frac{1}{u^2}\frac{du^2}{d\theta^2}\right\}^{\frac{3}{2}}}{1+\frac{1}{u}\frac{d^2u}{d\theta^2}}\ldots\ldots[3],$$

the radius of curvature at the point (r, θ), u being put for $\frac{1}{r}$.

[A useful formula for the ellipse is investigated at art. 590.]

589. **Chord of Curvature.**—If from C, the centre of curvature, a perp. CK be drawn to the radius vector SP, the angle included between this perp. and the radius of curvature CP, will be equal to the angle SPY, since the lines including this are perp. to those including the former angle. The perp. thus drawn from the centre of the circle to the *chord* of it from P passing through S, must necessarily *bisect* that chord PD—called the *polar chord of curvature:* hence, since



$PK=CP\sin C=\rho\sin SPY$, we have

$$\text{Polar chord of Curvature}=2\rho\sin SPY=2\rho\frac{p}{r}\ldots[4].$$

Note.—It may be worth while to notice, in reference to [1] above, that if Δs be an increment of the arc s of the curve, and normals be drawn from its extremities, these, being perpendicular to the tangents at the same extremities, will include the same angle as those tangents, namely, the angle $\Delta\tau$. If the arc Δs actually coincided with the arc of the equicurve circle, the two normals, up to the point where they intersect, would be equal, and that intersection would be the centre of curvature: and it is plain that this centre is approached nearer and nearer as $\Delta\tau$, and consequently Δs, diminishes;

so that the ratio $\frac{\Delta s}{\Delta\tau}$ approaches nearer and nearer to ρ, without limit

to the degree of nearness, as $\Delta\tau$ diminishes: hence, as before $\frac{ds}{d\tau}=\rho$.

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590. Of all curves the ellipse is that in reference to which (for the purposes of Astronomy and Geodesy) the radius and chord of curvature are the most important: we shall return for a moment to that curve.

By the formula at p. 329

$$\frac{1}{2}chd=\frac{b'^2}{a}=\left(\frac{b'}{b}\right)^2\frac{b^2}{a}=\rho\sin SPY \text{ by [4], where } \frac{b'}{b}=\frac{r}{p}=\frac{1}{\sin SPY}\text{ (p. 325)},$$

$$\therefore\ \rho=\frac{1}{\sin^3 SPY}\frac{b^2}{a}.$$

Now referring to the figure at p. 324, if λ be put for the angle PNF, we have

$$\frac{OF}{OP'}=\frac{\sin OP'F}{\sin OFP'}=\frac{\sin PFP'}{\sin\lambda}, \text{ that is, } \frac{c}{a}=\frac{\cos SPY}{\sin\lambda}=e, \therefore \cos SPY=e\sin\lambda,$$

$\therefore \sin SPY=\sqrt{(1-e^2\sin^2\lambda)}$, $\therefore$ by substitution, putting also $a(1-e^2)$ for $\frac{b^2}{a}$, we have

$$\rho=\frac{a(1-e^2)}{(1-e^2\sin^2\lambda)^{\frac{3}{2}}},$$

a very useful expression for the radius of curvature of the elliptic meridian of the earth, at a place of which the latitude is λ. It may be otherwise investigated, without a reference to [4], thus. By p. 328, and equations 3, 4, p. 316,

$$\rho=\frac{b'^3}{ab}=\frac{a^3b^3}{ab(a^2\sin^2\alpha'+b^2\cos^2\alpha')^{\frac{3}{2}}}=\frac{a^2b^2}{(a^2\sin^2\alpha'+b^2\cos^2\alpha')^{\frac{3}{2}}}.$$

Now if λ be the latitude of P, that is, the angle

$PNB=pOB$, then since $\alpha'=90°+pOB$, we have $\sin^2\alpha'=\cos^2\lambda$, and $\cos^2\alpha'=\sin^2\lambda$,

$$\therefore \rho=\frac{a^2b^2}{(a^2\cos^2\lambda+b^2\sin^2\lambda)^{\frac{3}{2}}}=\frac{a^2b^2}{\{a^2-(a^2-b^2)\sin^2\lambda\}^{\frac{3}{2}}}=\frac{a(1-e^2)}{(1-e^2\sin^2\lambda)^{\frac{3}{2}}}.$$

591. In order to show the practical application of this formula, let D, D', be the measured lengths in feet of two degrees of the earth's meridian, the middle points of those degrees being in the latitudes λ, λ', respectively. Then these measured arcs, being very small in comparison with the entire circumference of the osculating circles at the middle points, may be considered each as coincident with a degree of the corresponding circle, without any practical error.

Now the chief object of measuring degrees of the meridian is to determine the figure of the earth, or the ratio of its polar and equatorial diameters, or the amount of

$\frac{a-b}{a}$, called the earth's ellipticity, and which we know, from independent considerations, to be but a small fraction, $\frac{b}{a}$ differing but little from unity.

We thus know that e^2 can be but a small fraction, and therefore $e^2\sin^2\lambda$ a fraction still smaller; so small that the powers of it may be disregarded in a practical matter of this kind: hence, neglecting these powers,

$$\rho=a(1-e^2)(1-e^2\sin^2\lambda)^{-\frac{3}{2}}=a(1-e^2)(1+\frac{3}{2}e^2\sin^2\lambda).$$

Also length of $1°=\rho\frac{\pi}{180}=D$, $\therefore D=\frac{\pi}{180}a(1-e^2)(1+\frac{3}{2}e^2\sin^2\lambda)$, and

$$D'=\frac{\pi}{180}a(1-e^2)(1+\frac{3}{2}e^2\sin^2\lambda'), \therefore \text{ by actual division, } \frac{D}{D'}=1+\frac{3}{2}e^2(\sin^2\lambda-\sin^2\lambda'),$$

$$\therefore e^2=\frac{2}{3}\frac{D-D'}{D'(\sin^2\lambda-\sin^2\lambda')}=\frac{2}{3}\frac{D-D'}{D'\sin(\lambda+\lambda')\sin(\lambda-\lambda')},$$

which determines e^2 or $1-\frac{b^2}{a^2}$, and thence $\frac{b}{a}$, and $1-\frac{b}{a}$, the ellipticity.

The equatorial radius a may be found by measuring a degree on the

equator, and thence b the polar radius becomes known. By measurements of this kind, these two radii are found to be

3962·8 miles and 3949·6 miles respectively, the ellipticity

being $\frac{1}{298}$ (See Airy on the Figure of the Earth : *Ency. Met.*).

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592. **Evolutes and Involutes.**—If to every point of a given curve an osculating circle were to be applied, the centres of these circles would trace out a second curve called the *evolute* of the former: the original curve in relation to this is called the *involute*.

Denoting, as before, the centre of any one of the osculating circles indifferently by (α, β), it is plain that to find the locus of this point, or the evolute of the curve $y=F(x)$, we shall have merely to eliminate x, y, from the three equations at (385), namely,

$$y=F(x),\ \alpha=x-\frac{p'(p'^2+1)}{p''},\ \beta=\frac{p'^2+1}{p''},$$

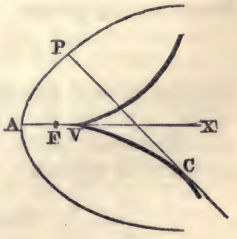
the resulting equation in α, β, will be that of the evolute. For example,

(1) Let the proposed curve be the common parabola $y^2=4mx$.

Here $p'=\frac{2m}{y}$, $\therefore p''=-\frac{4m^2}{y^3}$, $\therefore p'^2+1=\frac{4m^2}{y^2}+1=\frac{m}{x}+1$,

$$\frac{p'}{p''}=-\frac{y^2}{2m},\ \therefore\ \alpha=x+2m+\frac{y^2}{2m}=3x+2m,\ \therefore x=\frac{\alpha-2m}{3},$$

and $\beta=y-y-\frac{y^3}{4m^2}=-\frac{y^3}{4m^2}$, $\therefore y=(-4m^2\beta)^{\frac{1}{3}}$.



Substituting these values of x and y, in the given equation, we have

$$\beta^2=\frac{4}{27m}(\alpha-2m)^3,$$ the equation of the evolute.

This curve is called the semi-cubical parabola: by removing the origin to the point on the axis whose abscissa is $2m$, the equation takes

the simpler form $\beta^2=\frac{4}{27m}\alpha^3$. Because $\beta=0$, when $\alpha=0$,

the curve passes through the new origin, that is, the focus of the original parabola is midway between its vertex A and the vertex V of the evolute: moreover, since for every positive value of α, β has two values, equal and opposite, and increasing as α increases, while for every negative value of α, β is imaginary, it follows that the evolute consists of two equal branches of infinite extent, wholly situated to the right of the origin, one above the axis of x, and the other below it, and each receding from that axis more and more as in the figure: the abrupt termination of the two branches touching AX at V is called a *cusp*. That VX is a tangent may be easily proved from the equation of the curve giving

$\frac{d\beta}{d\alpha}=0$, at the point V: thus $\frac{d\beta}{d\alpha}=\frac{6}{27m}\frac{\alpha^2}{\beta}$, or C, representing

a constant factor, $\frac{d\beta}{d\alpha}=C\frac{\alpha^2}{\alpha^{\frac{3}{2}}}=C\alpha^{\frac{1}{2}}=0$, when $\alpha=0$.

The following are added as exercises for the student.

(2) Prove that in the circle the centre itself is the evolute.

(3) Show that the equation of the evolute of an ellipse is

$$(a\alpha)^{\frac{2}{3}}+(b\beta)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}},$$

and that it has the form in the annexed diagram, the points a, b, c, d, being cusps. [See the end of the ANSWERS.]



(4) The involute of the rectangular hyperbola whose equation in reference to the asymptotes is $xy=a^2$, has for equation

$$(\beta+\alpha)^{\frac{2}{3}}-(\beta-\alpha)^{\frac{2}{3}}=(4a)^{\frac{2}{3}}.$$

593. The following theorems show the geometrical connection between the involute and the evolute.

THEOREM 1. Normals to the curve are tangents to its evolute.

Let $y=F(x)$, and $\beta=f(\alpha)$, be the equations of the curve and its evolute. The tangent to the latter at any point (α, β), is

$$y-\beta=\frac{d\beta}{d\alpha}(x-\alpha).....[1],$$

where x, y are the current co-ordinates of the line; also the normal to the former at any point

$$(x', y) \text{ of the curve is } y-y'=-\frac{1}{p'}(x-x').....[2],$$

and this normal passes through the point (α, β), the centre of curvature of (x', y'). As, therefore, the straight lines [1], [2] both pass through the same point (α, β), they will be identical, provided their tangents of inclination be the same, that is, provided

$$\frac{d\beta}{d\alpha}=-\frac{1}{p'}. \quad \text{Now, if for brevity we put } \frac{p'^2+1}{p''}=q, \text{ we have (585),}$$

$$\alpha=x-p'q,\ \beta=y+q,\ \therefore \frac{d\alpha}{dx}=1-p''q-p'\frac{dq}{dx}=-p'^2-p'\frac{dq}{dx}=-p'\left(p'+\frac{dq}{dx}\right),$$

$$\text{also } \frac{d\beta}{dx}=p'+\frac{dq}{dx},\ \therefore \frac{d\beta}{dx}\div\frac{d\alpha}{dx}=\frac{d\beta}{d\alpha}=-\frac{1}{p'},$$

hence the normal and tangent coincide.

THEOREM 2. The difference between any two radii of curvature is equal to the arc of the evolute comprehended between them.

Regarding α as the independent variable, we have by differentiating the equation

$$(y-\beta)^2+(x-\alpha)^2=\rho^2.....[3],$$

$$(y-\beta)\frac{dy}{d\alpha}+(x-\alpha)\frac{dx}{d\alpha}-(y-\beta)\frac{d\beta}{d\alpha}-(x-\alpha)=\rho\frac{d\rho}{d\alpha}.$$

But [2] p. 566, remembering that $p'=\frac{dy}{dx}=\frac{dy}{d\alpha}\div\frac{dx}{d\alpha}$, $(y-\beta)\frac{dy}{d\alpha}+(x-\alpha)\frac{dx}{d\alpha}=0$,

$$\therefore -(y-\beta)\frac{d\beta}{d\alpha}-(x-\alpha)=\rho\frac{d\rho}{d\alpha}...[4], \text{ but [1] above, } y-\beta=\frac{d\beta}{d\alpha}(x-\alpha),$$

$$\therefore [3]\ (x-\alpha)^2\left\{\left(\frac{d\beta}{d\alpha}\right)^2+1\right\}=\rho^2, \text{ and [4], } -(x-\alpha)\left\{\left(\frac{d\beta}{d\alpha}\right)^2+1\right\}=\rho\frac{d\rho}{d\alpha}.$$

Dividing the second of these by the square root of the first, we have

$$-\sqrt{\left\{\left(\frac{d\beta}{d\alpha}\right)^2+1\right\}}=\frac{d\rho}{d\alpha}, \text{ that is (544), } -\frac{ds}{d\alpha}=\frac{d\rho}{d\alpha}, \therefore -s=\rho+C.$$

Also for any other arc s' terminating at a point where the radius is ρ',

$$-s'=\rho'+C, \therefore s-s'=\rho'-\rho,$$

that is, the difference of the two radii is equal to the arc of the evolute comprehended between them. From these theorems it follows that if a string be fastened to one extremity of the arc of the evolute, and be wrapped round it, and then stretched in the direction of the tangent, and continued up to the involute or original curve, this tangential portion of the string CP (fig. at p. 571) will be the radius of curvature of the point P of the involute which it meets; and if the string be now unwound, the point P will trace out an arc of the involute. It is on account of this property that the names involute and evolute are given to the curves thus connected; but it is right to notice that although every plane curve has but a single evolute, it has an infinite number of involutes, since the tangential portion of the thread wound round it may terminate at any point whatever.

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594. **Consecutive Lines and Curves.**—When the equation of a curve contains a parameter (584), that is, a constant, the value of which is unassigned or arbitrary, it is plain that by giving a series of different values to this parameter the equation will furnish a corresponding series of different curves, all, however, belonging to the same *family of curves*. Let $F(x, y, x')=0$ represent any such family of curves, x' being a parameter, and whatever value of x' fixes one of these curves, let any other, intersecting it, arise from changing x' into $x'+h$: this latter curve will, of course, continuously approach the former as h continuously diminishes from $h=h$ down to $h=0$, when the two curves will coalesce, and the points of intersection will settle into fixed positions: the object, at present, is to find these ultimate positions:—the intersections of the *consecutive curves*, as curves brought into coincidence in this manner are called. Now, by Taylor's theorem

$$F(x, y, x'+h)=F(x, y, x')+\frac{dF(x, y, x')}{dx'}h+\frac{d^2F(x, y, x')}{dx'^2}\cdot\frac{h^2}{1.2}+\ldots=0.$$

We equate this to 0, because, by the hypothesis, both

$$F(x, y, x')=0\ldots\ldots[1], \text{ and } F(x, y, x'+h)=0\ldots\ldots[2],$$

and consequently, dividing the development by h, $\frac{dF(x, y, x')}{dx'}=0\ldots\ldots[3]$.

Hence, when h by continuously diminishing becomes 0, and, therefore, the curves consecutive, their intersections, which would be *generally* given by the simultaneous equations [1], [2], are, in these special circumstances, given by [1] and [3]: in other words, the points (x, y), determined by the solution of the two simultaneous equations

$$u=F(x, y, x')=0, \frac{du}{dx'}=\frac{dF(x, y, x')}{dx'}=0\ldots\ldots[4],$$

are the points of intersection of the consecutive curves.

Suppose, for example, it were required to determine the point of intersection of consecutive normals to any plane curve $y'=F(x')$. The normal at any point (x', y') of the curve has for equation

$$y-y'=-\frac{1}{p'}(x-x'), \text{ or } (y-y')p'+x-x'=0.....[5].$$

This corresponds to the first of equations [4], x' being the parameter. Differentiating it with respect to x' of which y' is a function given by the equation of the curve, we have

$$(y-y')p''-p'^2-1=0...[6], \therefore y=y'+\frac{p'^2+1}{p''}, \therefore [5], x=x'-\frac{p'(p'^2+1)}{p''},$$

so that consecutive normals intersect at the centre of curvature as might have been anticipated. And we see that the position of this centre is determined simply by combining the equation [5], of the normal, with its differential, those of the proposed curve being regarded as the only variables. This suggests a ready extension of the present theory: from a curve consecutive to a fixed one we may pass to the curve consecutive to this latter, then to a third consecutive curve, and so on: if, for instance, three such consecutive curves have the same intersections, these will be determined by the simultaneous equations

$$F(x, y, x')=0, \frac{dF(x, y, x')}{dx'}=0, \frac{d^2F(x, y, x')}{dx'^2}=0.....[7],$$

from which the two quantities x, y, may be eliminated, and thus an equation of condition determining x' obtained. Take the case of consecutive normals considered above: differentiating [6] with respect to x', we have then the three simultaneous equations

$$(y-y')p'+x-x'=0, (y-y')p''-p'^2-1=0, (y-y')p'''-3p''p'=0.$$

And eliminating x, y, from these, we get

$$(p'^2+1)p'''-3p'p''^2=0, \text{ as at page 567,}$$

the condition which determines the points (x', y') of the curve $y'=F(x')$ of greatest and least curvature. A geometrical significance is given to these analytical indications as follows: conceive a normal drawn from an assigned point in a curve: we may so select a point on this normal that a circle described from it as centre shall pass through the assigned point, and shall also cut the curve *once* again as near to it as we please, but not in general more than once as near as we please, seeing that the centre must not be taken out of a fixed straight line: when the centre by continuously moving towards, at length reaches the centre of curvature, the two points of the curve coalesce, and the circle becomes that of ordinary curvature:—the centre of it is the point of intersection of the fixed normals, with its consecutive normal.

Again: we may conceive two neighbouring normals to be such as to intersect on an intermediate third normal, and as they approach, to preserve their intersection on this normal up to coincidence with it: these neighbouring normals must have the peculiarity that their three equations have place simultaneously for the point of intersection common to all: but, when this point takes up the position due to the three normals, when they all coalesce, or become consecutive, their general equations become replaced by the simultaneous conditions [7], which fixes the point (x', y') of

the curve; which point is such that the circle described as above passes through it and *two* neighbouring points of the curve, all three points coalescing when the centre is that of curvature.

595. **Enveloping Curves.**—The equations [4] after the differentiation implied in the second is performed, determine the point (or points) (x, y) in which the curve represented by the first equation is intersected by its consecutive curve, the parameter x' being any assigned constant whatever: hence, if from those equations we were to eliminate this arbitrary constant, the resulting equation must equally apply to every point of intersection between each curve of the family $F(x, y, x')=0$, and its consecutive curve. This resulting equation, therefore, must represent the locus of all the intersections, which locus will always be a curve touching every individual curve of the family at the points of their consecutive intersections. The evolute is a particular instance: this curve is the locus of the intersections of consecutive normals to the involute, and we have seen that this locus touches all the individual normals at those intersections: but to prove the proposition generally, take any one of the family of curves [4], x' being constant for that one, and differentiate its equation: we thus have

$$\left\{\frac{du}{dx}\right\}=\frac{du}{dx}+\frac{du}{dy}\frac{dy}{dx}=0\ldots\ldots[1].$$

Now the equation thus differentiated becomes that of the locus, when the value of x', as given by the second condition [4], and which is $\therefore$ a function of x and y, is substituted in it. Differentiating then this equation of the locus, x' being a function of x, we have

$$\left\{\frac{du}{dx}\right\}=\frac{du}{dx}+\frac{du}{dy}\frac{dy}{dx}+\frac{du}{dx'}\frac{dx'}{dx}=0\ldots\ldots[2],$$

but by the condition just referred to, $\frac{du}{dx'}=0$, at each point of consecutive intersection, that is, at every point of the locus: hence $\frac{dy}{dx}$ in [1], as deduced from the equation of the fixed curve for the point (x, y), is the same as $\frac{dy}{dx}$ in [2], as deduced from the equation of the locus, for the same point (x, y): consequently both curves touch at that point.

The following example will illustrate this theory of *envelopes*.

Ex. Between the sides of a given angle θ are drawn an infinite number of straight lines, each cutting off a triangle of constant surface s: required the curve to which all these lines are tangents, that is, their envelope.

The equation of any one of the lines indifferently is $y=\alpha x+\beta$, the sides of the given angle being taken for axes: the intercepts or portions cut off by the line are β, and $-\frac{\beta}{\alpha}$, so that the constant area is

$$s=-\frac{\beta^2}{2\alpha}\sin\theta,\ \therefore\ \alpha=-\frac{\beta^2}{2s}\sin\theta,\ \therefore\ y=-\frac{\beta^2}{2s}x\sin\theta+\beta\ldots\ldots[1].$$

This then is the general equation comprehending all the lines, β being

a variable parameter, and occupying the place of x' in the above theory. Differentiating [1] with respect to β, we have agreeably to [4], p. 573,

$$-\frac{\beta}{s}x\sin\theta+1=0,\ \therefore \beta=\frac{s}{x\sin\theta},\ \therefore [1],\ y=-\frac{s}{2x\sin\theta}+\frac{s}{x\sin\theta},\ \therefore xy=\frac{s}{2\sin\theta},$$

the variable parameter being eliminated. The envelope is, therefore, an hyperbola, having the sides of the given angle for asymptotes.

Examples for Exercise.

(1) Prove that the envelope of all the straight lines comprehended in the equation $y=\alpha x+\frac{m}{\alpha}$, where m is fixed, and α a variable parameter, is a parabola.

(2) Prove that the envelope of all the straight lines indicated by the equation $y=\alpha x+(a^2\alpha+b^2)^{\frac{1}{2}}$, where α is a parameter, is an ellipse.

(3) Show that the envelope of all parabolas, comprehended in the equation

$$y=\alpha x-\frac{1+\alpha^2}{2\rho}x^2,$$ is another parabola.

(4) A straight line of given length c is placed in all possible positions with its two extremities on two rectangular axes: show that the equation of the curve to which the varying line is always a tangent is $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$.

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596. **Singular Points of Curves.**—Any point at which a curve presents such peculiarities as distinguish it from the neighbouring points, without regard to the position of the axes of reference, is called a *singular point:* for instance, points of inflexion, and cusps (pp. 567, 571), are singular points.

Multiple Points.—If two or more branches of a curve unite in a single point, either by crossing each other there, or by simply touching, the point is called a *multiple point*, its degree of multiplicity being indicated by the number of branches issuing from it, or passing through it.

The analytical characteristic of such a point is evidently $\frac{dy}{dx}=\frac{0}{0}$ for the co-ordinates of the point; and the multiple values of this are readily found, without the calculus, by the rule given at (564), the examples to which might all have been introduced under the present head: we shall here add one or two others.

(1) Required the nature of the point at the origin of the curve $y^2(a^2-x^2)=x^4$.

By the rule referred to

$$a^2\left(\frac{y}{x}\right)^2-y^2-x^2=0,\ \therefore \left(\frac{y}{x}\right)=\frac{dy}{dx}=\pm 0.$$

Hence two branches of the curve pass through the origin, touching the axis of x on opposite sides of it.

(2) Required the nature of the point at the origin of the curve $y^2=ax^3$.

$$\text{By the rule, } \left(\frac{y}{x}\right)^2=ax,\ \therefore \left(\frac{y}{x}\right)=\frac{dy}{dx}=\pm 0.$$

Hence two branches, to each of which the axis of x is a tangent, spring from the origin, one on each side of that axis: they both proceed towards the *right*, for if x vanish negatively,

the result is $\left(\frac{y}{x}\right)=\pm 0\sqrt{-1}$, so that the two branches form a *cusp*:

the curve is the semi-cubical parabola figured at p. 571, the origin being at V. The equation of the curve itself shows that no part of it exists for negative values of x.

(3) Required the nature of the point $(a, 0)$ of the curve $y^2=x(x-a)^2$.

Removing the origin to the point, we must replace x by $x+a$, so that then the equation will be

$$y^2=(x+a)x^2, \text{ or } y^2=x^3+ax^2, \therefore \left(\frac{y}{x}\right)^2=x+a, \therefore \frac{dy}{dx}=\pm\sqrt{a}.$$

Hence two branches of the curve cross the axis of x at the point, the inclinations to it on opposite sides being equal.

Conjugate Points.—It sometimes happens that the equation of a curve is such as to be satisfied not only for the co-ordinates of every point in the curve, but also for the co-ordinates of an additional isolated point, entirely detached from the curve, and thus having no place in the continuous series of points belonging to the curve: such a point is called a *conjugate point*. For instance, the curve figured in the margin has for its equation $a(y-b)^2=x^3-cx^2$, the origin of the rectangular axes being at O. This equation is satisfied for $x=0$, since the corresponding value of y is the real value $y=b$: these co-ordinates, therefore, mark out a point P on the axis of y: but between the values $x=0$, and $x=c(=OC)$, there are evidently no real values for y; nor are there any corresponding to negative values of x: P is therefore a point entirely isolated from all real points in the plane of the curve:—it is a *conjugate* point.



The differential calculus, to be applicable to any particular point, always requires that that point be one of a continuous series of points: the point in question may terminate that series, or mark the limit or boundary of a continuous curve, but in the absence of all continuous connection with neighbouring points, the calculus cannot deal with it. In examining whether a singular point is a conjugate point or not, the legitimate way of proceeding is this:—

Observe whether the values x, y, when they reach the singular case, differ either of them from a real value, only by the imaginary zero $0\sqrt{-n}$, in whichever direction that zero be arrived at: if so, the real values, the imaginary symbol being suppressed, will be the co-ordinates of a conjugate point. But if the zero is real when reached in one direction, and imaginary when reached in the opposite direction, the point is a limit or a cusp. For example

(1) Required the nature of the point $(-a, 0)$ of the curve $y^2=(x+a)^2x$.

Here, for $x=-a$, we have $y=0\sqrt{-a}$, whether $x=-a$ be reached by decreasing or increasing values of x, showing that on each side of the point the ordinate is imaginary: the point is therefore a conjugate point.

(2) Required the nature of the point, whose abscissa is a, of the curve

$$(c^2y-x^3)^2=(x-b)^5(x-a), \; a<b, \text{ or } b-a=k.$$

Solving this for y, we have

$$c^2y=(x-b)^{\frac{5}{2}}(x-a)^3+x^3, \text{ which for } x=a, \text{ gives } y=\frac{a^3}{c^2}+0\sqrt{-1}.$$

and since the zero remains imaginary, whether $x=a$ be arrived at from a succeeding or a preceding value of x, we conclude that the co-ordinates

$$x=a,\ y=\frac{a^3}{c^2}, \text{ belong to a conjugate point.}$$

In each of these examples y passes through a continuous series of imaginary values, between the extremes of which there is one value in which the symbol of impossibility is of the form $0\sqrt{-n}$, which, being in value *nothing*, may be suppressed: the real value of y which then remains equally (in conjunction with the corresponding value of x) satisfies the equation of the curve, and therefore indicates a real isolated point.

Conceiving imaginary values of y thus to generate an imaginary curve —as the imaginary ordinates of an ellipse belong to the imaginary hyperbola having the same axes—we see that the real conjugate point is passed through by the imaginary curve:—the ordinate of the imaginary point being $y=m+0\sqrt{-n}$, and that of the real point $y=m$, the latter being *inferred* from the former, inasmuch as $0\sqrt{-n}$ may be suppressed without loss to the value of y.

What are here called imaginary curves in the plane of the co-ordinates, by a peculiar geometrical interpretation of the imaginary symbol $\sqrt{-1}$, may be replaced by real curves out of that plane. On this subject the student may consult Vol. ii. of Peacock's Algebra. What by the French analysts are called *courbes pointillées*, or punctuated curves—curves traced by isolated points or dots—may be inferred to be an assemblage of real points in the co-ordinate plane, each (or rather an imaginary point superimposed upon it) belonging to an imaginary curve in that plane, and for which the ordinate is $y=m+0\sqrt{-n}$, or belonging to a real curve out of that plane having this ordinate. To *continuous* values, whether real or imaginary, the calculus is, of course, always applicable.

[NOTE.—Many errors have crept into analysis from neglecting to take account of the *signs* with which variable quantities vanish in the extreme cases of a formula. Some of these were pointed out by the author in a Paper read before the British Association at the Cambridge Meeting of 1845.

It may be instructive to mention an instance here. No less a man than Abel has stated, in reference to the series,

$$\frac{1}{2}\theta=\sin\theta-\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta-\ldots\ldots\text{(p. 218)},$$

that "lorsque $\theta=\pi$, ou $-\pi$, la série se reduit à zéro, comme on voit aisément." Now up to these values of θ, the series is complete, without the correction introduced at p. 220; and when θ becomes π, $\sin\theta$ vanishes *positively*, $\sin 2\theta$ *negatively*, $\sin 3\theta$ *positively*, and so on: and when θ becomes -2π, the signs are the opposites of these, so that in the extreme cases we have

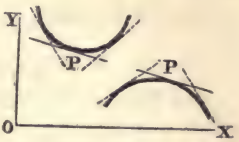
$$\frac{1}{2}\pi=\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\right)\times 0, \text{ and } -\frac{1}{2}\pi=\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\right)\times -0,$$

from which we infer that $\frac{1}{2}\pi \div 0 = \infty = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots$,

which accords with what we otherwise know to be true. See art. 207.]

Cusps are also singular points. They may be detected as explained in ex. 1, p. 571, and in ex. 3, p. 572. Points of inflexion have already been briefly noticed at p. 567, but their analytical indications offer themselves more readily from examining those by which convexity is distinguished from concavity: we shall, therefore, investigate the latter indications here.

Convexity and Concavity.—In order to ascertain whether at any proposed point P a curve is convex or concave towards the axis of x, let τ be the angle which a linear tangent at the point makes with that axis, then

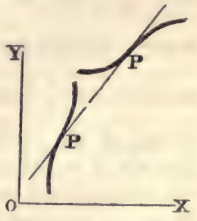


$$\tan\tau = \frac{dy}{dx}, \quad \therefore d\tan\tau = \sec^2\tau \frac{d\tau}{dx} = \frac{d^2y}{dx^2},$$

so that $\frac{d^2y}{dx^2}$ must necessarily have the same sign as $\frac{d\tau}{dx}$.

Now if $\frac{d\tau}{dx}$ be *positive*, we shall know that the angle τ increases as x increases, and diminishes as x diminishes, which can happen only when the curve proceeds from P (y being positive) upwards from the linear tangent—in both directions, as in the first diagram: hence $\frac{d^2y}{dx^2}$ *positive* indicates *convexity*.

But if $\frac{d\tau}{dx}$ be *negative*, then we shall be apprized that τ diminishes as x increases, and vice versâ, and, therefore, that the curve proceeds from P (y being positive) downwards from the linear tangent, in both directions, as in the second diagram: hence $\frac{d^2y}{dx^2}$ *negative* indicates *concavity*.



Points of Inflexion.—But if, in approaching P from a neighbouring point, and upon passing through it to a point on the other side of it, $\frac{d^2y}{dx^2}$ should change from + to −, or from − to +, the inference will, of course, be that the curve is convex on one side of the point, and concave on the other, so that P must then be a point of inflexion. As, however, a continuous quantity cannot pass from + to −, or from − to +, without becoming 0 or ∞ at the point of passage, it follows that a point of inflexion is indicated by $\frac{d^2y}{dx^2} = 0$, or $= \infty$, and by this coefficient changing sign at the point or points determined by these conditions. If the change be from + to −, as x increases, the curve is convex up to the point, and then becomes concave, as in the first diagram; and if the change be from − to +, the curve passes from concave to convex, as in the second diagram.

NOTE.—In the preceding examination, the point P has been considered

as *above* the axis of x, or to have a positive ordinate; but if the point be on the opposite side of the axis, y being negative, convexity and concavity will be respectively indicated by signs the opposite to those above, so that generally a curve is convex or concave to the axis of x, according as y and $\frac{d^2y}{dx^2}$ have like or unlike signs.

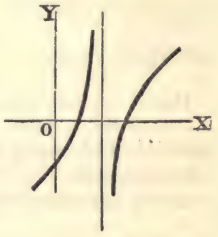
In the application of these tests it is necessary to ascertain that the point which appears to satisfy them is not at an infinite distance: if either the x or the y of it be infinite, then, of course, the tests go for nothing. The first of the following examples will show the necessity of attending to this.

Ex. (1). Required the peculiarities of the curve $(x-2)y=(x-1)(x-3)$,

$$y=\frac{(x-1)(x-3)}{(x-2)},\ \therefore\ \frac{dy}{dx}=\frac{x^2-4x+5}{(x-2)^2},\ \therefore\ \frac{d^2y}{dx^2}=-\frac{2}{(x-2)^3}.$$

Now from $x=0$ up to $x=2$, this coefficient is positive, but beyond this value of x it is negative: hence, the point whose abscissa is $x=2$, would be a point of inflexion, were it not that for $x=2$, $y=\infty$: we infer, therefore, that a parallel to the axis of y, at the distance $x=2$, is an asymptote. At all points within the limits $x=1$, $x=2$, y is positive, and the sign of the second differential coefficient, within this extent, being also positive, we infer that throughout this range the curve is convex to the axis of x.

Again: within the limits $x=2$, $x=3$, y is negative, and so likewise is the sign of the second diff. coef.: throughout this range, therefore, the curve is also convex to the axis of x: but beyond $x=3$, where it crosses the axis, it is concave, since y and $\frac{d^2y}{dx^2}$ have unlike signs. Lastly, for values of x less than 1, y is negative and $\frac{d^2y}{dx^2}$ positive, as also for all negative values of x: hence from $x=1$, where it crosses the axis of x, the curve proceeds to the left below the axis, without termination, and is throughout concave to it. The curve whose peculiarities we have been discussing is, in fact, an hyperbola, the proposed equation being $xy-x^2-2y+4x-3=0$ (see p. 354).



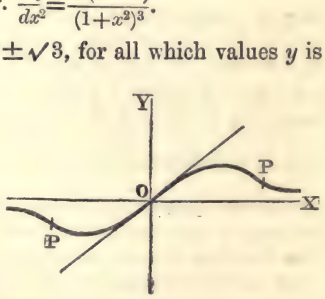
(2) Required the peculiarities of the curve $x=(1+x^2)y$.

$$y=\frac{x}{1+x^2},\ \therefore\ \frac{dy}{dx}=\frac{1-x^2}{(1+x^2)^2},\ \therefore\ \frac{d^2y}{dx^2}=\frac{2x(x^2-3)}{(1+x^2)^3}.$$

This becomes 0 for $x=0$, and for $x=\pm\sqrt{3}$, for all which values y is finite. For values of x immediately preceding $x=0$, that is, for small negative values of x, the second diff. coef. is positive: hence it is positive throughout the interval $x=0$, $x=-\sqrt{3}$, this latter being the only negative root of

$$2x^3-6x=0:$$

for the same interval y is negative, $\therefore$ the curve to the left of the origin, and up to $x=-\sqrt{3}$, is concave to the axis



of x. In like manner is it shown to be concave to the axis from $x=0$ to $x=+\sqrt{3}$, throughout which interval the second diff. coef. is negative, and y positive: hence the origin is a point of inflexion. Moreover, beyond $x=-\sqrt{3}$, in the negative direction, and $x=+\sqrt{3}$, in the positive direction, the curvature again changes, there being inflexions at these points (P, P), beyond which the curve proceeds indefinitely right and left, and is convex to the axis. Since y is $=0$, not only for $x=0$, but also for $x=\infty$, it follows that the curve, after crossing the axis at the origin, and becoming concave towards it in each direction, as far as the two other points of inflexion, as in the figure, again bends, and continually approaches the axis, which is thus an asymptote. The tangent to the curve at the origin has for inclination $\frac{dy}{dx}=1$; it is, therefore, inclined to the axis of x at an angle of 45°. The direction of this tangent is equally determined, without differentiation, by the rule at (564), from the equation of the curve

$$y+x^2y=x, \text{ which gives } \left(\frac{y}{x}\right)+xy=1, \text{ and this, for } x=0,\ y=0, \text{ is } 1.$$

These two examples must suffice as specimens of the way in which curves are traced from their equations, and their geometrical peculiarities discovered. For further examples on this and the other parts of the differential calculus, the student may refer to the collection of "Examples in the Differential Calculus," by Mr. Haddon, in Weale's Series.

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597. **Integration.**—In the articles already given under this head the integrals determined are, with one or two exceptions, what are called the fundamental integrals:—the elementary forms into which, when practicable, other integrals are to be decomposed, or reduced. The student will be prepared to expect that this reduction is to be effected, not by help of any extension of the principles of integration, but solely by our availing ourselves of the resources of common algebra: a chapter devoted to the integration of differential expressions, not included in the forms just adverted to, is in fact only a chapter on algebraic expedients, and algebraic transformations, in reference to a specific purpose. When an object is to be accomplished solely by these means, the successful mode of procedure, in every individual case, cannot be taught by precepts:—much must be left to the skill and penetration of the algebraist. There are, however, a few general methods of operation which, to certain classes of differentials, may always be applied with success: these are

1. Integration by decomposing Rational Fractions; 2, by Rationalization; 3, by Formulæ of Reduction; 4, by *Parts*.

These four methods we shall here briefly explain.

598. I. **Integration of Rational Fractions.**—An algebraic fraction is rational when numerator and denominator involve only positive integral powers of the variable. Every such fraction may be decomposed into others X_1, X_2, &c., such that X_1dx, X_2dx, &c., may always be integrated by the rules for the fundamental forms.

Whenever the highest exponent of x in the numerator is greater than the highest exponent of x in the denominator, actual division of the former by the latter will convert the fraction into an equivalent expression, of which a *part* only is fractional, the part prefixed being integral. The

fractional part will have the highest exponent of x in the numerator less than the highest in the denominator, because rational integral terms continue to occur in the quotient till this is brought about. Every rational fraction is thus reducible to one in these latter circumstances, united or not to non-fractional terms involving only positive integer powers of the variable. Terms of this kind, connected with dx, being always integrable by the rule for powers, we have only to inquire how the fractional part of the differential is to be integrated: the mode of decomposition explained at pp. 156-7, suggests the course to be adopted, thus:

By (3) p. 156, $\int \frac{2x+3}{x^2+x-2}\,dx=\frac{5}{3}\int\frac{dx}{x-1}+\frac{1}{3}\int\frac{dx}{x+2}=\frac{5}{3}\log(x-1)+\frac{1}{3}\log(x+2).$

By (4) ,, $\int \frac{3x+2}{x(x+1)^3}\,dx=2\int\frac{dx}{x}-2\int\frac{dx}{x+1}-2\int\frac{dx}{(x+1)^2}+\int\frac{dx}{(x+1)^3}=$

$$2\log x-2\log(x+1)+\frac{2}{x+1}-\frac{1}{2(x+1)^2}.$$

These two examples show the advantage of the decomposition taught in the ALGEBRA, but there is another, and, in most cases, an easier way of proceeding.

1. *When the roots of* $F(x)=0$ *are all unequal.*—In this case the components of the rational fraction $\frac{f(x)}{F(x)}$ may be found as follows. Let the roots be

$$a_1,\ a_2,\ \&c.,\ \text{and put}\ \frac{f(x)}{F(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x}+\frac{A_3}{x-a_3}+\ldots+\frac{A_n}{x-a_n}\ldots\ldots[1],$$

$$\therefore f(x)=A_1\frac{F(x)}{x-a_1}+A_2\frac{F(x)}{x-a_2}+A_3\frac{F(x)}{x-a_3}+\ldots+A_n\frac{F(x)}{x-a_n}.$$

Now since for $x=a_1$ each numerator $F(x)$ vanishes, it follows that for this value of x the above expression is reduced to the vanishing fraction which its first term then becomes, so that (561) we must have $f(a_1)=A_1F_1(a_1)$,

$$\therefore A_1=\frac{f(a_1)}{F_1(a_1)},\ \text{and similarly,}\ A_2=\frac{f(a_2)}{F_1(a_2)},\ A_3=\frac{f(a_3)}{F_1(a_3)},\ \&c.$$

Hence the roots of the equation $F(x)=0$ having been found, if these are all real and unequal, the several numerators of the partial fractions [1] into which $\frac{f(x)}{F(x)}$ is to be decomposed, may be got thus:—Divide $f(x)$ by the first diff. coef. furnished by $F(x)$, and in the quotient put for x the roots a_1, a_2, &c., successively.

For ex. Let $\int\frac{x^2-3}{x^3-7x+6}\,dx$ be proposed. Equating the den. to zero, the roots are found to be $a_1=1$, $a_2=2$, and $a_3=-3$, $\therefore$ for the numerators of the three partial fractions we have what $\frac{f(x)}{F(x)}=\frac{x^2-3}{3x^2-7}$ becomes when 1, 2, and -3 are successively put for x: that is, we get $A_1=\frac{1}{2}$, $A_2=\frac{1}{5}$, $A_3=\frac{3}{10}$, and consequently

$$\frac{x^2-3}{x^3-7x+6}=\frac{1}{2}\left(\frac{1}{x-1}\right)+\frac{1}{5}\left(\frac{1}{x-2}\right)+\frac{3}{10}\left(\frac{1}{x+3}\right),$$

$$\therefore \int \frac{x^2-3}{x^3-7x+6}\,dx=\frac{1}{2}\log(x-1)+\frac{1}{5}\log(x-2)+\frac{3}{10}\log(x+3).$$

Again: let the rational fraction be $\frac{8x^2-6x+8}{x^3-x^2+4x-4}$. Here, equating the den. to zero, the roots are $a_1=1$, $a_2=2\sqrt{-1}$, $a_3=-2\sqrt{-1}$, also

$$\frac{f(x)}{F_1(x)}=\frac{8x^2-6x+8}{3x^2-2x+4},\ \therefore A_1=2,\ A_2=3,\ A_3=3,\ \therefore \text{ putting } i \text{ for } \sqrt{-1},$$

$$\int \frac{8x^2-6x+8}{x^3-x^2+4x-4}\,dx=\int\left(\frac{2}{x-1}+\frac{3}{x-2i}+\frac{3}{x+2i}\right)dx=$$

$$2\log(x-1)+3\log(x-2i)+3\log(x+2i)=2\log(x-1)+3\log(x^2+4).$$

Lastly: let the rational fraction be $\frac{1}{x^3-x^2+2x-2}$. Equating the den. to zero, the roots are $a_1=1$, $a_2=\sqrt{-2}$, $a_3=-\sqrt{-2}$, $\therefore$ putting $i\sqrt{2}$ for $\sqrt{-2}$,

$$\frac{f(x)}{F_1(x)}=\frac{1}{3x^2-2x+2},\ \therefore A_1=\frac{1}{3},\ A_2=-\frac{1}{12}(2-i\sqrt{2}),\ A_3=-\frac{1}{12}(2+i\sqrt{2}),$$

$$\therefore \int \frac{dx}{x^3-x^2+2x-2}=\frac{1}{3}\log(x-1)-\frac{1}{12}\int\left\{\frac{2-i\sqrt{2}}{x-i\sqrt{2}}+\frac{2+i\sqrt{2}}{x+i\sqrt{2}}\right\}dx\ldots\ldots[1].$$

The latter integral, like that in the former example, involves the logs of two imaginary quantities, but by reducing the two fractions within the brackets to a com. den., and adding them, their sum is

$$\frac{4x-4i^2}{x^2-2i^2}=4\,\frac{x+1}{x^2+2};\ \text{the integral is}.\therefore \int\frac{x+1}{x^2+2}\,dx=\int\frac{x\,dx}{x^2+2}+\int\frac{dx}{x^2+2}$$

$$=\frac{1}{2}\log(x^2+2)+\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}},$$

$$\therefore [1],\ \int\frac{dx}{x^3-x^2+2x-2}=\frac{1}{3}\log(x-1)-\frac{1}{6}\log(x^2+2)-\frac{1}{3\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}.$$

It is obvious that, as in these examples, whenever the roots of $F(x)=0$ are all unequal, the integral will consist of only common algebraic quantities and logarithms; and that when two of the roots are imaginary, two of the logs will be logs of imaginary quantities. But no imaginary quantity can *of necessity* enter the integral, seeing that the differential is rational: hence the sum of the two logs adverted to will always give either the log of a real quantity, or else a circular function: and this latter will always be characterized by $\tan^{-1}$, because this is the only function of the class, the differential of which does not involve an irrational denominator. Of course we include $\cot^{-1}$ in $\tan^{-1}$, since

$$d\cot^{-1}X=-d\tan^{-1}X.$$

2. *When the roots of* $F(x)=0$ *are not all unequal.*—Let $F(x)=(x-a_1)^m\varphi(x)$, where $\varphi(x)$ is a polynomial not containing $x-a_1$ as a factor.

$$\text{Assume } \frac{f(x)}{F(x)}=\frac{A_m}{(x-a_1)^m}+\frac{A_{m-1}}{(x-a_1)^{m-1}}+\ldots+\frac{A_1}{x-a_1}+\frac{Q}{\varphi(x)}\ldots\ldots[1],$$

$$\therefore f(x)=A_m\varphi(x)+A_{m-1}(x-a_1)\varphi(x)+A_{m-2}(x-a_1)^2\varphi(x)+\ldots+Q(x-a_1)^m\ldots[2].$$

For $x=a_1$, $f(a_1)=A_m\varphi(a_1)$, $\therefore A_m=\frac{f(a_1)}{\varphi(a_1)}$: and differentiating [2], we have

$$f_1(a_1)=A_m\varphi_1(a_1)+A_{m-1}\varphi(a_1), \text{ which determines } A_{m-1}.$$

And in like manner, by repeated differentiations of [2], may A_{m-2}, A_{m-3}, &c., be found, and the last fraction in [1] may be decomposed, if $\varphi(x)$ have no equal factors, by the last case: if it have equal factors, by a repetition of the present method. For example,

(1) Let the rational fraction be

$$\frac{f(x)}{F(x)}=\frac{4x^3+6x^2-8x+4}{x^2(x^2+x-2)}=\frac{A_2}{x^2}+\frac{A_1}{x}+\frac{Q}{x^2+x-2}.$$

Here $A_2=\frac{f(0)}{\varphi(0)}=-2$, also $f_1(0)=-8=A_2(1)+A_1(-2)=-2-2A_1$, $\therefore A_1=3$.

$$\text{Again : by the last case, } \frac{f(x)}{F(x)}=\frac{Q}{x^2+x-2}+\&c.=\frac{A'_1}{x-1}+\frac{A'_2}{x+2}+\&c.,$$

$$\therefore A'_1=\frac{f(1)}{F_1(1)}=\frac{6}{3}=2,\ A'_2=\frac{f(-2)}{F_1(-2)}=\frac{12}{-12}=-1,$$

$$\therefore \int\frac{4x^3+6x^2-8x+4}{x^4+x^3-2x^2}\,dx=-\frac{2}{x}+3\log x+2\log(x-1)-\log(x+2).$$

(2) Let the fraction be

$$\frac{f(x)}{F(x)}=\frac{x^3}{(x-1)^2(x^2+1)}=\frac{A_2}{(x-1)^2}+\frac{A_1}{x-1}+\frac{Q}{x^2+1} \text{ : then}$$

$$A_2=\frac{f(1)}{\varphi(1)}=\frac{1}{2}, \text{ also } f_1(1)=3=A_2(2)+A_1(2)=1+2A_1, \therefore A_1=1.$$

$$\text{Again : by the last case, } \frac{f(x)}{F(x)}=\frac{Q}{x^2+1}+\&c.=\frac{A'_1}{x-i}+\frac{A'_2}{x+i}+\&c.,$$

$$\therefore A'_1=\frac{f(i)}{F_1(i)}=\frac{-i}{2(i-1)^2i}=\frac{1}{4i},\ A'_2=\frac{f(-i)}{F_1(-i)}=\frac{1}{-4i}, \therefore Q=\frac{1}{2}.$$

We here get Q to avoid logs of imaginary quantities: Q is evidently $A'_1(x+i)+A'_2(x-i)$. Or Q may be deduced from the original equation, after A_2 and A_1 are found.

$$\text{We thus have } \frac{f(x)}{F(x)}=\frac{1}{2(x-1)^2}+\frac{1}{x-1}+\frac{1}{2(x^2+1)}, \text{ and therefore}$$

$$\int\frac{x^3dx}{(x-1)^2(x^2+1)}=-\frac{1}{2(x-1)}+\log(x-1)+\frac{1}{2}\tan^{-1}x.$$

As an example of the ordinary way of proceeding, by indeterminate coefficients, to avoid logs of imaginary quantities, we shall add the following.

$$\text{To integrate } \frac{x}{x^3-1}dx, \text{ put } \frac{x}{(x-1)(x^2+x+1)}=\frac{A_1}{x-1}+\frac{A_2x+A_3}{x^2+x+1}.$$

Reducing to a com. den., and equating like powers of x, in numerators,

$$A_1+A_2=0,\ A_1+A_2-A_3=1,\ A_1-A_3=0,\ \therefore A_1=\frac{1}{3},\ A_2=-\frac{1}{3},\ A_3=-\frac{1}{3},$$

$$\therefore \int\frac{x}{x^3-1}dx=\frac{1}{3}\int\frac{dx}{x-1}-\frac{1}{3}\int\frac{x-1}{x^2+x+1}dx. \quad \text{The latter is } \frac{1}{3}\int\frac{x-1}{(x+\frac{1}{2})^2+\frac{3}{4}}dx,$$

which, putting z for $x+\frac{1}{2}$, is $\frac{1}{3}\int\frac{zdz}{z^2+\frac{3}{4}}-\frac{1}{2}\int\frac{dz}{z^2+\frac{3}{4}}=$

$$\frac{1}{6}\log\left(z^2+\frac{3}{4}\right)-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}z.$$

Restoring, therefore, the value of z, we have

$$\int\frac{x}{x^3-1}dx=\frac{1}{3}\left\{\log(x-1)-\frac{1}{2}\log(x^2+x+1)+\sqrt{3}\,.\tan^{-1}\frac{2x+1}{\sqrt{3}}\right\}.$$

The following are left for the student to integrate.*

$$\frac{5x+1}{x^2+x-2}dx,\ \frac{5x^3+1}{x^2-3x+2}dx,\ \frac{2a}{a^2-x^2}dx,\ \frac{3x-5}{x^2-6x+8}dx,\ \frac{2ax}{(x+a)^2}dx.$$

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599. II. **Integration of Irrational Functions.**—Many functions involving radicals may be rationalized by certain obvious substitutions, as in the following example.

Irrational Monomials.—Let it be required to integrate

$\frac{x^{\frac{1}{2}}-a}{x^{\frac{1}{3}}-x^{\frac{1}{2}}}dx$. Then, since the common den., given by the reduction of the fractional exponents, is 6, put z^6 for x, by which substitution the form is rationalized; for we have

$$\frac{x^{\frac{1}{2}}-a}{x^{\frac{1}{3}}-x^{\frac{1}{2}}}dx=\frac{z^3-a}{z^2-z^3}6z^5dz=\frac{z^3-a}{1-z}6z^3dz,$$

which, by actual division, gives a mixed quantity, the fractional part of which, as well as the integral part, is rational: the integration may therefore readily be performed, and then the value of z restored. By the method here exemplified it is obvious that, whatever be the fractional exponents (p, q, &c.), every expression may be rendered rational, if it be included in the general form

$$\frac{ax^p+bx^q+\&c.}{a'x^{p'}+b'x^{q'}+\&c.}dx,$$

where the irrational quantities are all *monomial*. When they are *binomial*, the general methods of proceeding are as follows:—

Irrational Binomials.—Integration of the form $x^m(a+bx^n)^p dx$.

Cases coming under this general form—where the exponents are whole or fractional, positive or negative—are of very frequent occurrence. The

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* As this work has already exceeded the prescribed limits, space cannot be afforded for many examples for exercise in this concluding portion of the volume: this is of less consequence to the student, as the shilling volume of "Examples on the Integral Calculus," in Weale's series, furnishes an abundance suitable for the purpose.

form may always be rendered rational whenever the exponents satisfy one or other of two conditions called *criteria of integrability:* these are as follow:—

First Criterion of Integrability.—The condition for this case is, that whatever be the individual exponents m, n, p, we must always have

$$(m+1)\div n = \text{a } \textit{positive integer} = N \ldots [1].$$

Substitute z for $a+bx^n$, then for x we have

$$x=\left(\frac{z-a}{b}\right)^{\frac{1}{n}},\ \therefore\ x^m=\left(\frac{z-a}{b}\right)^{\frac{m}{n}},\ \therefore\ x^{m-1}dx=\frac{1}{nb^{\frac{m}{n}}}(z-a)^{\frac{m}{n}-1}dz.\quad \text{Mult. by } x,$$

$$\therefore [1],\ x^m dx=\frac{1}{nb^N}(z-a)^{N-1}dz,\ \therefore\ x^m(a+bx^n)^p dx=\frac{1}{nb^N}(z-a)^{N-1}z^p dz.$$

The original is thus converted into a form which may always be integrated: this is plain, for $N-1$ is either a positive integer or 0, and in the former case the binomial may be developed into a finite series of monomials.

NOTE.—Should $N-1$ be a *negative* integer, and p also an integer, the coefficient of dz would evidently be a rational fraction.

And to a rational fraction the expression may always be reduced in this case, though p be a fraction: for let $\frac{q}{t}$ be the fraction p; then putting y^t for z, we have $ty^{t-1}dy$ for dz, so that, writing $-k$ for the negative exponent $N-1$, we shall have to integrate

$$(y^t-a)^{-k}y^{qt}ty^{t-1}dy=t\frac{y^q}{(y^t-a)^k y^{t-1}}dy,$$

which, the exponents being whole numbers, is a rational fraction, whether q be positive or negative.

Second Criterion of Integrability.—The condition for this case is that

$$\frac{m+1}{n}+p=\text{a } \textit{negative integer}=-N \ldots [2].$$

Multiply the first factor by x^{np}, and divide the second by the same,

$$\therefore\ x^m(a+bx^n)^p dx=x^{m+np}(ax^{-n}+b)^p dx.$$

This is an expression still of the same form, so that if we substitute z for $ax^{-n}+b$, the transformed expression can differ from that arrived at in the former case, only in this, namely, that a, b, will be interchanged, that $-n$ will appear instead of n, and that $m+np$ will replace m. Hence the transformed expression here will be

$$x^m(a+bx^n)^p dx=-\frac{1}{na^N}(z-a)^{N-1}z^p dz.$$

Ex. 1. Required the integral of $x^3(a+bx^2)^{\frac{1}{2}}dx$.

$$\text{Here } m=3,\ n=2,\ p=\frac{1}{2},\ \text{and } (m+1)\div n=2=N.$$

Hence the first criterion is satisfied, and we shall now work the example by the process, and not by mere substitution in the formula.

Put $a+bx^2=z$, $\therefore x=\left(\frac{z-a}{b}\right)^{\frac{1}{2}}$, $\therefore x^3=\left(\frac{z-a}{b}\right)^{\frac{3}{2}}=\frac{1}{b^{\frac{3}{2}}}(z-a)^{\frac{3}{2}}$,

$$\therefore x^2dx=\frac{1}{2b^{\frac{3}{2}}}(z-a)^{\frac{1}{2}}dz,\ \therefore x^3dx=\frac{1}{2b^2}(z-a)dz,$$

$$\therefore \int x^3(a+bx^2)^{\frac{1}{2}}dx=\frac{1}{2b^2}\int(z-a)z^{\frac{1}{2}}dz=\frac{1}{b^2}\left(\frac{1}{5}z^{\frac{5}{2}}-\frac{a}{3}z^{\frac{3}{2}}\right)$$

$$=\frac{(a+bx^2)^{\frac{3}{2}}}{b^2}\left(\frac{a+bx^2}{5}-\frac{a}{3}\right)=\frac{(a+bx^2)^{\frac{3}{2}}}{b^2}\cdot\frac{bx^2-2a}{15}$$

(2) Required the integral of $\frac{\sqrt{(a^2-x^2)}}{x^6}dx$, or $x^{-6}(a^2-x^2)^{\frac{1}{2}}dx$.

Here $m=-6$, $n=2$, $p=\frac{1}{2}$, and $\frac{m+1}{n}+p=-2=-N$.

Hence the second criterion is satisfied: we shall solve the ex. at length.

Mult. and div. by x, $\therefore x^{-6}(a^2-x^2)^{\frac{1}{2}}dx=x^{-5}(a^2x^{-2}-1)^{\frac{1}{2}}dx$.

Substitute z for $a^2x^{-2}-1$, $\therefore x^{-1}=\frac{(z+1)^{\frac{1}{2}}}{a}$, $\therefore x^{-5}=\frac{(z+1)^{\frac{5}{2}}}{a^5}$, $\therefore x^{-6}dx=-\frac{(z+1)^{\frac{3}{2}}}{2a^5}dz$.

Divide by x^{-1}, $\therefore x^{-5}dx=-\frac{z+1}{2a^4}dz$, $\therefore x^{-5}(a^2x^{-2})^{\frac{1}{2}}dx=-\frac{z+1}{2a^4}z^{\frac{1}{2}}dz$,

and $-\frac{1}{2a^4}\int(z+1)^{\frac{1}{2}}zdz=-\frac{1}{a^4}\left(\frac{1}{5}z^{\frac{5}{2}}+\frac{1}{3}z^{\frac{3}{2}}\right)=-\frac{(3z+5)z^{\frac{3}{2}}}{15a^4}$,

$$\therefore \int x^{-6}(a^2-x^2)^{\frac{1}{2}}dx=-\frac{(3a^2x^{-2}+2)(a^2x^{-2}-1)^{\frac{3}{2}}}{15a^4}=-\frac{(3a^2+2x^2)(a^2-x^2)^{\frac{3}{2}}}{15a^4x^5}.$$

And in a similar way may the student integrate the following:—

$$x(a+bx)^{\frac{3}{2}}dx,\ x^3(a^2+x^2)^{-\frac{1}{2}}dx,\ x^2(a^2+x^2)^{-\frac{5}{2}},\ x^{-2}(a+x^3)^{-\frac{5}{3}}dx.$$

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600. III. **Integration by Successive Reduction.**—When neither of the criteria of integrability is satisfied by a differential of the preceding form, it cannot be immediately converted into an elementary differential, as in the former case. The method of proceeding is then, by successive transformations, to reduce the differential to a form which is integrable by means of the fundamental rules: these transformations are effected by repeated applications of the formula for *integrating by parts:* that is, by the repeated application of

$$\int udv=uv-\int vdu \text{ to } \int x^m(a+bx^n)^pdx.$$

Put $(a+bx^n)^p=u$, and $x^mdx=dv$, then $v=\frac{x^{m+1}}{m+1}$,

$$\therefore \int x^m(a+bx^n)^pdx=(a+bx^n)^p\frac{x^{m+1}}{m+1}-\frac{pnb}{m+1}\int x^{m+1}(a+bx^n)^{p-1}x^{n-1}dx.$$

Or, as in last article, putting z for $a+bx^n$,

$$\int x^mz^pdx=z^p\frac{x^{m+1}}{m+1}-\frac{pnb}{m+1}\int x^{n+n}z^{p-1}dx\ldots[1.]$$

Again: since $z^p=z^{p-1}(a+bx^n)=az^{p-1}+bz^{p-1}x^n$, it follows that

$$\int x^m z^p dx=a\int x^m z^{p-1}dx+b\int x^{m+n}z^{p-1}dx \ldots [2.]$$

And subtracting this from [1], and transposing, there results,

$$a\int x^m z^{p-1}dx=z^p\frac{x^{m+1}}{m+1}-\frac{b(pn+m+1)}{m+1}\int x^{m+n}z^{p-1}dx,$$

whatever be p. Put $p+1$ for p, then this becomes, after dividing by a,

$$\int x^m z^p dx=z^{p+1}\frac{x^{m+1}}{a(m+1)}-\frac{b(pn+n+m+1)}{a(m+1)}\int x^{m+n}z^p dx \ldots [3.]$$

The formulæ marked [1], [3], may obviously be useful when m is negative and n positive: they are not applicable, however, when $m+1=0$, as then the denominators vanish; but in this case formulæ of reduction are not wanted, because, since $\frac{m+1}{n}=0$, the first criterion of integrability is satisfied.

By transposing the integrals in [3], we have

$$\int x^{m+n}z^p dx=z^{p+1}\frac{x^{m+1}}{b(pn+n+m+1)}-\frac{a(m-1)}{b(pn+n+m+1)}\int x^m z^p dx,$$

whatever be m. Put $m-n$ for m, then this is changed into

$$\int x^m z^p dx=z^{p+1}\frac{x^{m-n+1}}{b(pn+m+1)}-\frac{a(m-n+1)}{b(pn+m+1)}\int x^{m-n}z^p dx \ldots [4.]$$

If, instead of subtracting [2] from [1], we had multiplied it by $\frac{pn}{m+1}$, and then added, we should have got

$$\left(1+\frac{pn}{m+1}\right)\int x^m z^p dx=z^p\frac{x^{m+1}}{m+1}+\frac{apn}{m+1}\int x^m z^{p-1}dx,$$

so that, dividing by the first coefficient, we have

$$\int x^m z^p dx=z^p\frac{x^{m+1}}{pn+m+1}+\frac{apn}{pn+m+1}\int x^m z^{p-1}dx \ldots [5.]$$

And multiplying by the denominator, and transposing,

$$\int x^m z^{p-1}dx=-z^p\frac{x^{m+1}}{apn}+\frac{pn+m+1}{apn}\int x^m z^p dx,$$

which, putting $p+1$ for p, gives

$$\int x^m z^p dx=-z^{p+1}\frac{x^{m+1}}{a(p+1)n}+\frac{(p+1)n+m+1}{a(p+1)n}\int x^m z^{p+1}dx \ldots [6.]$$

Finally, dividing [1] by the coefficient of the last term, and transposing the integrals, we have

$$\int x^{m+n}z^{p-1}dx=z^p\frac{x^{m+1}}{bpn}-\frac{m+1}{bpn}\int x^m z^p dx,$$

which, by putting $m-n$ for m, and $p+1$ for p, becomes

$$\int x^m z^p dx=z^{p+1}\frac{x^{m-n+1}}{b(p+1)n}-\frac{m-n+1}{b(p+1)n}\int x^{m-n}z^{p+1}dx \ldots [7.]$$

We thus get the following collection of formulæ.

Formulæ for the Reduction of $\int x^m(a+bx^n)^p\,dx$, or $\int x^m z^p\,dx$.

I. $$\int x^m z^p dx = \frac{x^{m+1}}{pn+m+1} z^p + \frac{apn}{pn+m+1}\int x^m z^{p-1} dx.$$

II. $$= -\frac{x^{m+1}}{a(p+1)n} z^{p+1} + \frac{(p+1)n+m+1}{a(p+1)n}\int x^m z^{p+1} dx.$$

III. $$= \frac{x^{m-n+1}}{b(pn+m+1)} z^{p+1} - \frac{a(m-n+1)}{b(pn+m+1)}\int x^{m-n} z^p dx.$$

IV. $$= \frac{x^{m+1}}{a(m+1)} z^{p+1} - \frac{b(pn+n+m+1)}{a(m+1)}\int x^{m+n} z^p dx.$$

V. $$= \frac{x^{m+1}}{m+1} z^p - \frac{bpn}{m+1}\int x^{m-n} z^{p-1} dx.$$

VI. $$= \frac{x^{m-n+1}}{b(p+1)n} z^{p+1} - \frac{m-n+1}{b(p+1)n}\int x^{m-n} z^{p+1} dx.$$

Should the particular values of m, n, p, in any example, be such as to cause the denominator in any of these formulæ to vanish, that formula will, of course, be inapplicable: but in these circumstances, we should always have either

$(m+1)\div n=0$, that is, $N-1=-1$, or else $\frac{m+1}{n}+p=0$, that is, $N-1=-1$;

and therefore, as shown in the NOTE, p. 586, the integral is then always reducible to that of a rational fraction. To exemplify the use of the above formulæ, let $x^5(a+bx^2)^{-\frac{1}{2}}dx$ be proposed for integration. Then, since $m=5$, $n=2$, and $p=-\frac{1}{2}$, we have by III.,

$$\int x^5(a+bx^2)^{-\frac{1}{2}}dx = \int x^5 z^{-1} dx = \frac{x^4}{5b} z^{\frac{1}{2}} - \frac{4a}{5b}\int x^3 z^{-\frac{1}{2}} dx.$$

Again: by III., m being now $=3$, $\int x^3 z^{-\frac{1}{2}}dx = \frac{x^2}{3b} z^{\frac{1}{2}} - \frac{2a}{3b}\int x z^{-\frac{1}{2}} dx$.

Now $\int x z^{-\frac{1}{2}} dx = \int (a+bx^2)^{-\frac{1}{2}} x dx = \frac{1}{b}(a+bx^2)^{\frac{1}{2}}$: hence

$$\int \frac{x^5 dx}{\sqrt{(a+bx^2)}} = \frac{x^4}{5b}\sqrt{(a+bx^2)} - \frac{4ax^2}{15b^2}\sqrt{(a+bx^2)} + \frac{8a^2}{15b^3}\sqrt{(a+bx^2)}$$

$$= \frac{\sqrt{(a+bx^2)}}{15b}\left(3x^4 - \frac{4a}{b}x^2 + \frac{8a^2}{b^2}\right).$$

The formula III. was chosen because it was easy to foresee that two reductions by it would bring the proposed to an integrable form, but V. or VI. would have done the same thing.

The following integrals, reducible by III. and IV., m being a positive integer, are in frequent request.

$$\int \frac{x^m dx}{\sqrt{(a+bx^2)}} = \frac{x^{m-1}}{bm}\sqrt{(a+bx^2)} - \frac{a(m-1)}{bm}\int \frac{x^{m-2} dx}{\sqrt{(a+bx^2)}}$$

$$\int \frac{x^m dx}{\sqrt{(a-bx^2)}} = \frac{x^{m-1}}{-bm}\sqrt{(a-bx^2)} + \frac{a(m-1)}{bm}\int \frac{x^{m-2} dx}{\sqrt{(a-bx^2)}}$$

$$\int \frac{dx}{x^m\sqrt{(a+bx^2)}} = \frac{\sqrt{(a+bx^2)}}{a(1-m)x^{m-1}} - \frac{b(2-m)}{a(1-m)} \int \frac{dx}{x^{m-2}\sqrt{(a+bx^2)}}$$

$$\int \frac{dx}{x\sqrt{(a-bx^2)}} = \frac{\sqrt{(a-bx^2)}}{a(1-m)x^{m-1}} + \frac{b(2-m)}{a(1-m)} \int \frac{dx}{x^{m-2}\sqrt{(a-bx^2)}}.$$

We here see that at every repetition of the process m is reduced by 2; hence, when m is even, it is ultimately reduced to 0, and when odd, to 1; and these ultimate forms may be directly integrated by the fundamental rules.

To these we shall add the integral of the trinomial form $(a+bx\pm cx^2)^{\frac{1}{2}}\,dx$, to which many others may be reduced.

By the rule for solving a quadratic equation, we know that

$$a+bx+cx^2 = c\left\{\left(x+\frac{b}{2c}\right)^2 - \frac{b^2-4ac}{4c^2}\right\} = c(y^2+A),\ \therefore dx=dy,$$

$$\therefore \int \frac{dx}{\sqrt{(a+bx+cx^2)}} = \frac{1}{\sqrt{c}} \int \frac{dy}{\sqrt{(y^2+A)}} = \frac{1}{\sqrt{c}} \log\{\sqrt{(y^2+A)}+y\} \text{ (p. 508)}$$

$$= \frac{1}{\sqrt{c}} \log\left\{\left(x^2+\frac{b}{c}x+\frac{a}{c}\right)^{\frac{1}{2}} + x + \frac{b}{2c}\right\}.$$

And $$\int \frac{dx}{\sqrt{(a+bx-cx^2)}} = \frac{1}{\sqrt{c}} \int \frac{dy}{\sqrt{(A-y^2)}} = \frac{1}{\sqrt{c}} \sin^{-1}\frac{y}{\sqrt{A}} = \frac{1}{\sqrt{c}} \sin^{-1}\frac{2cx-b}{(4ac+b^2)^{\frac{1}{2}}}.$$

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601. IV. **Integration by Parts.**—This method has already been employed in obtaining the preceding formulæ of reduction: we shall here show its application to certain logarithmic, exponential, and trigonometrical forms.

1. *To integrate $X\log^n x\,dx$, where X is a function of x.*

In the formula $\int u\,dv = uv - \int v\,du$, put $dv = X\,dx$, and $u = \log^n x$,

then $$\int X\,dx\,.\log^n x = \log^n x \int X\,dx - \int \left(\int X\,dx\,.\,n\log^{n-1}x\,\frac{dx}{x}\right),$$

or, putting, for brevity, $\int X\,dx = X_1$,

$$\int X\log^n x\,dx = X_1\log^n x - n\int \frac{X_1}{x}\log^{n-1}x\,dx \ldots [1.]$$

Now, if n be a positive integer, the repeated application of this formula will finally reduce the integration to that of $\frac{X_m}{x}\,dx$, so that the proposed form can always be integrated, provided we can integrate in succession the functions

$$X\,dx = dX_1,\ \frac{X_1}{x}dx = dX_2,\ \frac{X_2}{x}dx = dX_3,\ \ldots,\ \frac{X_m}{x}dx.$$

As an example, let it be required to integrate $(a^2+x^2)^{-\frac{1}{2}}\,x\log x\,dx$.

Here $n=1$, $X=(a^2+x^2)^{-\frac{1}{2}}x$, $\therefore Xdx = dX_1 = (a^2+x^2)^{-\frac{1}{2}}x\,dx$, $\therefore X_1 = (a^2+x^2)^{\frac{1}{2}}$

$$\therefore X_2 = \int \frac{(a^2+x^2)^{\frac{1}{2}}}{x}dx = \int \frac{a^2+x^2}{x(a^2+x^2)^{\frac{1}{2}}}dx = \int \frac{a^2dx}{x(a^2+x^2)^{\frac{1}{2}}} + \int \frac{x\,dx}{(a^2+x^2)^{\frac{1}{2}}}.$$

The first of these is $-a \log \frac{\sqrt{(a^2+x^2)}+a}{x}$, and the second $\sqrt{(a^2+x^2)}$ (pp. 508, 504),

$$\therefore [1], \int \frac{x \log x dx}{\sqrt{(a^2+x^2)}} = \log x \,.\, \sqrt{(a^2+x^2)} - \sqrt{(a^2+x^2)} + a \log \frac{\sqrt{(a^2+x^2)}+a}{x}.$$

One of the most useful cases of [1] is that where $X=x^m$. This form has already been integrated at p. 509.

If, however, $m=0$, and $n=-1$, the form is $\int \frac{dx}{\log x}$, which, simple as it is in appearance, has never yet been integrated, except by series, that is, by developing $(\log x)^{-1}$ (see art. 602).

2. *To integrate* $x^m a^x dx$. Putting in the formula for parts,

$$u=x^m,\ dv=a^x dx, \text{ and } \therefore v=\frac{a^x}{\log a}, \text{ we have}$$

$$\int x^m a^x dx = \frac{x^m a^x}{\log a} - \frac{m}{\log a} \int x^{m-1} a^x dx,$$

a formula which, by repeated applications, gives

$$\int x^m a^x dx = \frac{a^x}{\log a} \left\{ x^m - \frac{m}{\log a} x^{m-1} + \frac{m(m-1)}{\log^2 a} x^{m-2} - \ldots \pm \frac{1.2.3 \ldots m}{\log^m a} \right\},$$

the upper being the sign of the last term when m is even; the lower, when m is odd. When m is negative, the terms within the brackets do not come to an end; in this case the substitutions in the formula for parts must be

$$u=x^x,\ dv=x^{-m}dx, \text{ and } \therefore v=\frac{-1}{(m-1)x^{m-1}}; \text{ whence}$$

$$\int \frac{a^x dx}{x^m} = -\frac{a^x}{(m-1)x^{m-1}} + \frac{\log a}{m-1} \int \frac{a^x dx}{x^{m-1}},$$

by repeated application of which, the final integral becomes $\int \frac{a^x dx}{x}$, which, however, like the simple integral above, can be found only by series, that is, by developing a^x, and then integrating term after term (see art. 602).

3. *To integrate* $\sin^m x \cos^n x dx \ldots [A]$.

First, let m be a positive and odd integer: then replacing it by $2p+1$, and putting

$(1-\cos^2 x)^p \sin x$ for its equal, $\sin^{2p+1} x$, we have

$$\int \cos^n x (1-\cos^2 x)^p \sin x dx = -\int \cos^n x (1-\cos^2 x)^p d \cos x = -\int z^n (1-z^2)^p dz,$$

and since p is a positive integer, this may be converted by the binomial theorem into a series of monomials, all integrable by the rule for powers, whatever be the exponent n.

Next, let n be a positive and odd integer$=2p+1$, then we have

$$\int \sin^m x (1-\sin^2 x)^p \cos x dx = \int \sin^m x (1-\sin^2 x)^p d \sin x = \int z^m (1-z^2)^p dz,$$

which, as before, is integrable, whatever be the exponent m.

Lastly, let $m+n$ be an even negative integer$=-2p$, then putting

$$\sin^m x \cos^n x = \frac{\sin^m x}{\cos^m x} \cos^{m+n} x = \tan^m x \cos^{-2p} x = \tan^m x \sec^{2p} x,$$

we have $\int \sin^m x \cos^n x dx = \int \tan^m x (1+\tan^2 x)^p dx = \int \tan^m x (1+\tan^2 x)^{p-1} \sec^2 x dx$

$$= \int \tan^m x (1+\tan^2 x)^{p-1} d \tan x = \int z^m (1+z^2)^{p-1} dz,$$

which may be integrated as in the former cases.

But if the exponents do not satisfy either of the above conditions, then putting the differential under the form

$$\sin^{m-1} x \cos^n x \sin x dx, \text{ and assuming } \sin^{m-1} x = u, \cos^n x \sin x dx = dv,$$

we have $du = (m-1) \sin^{m-2} \cos x dx$, and $v = -\frac{\cos^{n+1} x}{n+1}$, and the formula for parts, $\int u dv = uv - \int v du$, gives

$$\int \sin^m x \cos^n x dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{m+1} \int \sin^{m-2} x \cos^{n+2} x dx,$$

which is useful when n is negative and m positive. It may be put in another form thus. In the last integral put

$$\cos^n x (1-\sin^2 x) \text{ for } \cos^{n+2} x,$$

$$\therefore \int \sin^{m-2} x \cos^{n+2} x dx = \int \sin^{m-2} \cos^n x dx - \int \sin^m x \cos^n x dx,$$

so that by transposition, &c., we have

$$\int \sin^m x \cos^n x dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x dx \ldots [1].$$

If we apply to the cosine the like reductions to those just applied to the sine, we shall, in like manner, get

$$\int \sin^m x \cos^n x dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x dx \ldots [2].$$

And these two formulæ, by successive application, enable us, by diminishing the exponents each time by 2, to arrive ultimately, when these exponents are both positive integers, at one or other of the four differentials

$$dx, \sin x dx, \cos x dx, \sin x \cos x dx = \sin x d \sin x,$$

of which the integrals are, x, $-\cos x$, $\sin x$, $\frac{1}{2} \sin^2 x$.

But if either m or n be a negative, and the other a positive integer, the ultimate differentials will be one or other of the four forms

$$\frac{dx}{\sin x}, \frac{dx}{\cos x}, \frac{\sin x}{\cos x} dx, \frac{\cos x}{\sin x} dx,$$

which are integrated as follows :—

1. $\int \frac{dx}{\sin x} = \int \frac{\sin x}{\sin^2 x} dx = \int \frac{\sin x}{1-\cos^2 x} dx = -\int \frac{dz}{1-z^2} = \frac{1}{2} \log \frac{1-z}{1+z}.$

2. $\int \frac{dx}{\cos x} = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{\cos x}{1-\sin^2 x} dx = \int \frac{dz}{1-z^2} = \frac{1}{2} \log \frac{1+z}{1-z}.$

3. $\int \frac{\sin x}{\cos x} dx = -\int \frac{d \cos x}{\cos x} = -\log \cos x.$

4. $\int \frac{\cos x}{\sin x} dx = \int \frac{d \sin x}{\sin x} = \log \sin x.$

If, however, m and n are both negative, the formulæ [1], [2] will be

$$\int \frac{dx}{\sin^m x \cos^n x} = \frac{1}{m+n} \frac{1}{\sin^{m+1}x \cos^{n-1}x} + \frac{m+1}{m+n} \int \frac{dx}{\sin^{m+2}x \cos^n x}$$

$$,, \qquad = \frac{-1}{m+n} \frac{1}{\sin^{m-1}x \cos^{n+1}x} + \frac{n+1}{m+n} \int \frac{dx}{\sin^m x \cos^{n+2}x}.$$

And putting $m-2$ for m in the first of these, and $n-2$ for n in the second, and then transposing, &c., as at p. 592, we have

$$\int \frac{dx}{\sin^m x \cos^n x} = \frac{-1}{m-1} \frac{1.}{\sin^{m-1}x \cos^{n-1}x} + \frac{m+n-2}{m-1} \int \frac{dx}{\sin^{m-2}x \cos^n x},$$

$$,, \qquad = \frac{1}{n-1} \frac{1}{\sin^{m-1}x \cos^{n-1}x} + \frac{m+n-2}{n-1} \int \frac{dx}{\sin^m x \cos^{n-2}x},$$

which ultimately reduce to the same integrals, as in the former cases, or else to

$$\int \frac{dx}{\sin x \cos x} = \int \left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right) dx = \log \frac{\sin x}{\cos x} = \log \tan x \text{ (p. 592).}$$

Hence the form $[A]$ can always be integrated without series, provided m and n, whether positive or negative, are integral. We shall merely add, in conclusion, that when m and n are positive whole numbers, the forms

$$\sin^m x dx, \text{ and } \cos^n x dx,$$

may be immediately integrated by means of the following relations (259-60):—

$$\sin^2 x = -\frac{1}{2}\cos 2x + \frac{1}{2},$$

$$\sin^3 x = -\frac{1}{4}\sin 3x + \frac{3}{4}\sin x,$$

$$\sin^4 x = \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x + \frac{3}{8},$$

&c. &c.

$$\cos^2 x = \frac{1}{2}\cos 2x + \frac{1}{2},$$

$$\cos^3 x = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x,$$

$$\cos^4 x = \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8},$$

&c. &c.

The following are a few examples for exercise on the preceding articles:—

$$\sin^3 x dx,\ \sin^3 x \cos^2 x dx,\ \sin^4 x \cos^3 x dx,\ \sin^5 x \cos^4 x dx,$$

$$\cos^4 x dx,\ \sin^6 x dx,\ \sin^8 x \cos^3 x dx,\ \frac{dx}{\sin^3 x \cos^2 x}.$$

The formula for parts will apply successfully to many other differential forms besides those considered above; and in particular examples, algebraic artifice will often conduct more readily to the desired result than any general rule: but since the INTEGRAL TABLES of Meyer Hirsch give at once the integrals of nearly all the forms which have been integrated —just as a table of logs gives the log corresponding to a number, thus saving the labour of computation—we refer to that very useful work for ample practical information of this kind.*

* These Tables (the English reprint) are now becoming rather scarce: they may, however, be had of Mr. Maynard, mathematical bookseller, Earl's Court, Leicester Square.

602. **Integration by Series.**—In the preceding articles we have given all the methods of integration in finite terms, to which general processes have as yet been applied. Considering the endless variety of forms which algebraic and transcendental expressions may assume, these general rules are but very inadequate for all the demands of analysis. But, as previously remarked, algebraic artifice and ingenuity will often transform an individual differential into a shape which will bring it under the control of established methods; though, for the successful treatment of it in this way, no formal directions can be given. As a last resource, we must develope the proposed expression in an infinite series; integrate its terms separately, and thus be content with an approximation to the value of the required integral. As examples of this mode of proceeding we shall take the two forms left unintegrated at p. 591.

(1) To integrate $\frac{dx}{\log x}$ by series. Put $\log x=z$, or $x=e^z$: then, since

$$e^z=1+z+\frac{z^2}{2}+\frac{z^3}{2.3}+\frac{z^4}{2.3.4}+\ldots,$$

$$\therefore x=1+\log x+\frac{(\log x)^2}{2}+\frac{(\log x)^3}{2.3}+\frac{(\log x)^4}{2.3.4}+\ldots,$$

$$\therefore \int\frac{dx}{\log x}=\int\frac{dx}{x}\left\{0+\frac{1}{\log x}+1+\frac{\log x}{2}+\frac{(\log x)^2}{2.3}+\ldots\right\}$$

$$=\log(\log x)+\log x+\frac{1}{2}.\frac{(\log x)^2}{2}+\frac{1}{3}.\frac{(\log x)^3}{2.3}+\ldots$$

(2) To integrate $\frac{a^x dx}{x}$ by series. Here developing a^x, we have

$$\int\frac{a^x dx}{x}=\int\frac{dx}{x}\left\{1+\log a\,.\,x+\frac{(\log a)^2}{2}x^2+\frac{(\log a)^3}{2.3}x^3+\ldots\right\}$$

$$=\log x+\log a\,.\,x+\frac{1}{2}.\frac{(\log a)^2}{2}x^2+\frac{1}{3}.\frac{(\log a)^3}{2.3}x^3+\ldots.$$

In a similar way we may find

$$\int\frac{dx}{1+x^2}=\int(1-x^2+x^4-x^6+\ldots)dx=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\ldots=\tan^{-1}x.$$

See also p. 512.

603. **Series of John Bernoulli.**—For exhibiting the general value of an integral in an infinite series, the following, known as John Bernoulli's series, is that usually adopted: it is obtained thus:—Let X represent any function of x, and put $\int X dx=F(x)$; then, by Taylor's theorem, we have

$$F(x-h)=\int X dx-Xh+\frac{dX}{dx}\frac{h^2}{2}-\frac{d^2X}{dx^2}\frac{h^3}{2.3}+\ldots$$

As h is arbitrary, put $h=x$: then $F(x-h)=F(0)$, which is therefore what $\int X dx$ becomes when $x=0$: to mark this state, write it $[\int X dx]$,

$$\therefore \int X dx=Xx-\frac{dX}{dx}\frac{x^2}{2}+\frac{d^2X}{dx^2}\frac{x^3}{2.3}-\ldots+[\int X dx],$$

which is the desired series, and in which the constant annexed to it, and

written within brackets,—being what the integral becomes when $x=0$,—is the arbitrary constant. It is plain that for a definite integral, where x takes assigned values, the series is useless, unless for those values it be convergent. But it is difficult to ascertain when this is the case, seeing that the series does not proceed according to either the ascending or descending powers of x, inasmuch as this variable enters also the several coefficients. This inconvenience is avoided in the following analogous series. Applying to the proposed integral the theorem of Maclaurin, instead of that of Taylor, we have

$$\int X dx=[\int X dx]+[X]x+\left[\frac{dX}{dx}\right]\frac{x^2}{2}+\left[\frac{d^2X}{dx^2}\right]\frac{x^3}{2.3}+\ldots,$$

the brackets implying that the enclosed functions are to be taken in the state of $x=0$: the term here placed first is the arbitrary constant. The best way of applying this series is to proceed as follows: first develope X by Maclaurin's theorem; we thus get

$$X=[X]+\left[\frac{dX}{dx}\right]x+\left[\frac{d^2X}{d^2x}\right]\frac{x^2}{2}+\ldots;$$

then multiply the several terms by x, $\frac{1}{2}x$, $\frac{1}{3}x$, &c., and annex the arbitrary constant: actual integration is thus dispensed with. We can afford room for but one example.

Required the integral of $a^x dx \div (1-x)$. By division,

$$1\div(1-x)=1+x+x^2+\ldots \quad \text{Also } a^x=1+\log a\,.\,x+\frac{(\log a)^2}{2}x^2+\ldots$$

$$\text{The product of these is } 1+(1+\log a)x+\left(1+\log a+\frac{(\log a)^2}{2}\right)x^2+\ldots$$

$$\therefore \int\frac{a^x dx}{1-x}=x+(1+\log a)\frac{x^2}{2}+\left(1+\log a+\frac{(\log a)^2}{2}\right)\frac{x^3}{3}+\ldots.$$

This process of course applies only when X is developable, from the commencement, according to the ascending powers of x.

604. *Definite Integration by Series.*—In order to find $\int X dx=F(x)$, between the limits $x=a$, and $x=b$, in a series, put $b-a=h$, $\therefore b=a+h$; and by Taylor's theorem, putting $[X]_a$ for what the enclosed function becomes when $x=a$, we have

$$F(b)-F(a)=\int_a^b X dx=[X]_a h+\left[\frac{dX}{dx}\right]_a\frac{h^2}{2}+\left[\frac{d^2X}{dx^2}\right]_a\frac{h^3}{2.3}+\ldots.$$

If h, the interval between the limits, be sufficiently small to render this convergent—as it is here assumed to be—the definite integral may be approximated to by summing the leading terms: but if the interval h be too great for this, it must be subdivided into a sufficient number of smaller intervals, and it is best to make these all equal; we shall then have to substitute for x, in the coefficients, a, $a+h'$, $a+2h'$, &c., up to $b-h'$—each series proceeding according to the powers of h'—one of the equal component intervals: the aggregate of these series, dismissing those advanced powers of h' that may be disregarded on account of their smallness, will be of the form

$$\int_a^b X dx=Ah+A_1h^2+A_2h^3+\ldots.$$

But a better way will be to approximate to the value of the definite integral by the method of equidistant ordinates, explained at (312).

[For fuller information on the subject of Definite Integrals, see De Morgan's Calculus, and Vol. II. of Price's Infinitesimal Calculus: also a very ingenious Paper by Mr. Rawson, in the "Manchester Memoirs," Vol. VII., New Series.]

605. **Successive Integration.**—In all the preceding examples of integration it is the *first* differential coefficient that is given to determine the primitive function from which it has been derived. But if, instead of the first, it be the nth diff. coef. which is given, then, by a first integration, we shall arrive at the preceding, or $n-1$th diff. coef.; by a second integration, at the $n-2$th coef., and so on, till we reach the original function. Since at each integration a constant is introduced, it follows that the complete primitive ought to contain as many arbitrary constants as there have been integrations to arrive at it. Thus, let

$$\frac{d^n y}{dx^n}=X, \therefore d\left(\frac{d^{n-1}y}{dx^{n-1}}\right)=Xdx, \therefore \frac{d^{n-1}y}{dx^{n-1}}=\int Xdx=X_1+C_1.$$

$$\text{Also } d\left(\frac{d^{n-2}y}{dx^{n-2}}\right)=X_1dx+C_1dx, \therefore \frac{d^{n-2}y}{dx^{n-2}}=\int X_1dx+\int C_1dx=X_2+C_1x+C_2,$$

and after n integrations, we shall in this way get

$$\int^n Xdx^n=X_n+C_1\frac{x^{n-1}}{2.3\ldots(n-1)}+C_2\frac{x^{n-2}}{2.3\ldots(n-2)}+\ldots+C_{n-2}\frac{x^2}{2}+C_{n-1}x+C_n,$$

an expression involving n arbitrary constants.

Hence, if we multiply X, and the successive results, by dx, and integrate at each step—omitting the arbitrary constants—we shall at length get X_n, to which the remaining terms, involving the several constants, may be appended, conformably to this expression.

For example: let $\int^3 \sin x dx^3$ be proposed; then we have

$$\int \sin xdx=-\cos x, \ -\int \cos xdx=-\sin x, \ -\int \sin xdx=\cos x=X_3,$$

$$\therefore \int^3 \sin xdx^3=\cos x+C_1\frac{x^2}{2}+C_2x+C_3.$$

$$\text{Again: Required the curve whose equation is } \frac{d^3y}{dx^3}=C_1, \text{ or } \frac{d^4y}{dx^4}=0.$$

$$\text{Omitting constants, } X_4=0, \therefore y=C_1\frac{x^3}{2.3}+C_2\frac{x}{2}+C_3x+C_4.$$

Hence the curve is a parabolic curve of the third order.

We shall add one example more: an example in which an infinite series is necessary.

Let the integral be $\int^4(1+x^2)^{-\frac{1}{2}}dx^4$: then by the binomial theorem,

$$(1+x^2)^{-\frac{1}{2}}=1-\frac{1}{2}x^2+\frac{1.3}{2.4}x^4-\frac{1.3.5}{2.4.6}x^6+\ldots.$$

Now, instead of proceeding here as in the former examples, integrating step by step, we shall get the final integral X_4 at once, if we replace x^0, x^2, x^4, x^6, &c., by

$$\frac{x^4}{1.2.3.4}, \frac{x^6}{3.4.5.6}, \frac{x^8}{5.6.7.8}, \frac{x^{10}}{7.8.9.10}, \text{ &c.,}$$

as a little reflection on the effect of successive integration will show:

$$\therefore \int \frac{dx^4}{\sqrt{(1+x^2)}} = \frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6} + \frac{1.3x^8}{2.4.5.6.7.8} - \frac{1.3.5x^{10}}{2.4.6.7.8.9.10} + \ldots$$
$$+ C_1\frac{x^3}{2.3} + C_2\frac{x^2}{2} + C_3x + C_4.$$

This method of deriving the final integral at once, when X is developable in a series according to Maclaurin's theorem, was first given, in a general rule for the purpose, in the author's separate treatise on the Integral Calculus, p. 91 (1831). The plan is to replace x^0, x^1, x^2, x^3, &c., in the development of X, by

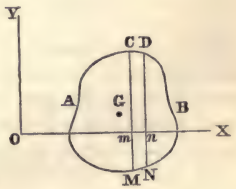
$$\frac{x^n}{2.3\ldots n}, \frac{x^{n+1}}{2.3\ldots(n+1)}, \frac{x^{n+2}}{3.4\ldots(n+2)}, \frac{x^{n+3}}{4.5\ldots(n+3)}, \text{ &c.;}$$

and then, annexing the terms containing the arbitrary constants, we shall have the complete integral $\int^n X dx^n$.

X. APPLICATIONS OF THE CALCULUS TO MECHANICS.

We shall terminate this treatise by giving a compendious view of the way in which the principles of the calculus are applied to statical and dynamical inquiries.

606. **The Centre of Gravity.**—Let ABC represent either a plane surface or a solid body:—if the former, let it be regarded as situated in the horizontal plane (of x, y). The co-ordinates X, Y, of the centre of gravity G, or of the point where the vertical line from G pierces the horizontal plane—the plane of x, y—are exhibited at p. 384. Now let CN be an increment of the body or plane, comprised between two planes, CM, DN, perp. to the axis of x: then the corresponding increment of the numerator of X (p. 384) will be the sum of all the particles in the slice CN, each multiplied by its abscissa. If we call the increment mn, h, it is plain that however small we take h, that is, however slender the slice CN may be, the sum just alluded to will always be comprised between these two, namely, the sum of the same particles when each is multiplied by $Om=x$, and the sum when each is multiplied by $On=x+h$; that is, putting S for the numerator of X, and ΔS, ΔB, for the corresponding increments of this and of the body, ΔS will always be intermediate between $x\Delta B$ and $(x+h)\Delta B$; but the ratio of these is



$$\frac{(x+h)\Delta B}{x\Delta B} = 1 + \frac{h}{x} = 1 \text{ in the limit, or when } h \text{ becomes } 0;$$

hence the ratio of the intermediate quantity ΔS to either must, in the limit, be 1; that is,

$$\text{in the limit, } \frac{\Delta S}{x\Delta B} = 1, \therefore \frac{dS}{xdB} = 1, \therefore dS = xdB, \therefore S = \int xdB,$$

where S is the sum of the infinite number of elements constituting the numerator of X: hence

$$X = \frac{\int xdB}{B}, \text{ and in like manner, } Y = \frac{\int ydB}{B} \ldots [I.],$$

where it is to be observed that the integrals are to be taken between the assigned limits, and where B, though called above *body*, may stand for either line, surface, or volume. If its figure be such that it cannot be analytically represented, then the co-ordinates of the centre of gravity can be expressed only thus, namely:—

$$X=\frac{\Sigma(xm)}{\Sigma(m)},\quad Y=\frac{\Sigma(ym)}{\Sigma(m)},$$

where m is any particle of the body, $\Sigma(m)$ being the sum of the particles, or B.

The foregoing reasoning merely amounts to this. A thin vertical slice of the body is conceived to be taken: the particles in it are regarded, by way of a first approximation, as all at the same horizontal distance from the origin: this supposition approaches nearer and nearer to the truth as the thickness of the slice is diminished, and actually reaches the truth when the slice, by continuous thinning, becomes a mere section:—a plane or line of particles; or, when B is a line, a single particle (dB): see p. 561, foot-note. We shall give an example or two of the application of the preceding formulæ.

(1) To find the centre of gravity of the parabola, $y=2(ax)^{\frac{1}{2}}$.

For a plane curve, $dB=ydx$, or $B=\int ydx$ (p. 511), therefore

$$X=\frac{\int yxdx}{\int ydx}=\frac{2a^{\frac{1}{2}}\int x^{\frac{3}{2}}dx}{2a^{\frac{1}{2}}\int x^{\frac{1}{2}}dx}=\frac{3}{5}x,\text{ which, from } x=0 \text{ to } x=x, \text{ is } \frac{3}{5}x.$$

NOTE.—It should not be passed by unnoticed—as in most books it is—that, agreeably to the general expression for X, we ought to have used $2y$ instead of y; for *here* B is only *half* the parabola: but as the 2 would occur equally as a factor in numerator and denominator, it is suppressed. In like manner, π is suppressed in num. and den. of ex. 3.

(2) To find the centre of gravity of a circular segment, $y=(2rx-x^2)^{\frac{1}{2}}$.

$$X=\frac{\int yxdx}{\int ydx}=\frac{\int(2rx-x^2)^{\frac{1}{2}}xdx}{\frac{1}{2}\text{ area of seg.}}$$

Now, by adding and subtracting the first of the terms following, we have

$$\int(2rx-x^2)^{\frac{1}{2}}xdx=\int(2rx-x^2)^{\frac{1}{2}}rdx-\int(2rx-x^2)^{\frac{1}{2}}(rdx-xdx)$$

$$=r\times\frac{1}{2}\text{ area of seg. }-\frac{1}{3}(2rx-x^2)^{\frac{3}{2}}=r\times\frac{1}{2}\text{ area of seg.}-\frac{y^3}{3},$$

$$\therefore X=r-\frac{2y^3}{3\text{ area of seg.}}.\quad\text{For the semicircle } (y=0,\ y=r),\ X=\frac{2r^3}{3\text{ area}}=\frac{4r}{3\pi}.$$

(3) To find the centre of gravity of a segment of a spheroid.

The equation of the generating ellipse being $y^2=\frac{b^2}{a^2}(2ax-x^2)$,

$$\therefore X=\frac{\int y^2xdx}{\int y^2dx}=\frac{\int(2ax-x^2)xdx}{\int(2ax-x^2)dx}=\frac{\frac{2}{3}ax^3-\frac{1}{4}x^4}{ax^2-\frac{1}{3}x^3}=\frac{8ax-3x^2}{12a-4x}.$$

For the hemispheroid $(x=0,\ x=a)$, $X=\frac{5a}{8}$.

Since the expression for X is independent of b, it remains the same

when $b=a$: hence, if a sphere be described on either axis of a spheroid, any two segments—one of each—cut off by a plane perp. to this axis, will have the same centre of gravity.

The following are added for exercise.

(1) The distance of the centre of gravity of a semi-ellipse from the base, or minor axis, is $X=4a\div 3\pi$.

(2) The dist. of the centre of gravity of a paraboloid, of alt. a, from the vertex, is $X=\frac{2}{3}a$.

(3) The dist. of the centre of figure from the centre of gravity of a circular arc, is $X=r$ chord $\div$ arc.

(4) The dist. of centre from centre of gravity of a circular sector is $\frac{2}{3}r$ chd. $\div$ arc.

(5) The centre of gravity of the surface of a spheric segment is at the middle of its height.

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607. In all the preceding examples, the bodies are symmetrical as respects the axis of x, on which axis, therefore, the centre of gravity is situated. But when such is not the case, and this centre is out of the axis of x, then the Y of it also must be found by the second of the formulæ [I.]. And here it must be remembered that the dB is not the same as in the expression for X:—in *this* the slice is parallel to the axis of y, in the other it is parallel to the axis of x: in the one case x is supposed to vary uniformly, and in the other, y to vary uniformly. But the expression for Y may be applied to the *same* slice thus:—Let B be regarded as a *surface;* then $dB=ydx$ is a *line* (CM) of particles: differentiating this variable line, y alone varying, we have $dydx$; which symbolizes a single particle, and $\therefore$ $ydydx$ is the general expression for any particle in the line multiplied by its distance from the axis of x: it is therefore the integral of this that forms the numerator in the expression for Y.

Integrating then in reference to y, we get $\frac{1}{2}y^2dx$: this refers to a single entire line of particles, and $\therefore$ $\frac{1}{2}\int y^2dx$ to *all* these lines, or to the whole surface, when taken between the limits of it.

Hence for a surface, $Y=\dfrac{\frac{1}{2}\int y^2dx}{\int ydx}$. For a curve, $Y=\dfrac{\int yds}{s}$...[II.].

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608. **The Theorems of Guldinus.**—The expressions just deduced furnish two remarkable theorems for determining surfaces and volumes of revolution; for they immediately give

For a curve, $2\pi Ys=2\pi\int yds$. For a surface, $2\pi Y\int ydx=\pi\int y^2dx$.

Now $2\pi Y$ is the circumference of the circle of which Y is the radius: it expresses, therefore, the circumference that would be described by the centre of gravity of the line s, if that line were to revolve round the axis of x. But $2\pi\int yds$ expresses the area of the surface which would be actually generated by this revolution (p. 517): hence,

1. The surface generated by the revolution of a curve round an axis is equal to the length of that curve multiplied by the circumference described by its centre of gravity.

Again: $2\pi Y$, in the second of the expressions referred to, is equal to the circumference which would be described by the centre of gravity of the plane surface, if it were to revolve round the axis of x; and $\pi\int y^2dx$ is the expression for the volume actually generated (543): hence,

2. The volume generated by the revolution of a plane surface round an axis is equal to the area of that surface multiplied by the circumference described by its centre of gravity. The application of these theorems to the determination of surfaces and volumes of revolution constitutes what is called the *Centrobaryc Method.*

But instead of determining surfaces and volumes in this way, the theorems may be conveniently employed, when these surfaces and volumes are already known, to find the centres of gravity of their generating lines and planes: thus, let it be required to find the centre of gravity of a semicircle, and also that of a semicircumference. Then, since a semicircle, revolving about its diameter, generates a sphere whose volume is $\frac{4\pi r^3}{3}$, and a surface whose area is $4\pi r^2$, we have only to divide the former expression by 2π times the area of the semicircle, which is $\frac{\pi r^2}{2}$, and the latter expression by 2π times the arc of the semicircle, which is πr, to get the distances of the respective centres of gravity from the fixed diameter; which distances are, therefore,

For the semicircle, $Y=\frac{4r}{3\pi}$. For the arc of it, $Y=\frac{2r}{\pi}$.

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609. **Moment of Inertia: Radius of Gyration.**—The moment of inertia of a body turning about an axis is expressed by the sum of the products of all the particles of the body into the squares of their respective distances (r) from the axis of rotation (480). Putting then M for the mass of the body, or the sum of all its particles, let dM denote a *line* of these particles parallel to (or equidistant from) the fixed axis, if the body be a plane; a *surface* of them, if the body be a solid; and merely a single particle, if it be a line: then the moment of inertia of the entire mass will be expressed by the integral $\int r^2dM$, taken between the limits of the body. If the mass could be condensed into a single point, at a distance k from the axis, such that

$$k^2M=\int r^2dM, \text{ and } \therefore \text{ such that } k=\sqrt{\frac{\int r^2dM}{M}},$$

k would be *the radius of gyration* of the rotating mass (480).

Note.—If the figure of the mass be such that it cannot be analytically represented, then the moment of inertia can be expressed only in the form $k^2M=\Sigma(r^2.m)$, m being any particle of it.

The evaluation of this expression, when m is a mere particle, is in general impracticable; but when M consists of several distinct masses, all rotating about a common axis, each mass may be regarded as condensed into a single particle at its centre of gyration; and the moment of inertia of the whole, putting K for the radius of gyration, will then be

$$K^2M=\Sigma(r^2.m)=k^2.m+k_1^2.m_1+k_2^2.m_2+\ldots, \text{ the no. of terms being finite.}$$

610. If the body were free, its resistance to *progressive* motion would

be merely the mass moved; but if it be constrained to turn about an axis, its resistance to *angular* motion is the above expression. In reference to motion about an axis, therefore, k^2M always supplies the place of the mass moved, as if M were condensed in a point distant k from the axis. Just as the mass M is conceived to be concentrated in a point at the distance k from the axis, if M_1 be another mass condensed into a point at the distance a, the resistance of *this* to angular motion would be a^2M_1: if this resistance be the same as the former, then $k^2M=a^2M_1$, so that, for such to be the case, the new mass M_1, condensed into the assigned point, must be such that $M_1=\frac{k^2M}{a^2}$; and such a substitution of M_1 for M will not modify the resistance to angular motion.

And here we must make a remark in anticipation of a perplexity which the student might otherwise feel in reference to the centres of gyration and of oscillation, into one or other of which points the mass of the rotating body is regarded as concentrated in the subsequent investigations. It is to be noticed that, as respects the first point, we have regard solely to the *mass*, dismissing all considerations as to gravity, or any other accelerating influence,—no such influence being supposed to act; in which circumstances, a body once put in motion, would move uniformly for ever. But in speaking of the centre of oscillation, we take matter as we find it, obeying the force of gravity, since without some such force there could be no oscillation; so that *acceleration* and *time* then become elements of consideration.

(1) A slender rod of length L, revolving about its extremity.

Putting r for any distance on the line from its fixed extremity, we have

$$k^2=\frac{\int r^2dr}{L}=\frac{r^3}{3L}, \text{ which for the whole line, } r=L, \text{ is } \frac{1}{3}L^2;$$

and this, multiplied by the mass of the rod, is the moment of inertia.

If the rod revolve about a point in it, at distances a, b, from the extremities, the preceding integral must be taken between the limits

$$r=-b,\ r=a, \text{ giving } k^2=\frac{a^3+b^3}{3L}. \quad \text{If } a=b=\frac{1}{2}L, \text{ then } k^2=\frac{1}{12}L^2.$$

(2) Let the rod vibrate lengthwise, about a fixed axis directly above its middle point. Then, calling the perpendicular distance of the point of suspension a, and any distance along the rod from its middle point, x, the distance r of a particle at x from the point of suspension,

$$\text{is, } r^2=a^2+x^2, \therefore k^2=\frac{\int(a^2+x^2)dx}{L},$$

$$\text{the integral being taken from } x=-\frac{1}{2}L, \text{ to } x=+\frac{1}{2}L, \therefore k^2=a^2+\frac{1}{12}L^2.$$

(3) A circle turning about an axis, perp. to its plane, through the centre.

Putting R for the radius of the circle, its area will be πR^2; and at any distance r the area is πr^2, $\therefore dM=2\pi rdr$; this is a ring of particles round the axis, at distance r,

$$\therefore k^2=\frac{2\pi\int r^3dr}{\pi R^2}=\frac{r^4}{2R^2}, \text{ which, when } r=R, \text{ is } k^2=\frac{1}{2}R^2.$$

(4) Let the circle turn about a diameter: then $2y$ being a chord parallel to it,

$$dM=2ydx, \therefore k^2=\frac{2\int x^2ydx}{\pi R^2}=\frac{2\int x^2(R^2-x^2)^{\frac{1}{2}}dx}{\pi R^2}.$$

The numerator here satisfies the second criterion of integrability (p. 586), and taken between the limits

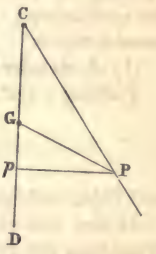
$$x=-R,\ x=R, \text{ the integral is } k^2=\frac{1}{4}R^2.$$

(5) A volume of revolution. The most convenient way of finding k^2, for a volume of revolution, is to regard the whole moment of inertia as generated by that of the generating circle of the solid: this generating moment $(k^2.2\pi y^2)$, by ex. 4, is $\frac{1}{2}\pi y^4$, the fixed axis being that of x: hence the moment of inertia generated is

$$k^2M=\frac{1}{2}\pi\int y^4dx, \therefore k^2=\frac{1}{2}\pi\frac{\int y^4dx}{M}.$$

If s be the arc, the k^2 of the surface generated is $k^2=\dfrac{2\pi\int y^3ds}{s}$.

611. When the axis about which a body turns passes through the centre of gravity, the moment of inertia is always less than for any other axis parallel to this. For conceive two axes, perp. to the plane of the paper, to pass through the points C, G, the latter being the centre of gravity of the body; and let P be a particle of the body projected on that plane, which may be allowed, as its distance from the axis remains unaltered. Join $PC(=r)$ and $PG(=r')$; draw also Pp, perp. to the line through C, G; then for this, and every point P, we have



$P.CP^2=P(CG^2+GP^2+CG.GP)$. But (421), $\Sigma(P.GP)=0$;

$$\therefore \Sigma(P.CP^2)=\Sigma(P.CG^2)+\Sigma(P.GP^2), \therefore \int r^2dM=M.CG^2+\int r'^2dM\ldots[1],$$

where it is to be observed that, in the first integral, dM is the continuous arc of particles, of which P is one, all equidistant from C; and in the second, that it is the continuous arc of particles, of which P is one, all equidistant from G: the integral extending to all the particles, or to the entire body in both cases. Hence, if to the integral $\int r'^2dM$, in reference to an axis through the centre of gravity G, we add the constant $M.CG^2$, the result will be the integral $\int r^2dM$, in reference to an axis parallel to the former through C.

The radius $k'=\sqrt{\dfrac{\int r'^2dM}{M}}$ is called the *principal radius of gyration.*

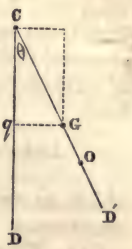
Putting then h for CG, we have from [1], $k^2=h^2+k'^2\ldots[2]$.

As an example, take the case of a slender rod L, revolving about an axis at the distance h from its middle: then (ex. 1),

$k^2=\frac{1}{12}L^2+h^2$: this will agree with the former determination by putting

$a+b$ for L, and $a-\frac{1}{2}(a+b)$ for h.

612. **Centre of Oscillation.**—When a heavy body oscillates about a horizontal axis by the force of gravity, it is regarded as a compound pendulum; and it is an important inquiry to seek what must be the length of a simple pendulum that would perform its oscillations in the same time: in other words, to find at what distance from the axis a weighty point must be suspended by a thread without weight, so that its vibrations may be identical with those of the body. Conceive the line CG, in last diagram, by a displacement of the body M, in a direction perp. to the axis through C, to be drawn out of the vertical, as in the annexed figure. Put $CG=h$, then the perp. $Gq=h\sin\theta$; and since the sum of the moments about C, of the several particles of the body, is the same as the moment of the entire mass M, supposed to be concentrated at G (415), it is $Mgh\sin\theta$; also the moment of inertia of the mass is $M(h^2+k'^2)$. Hence the moving force of gravity on the body, to turn it about C, is $Mgh\sin\theta$; and the resistance to this angular motion, or the mass moved, is expressed by $M(h^2+k'^2)$: therefore the angular acceleration is



$$\frac{Mgh\sin\theta}{M(h^2+k'^2)}=\frac{h}{h^2+k'^2}g\sin\theta\ (480).\quad \text{Make } CO=\frac{h^2+k'^2}{h}\ldots[1],$$

then, if the whole mass were concentrated in O, and connected to C by a thread without weight, the moving force producing rotation would be $MgCO\sin\theta$; and the resistance to this motion, that is, the moment of inertia, would be $M.CO^2$: and the angular acceleration would $\therefore$ be

$$\frac{MgCO\sin\theta}{M.CO^2}=\frac{1}{CO}g\sin\theta,\ \text{which is}=\frac{h}{h^2+k'^2}g\sin\theta,\ \text{as before.}$$

The point O, whose distance l from C is thus found to be

$$l=\frac{h^2+k'^2}{h}=h+\frac{k'^2}{h},$$

is called the *centre of oscillation of the body*, and l is $\therefore$ the length of the simple pendulum performing its oscillations in the same time.

The length $l=CO$ may be expressed otherwise thus:—

$$\frac{Mgh\sin\theta}{M(h^2+k'^2)}\ \text{may be written}\ gh\sin\theta\frac{\Sigma(m)}{\Sigma(r^2.m)},\ \text{so that we have}$$

$$gh\sin\theta\frac{\Sigma(m)}{\Sigma(r^2.m)}=\frac{g\sin\theta}{CO},\ \therefore\ CO=\frac{\Sigma(r^2.m)}{h\Sigma(m)}\ldots[2].$$

And it may be noticed here that if k be the radius of gyration, since $k^2=(h^2+k'^2)$, [1] shows that

$$\frac{h}{k^2}=\frac{1}{CO},\ \therefore\ h.CO=k^2;$$

so that the radius of gyration is a mean proportional between the distance of the centre of gravity, and the distance of the centre of oscillation, from the axis of suspension, $\therefore$ if two centres coincide all three coincide.

For an example of the application of the preceding expression for CO, see p. 616.

[The theory of the simple pendulum must be deferred till we have established the differential expressions for velocity and acceleration. See p. 614].

613. *The centres of oscillation and suspension are interchangeable;* that is, the body will vibrate in the same time whether it be suspended from C or from O. For suppose it to be suspended from O, and put $OG=h'$, so that l, the length of the simple pendulum corresponding to the centre of suspension C, is $l=h+h'$. The lengths l, l', corresponding to the centres of suspension C, O, are given by the formula deduced above: hence we have

$$l=h+h',\ l=h+\frac{k'^2}{h},\ l'=h'+\frac{k'^2}{h'},\ \therefore h'-\frac{k'^2}{h}=0,\ \therefore h'=\frac{k'^2}{h},\ \therefore l'=\frac{k'^2}{h}+h=l,$$

which, as we shall presently see, is a property of considerable practical importance.

614. **The Centre of Percussion.**—We have already seen that the resistance to angular motion about a fixed axis, or the moment of inertia, is expressed by

$$\Sigma(r^2.m),\ \text{that is, by}\ r_1.r_1.m_1+r_2.r_2.m_2+r_3.r_3.m_3+\ldots,$$

m denoting a particle of mass, and r its distance from the fixed axis, or its leverage. Now, if q be such a distance from the fixed axis that $\Sigma(r^2m)$ may also be equal to

$$q.r_1.m_1+q.r_2.m_2+q.r_3.m_3+\ldots=q\Sigma(r.m),$$

this quantity will equally represent the resistance to angular motion: it denotes, therefore, a moving force or pressure, which, if opposed directly to the resistance of inertia, will neutralize or destroy that resistance. Hence, if in the plane in which CD vibrates, and perpendicularly to CD at the distance q from C, the body strike an immovable obstacle, that obstacle will receive the full force of the shock, and none of it will be expended in straining the axis of suspension:—the mass will not incline to either side, but will, for the instant, be in equilibrio. The point thus determined, and whose distance from the fixed axis is

$$\Sigma(r^2.m)\div\Sigma(r.m),\ \text{is called the}\ \textit{centre of percussion:}$$

it is the point in which the moving force of the body may be regarded as concentrated, and we see that it is at the same distance from the axis of suspension as the centre of oscillation, for that distance is

$$(h^2+k'^2)M\div hM=\Sigma(r^2.m)\div\Sigma(r.m),\ (612).$$

The blow with which a body, having angular motion, strikes an obstacle, will be the hardest when it comes in contact with the resistance at the centre of percussion: thus, the hardest blow with a straight stick will be when the object is struck at $\frac{2}{3}$ the length of the stick from the hand, or extremity about which it turns—

$$\text{see ex. 1, p. 601, where}\ CO=k^2\div h=\frac{1}{3}L^2\div\frac{1}{2}L.$$

615. In the preceding articles the peculiar character of the several inquiries renders it necessary that we should view body or mass as made up of *particles*, or *molecules:* these are sometimes called *elements* of the body, as well as the differential slices of it adverted to at (547). Now what in common language is called a particle, or a physical atom, is not in strictness that which analysis symbolizes: physical particles necessarily have dimensions, and therefore magnitude; and a finite number of them will make up any finite body, however great. In analytical investigations the so-called particle is a *point.*

By Euclid, nothing is attributed to a point but position: he never in-

troduces it but in connection with magnitude;—magnitude either in actual existence, or in prospect: it is the same in analysis, with this additional information, namely, the *directions*, which the dimensions of the present or prospective magnitude take, are symbolized also. Thus, if (x, y, z) denote the position of a point in space, in reference to three rectangular co-ordinates, the calculus supplies us with the notation $dxdydz$, as symbolical not only of the point itself, but also of the directions which the dimensions are to take when magnitude is to originate from it. The symbols dx, dy, dz, do not mean little lengths:—they do not imply any lengths at all, but only the directions which foreseen lengths must take: magnitude is always in view, and magnitude generated in a certain way, though not yet in actual existence. In most physical investigations the student will no doubt find this strict rigour departed from, and $dxdydz$ employed to represent a physical particle of minute dimensions; but he should regard this form of expression as adopted more for brevity and convenience, than from necessity:—a mathematical particle is a mathematical point, and analysis does not require that we should give dimensions to it.

In the calculus, dx, or ds, is a zero-line; $dxdy$, a zero-surface; and $dxdydz$, a zero-volume; and these zeros are all as distinct from one another, and, to the analyst, as significant, as the finite geometrical quantities to which they relate.

There is another matter too which must be considered in physical applications. In the theory of curves (and the same may be said of the theory of surfaces) the analytical equations refer exclusively to the boundaries of the figure: (x, y) always denotes a point in the outline, and every point in the interior is excluded, in virtue of the condition which fixes the relationship of x and y, one of these quantities being an assigned function of the other: but for a point not in the boundary of the figure, no relation exists between the co-ordinates, x being the abscissa of every point in y: for such a point, therefore, the variations or differentials of x and y are quite independent.

We have thought it well to offer these remarks here, because students frequently entertain the impression that, in applying the calculus to physical inquiries, strict geometrical rigour must of necessity be dispensed with, and approximation only to the truth tolerated; which is a mistake. In order to illustrate and confirm what has just been said, we give the following short article on Attractions.

616. **On the Attraction of Bodies.**—Imagine any curve on the plane of the paper:—a circle, an ellipse, a parabola, &c.; and that the solid body here to be considered is generated by the rotation of this plane curve about the diameter or axis which lies across the page. Our object is to find an expression for the attraction of the body—supposing it to exercise such influence—on an assigned point in the axis, or in the axis prolonged.

Let P be any point (or particle) in the solid; call the perp. from P to the fixed axis, y; and the distance of the foot of y from the assigned point attracted, x: let also θ be the angle of rotation through which P has passed from its original situation on the plane of the paper; that is, let it be the angle which y makes with that plane. Then, having regard to the generation of the body, the symbolical representation of the point P will be $yd\theta dydx$; and as, in nature, the attraction on a point varies directly

as the mass (which, for uniform density, is as the magnitude) of the attracting particle, and inversely as the square of the distance,

the attractive influence of P on the proposed point may be expressed by $\frac{yd\theta dydx}{y^2+x^2}$.

Consequently the component of it, in the direction of the axis, is found by multiplying this expression by the cosine of the angle which the line from the attracted point to P makes with the axis,

which cosine is $\frac{x}{(y^2+x^2)^{\frac{1}{2}}}$: hence,

$$\frac{xyd\theta dydx}{(y^2+x^2)^{\frac{3}{2}}} = \text{the attraction of a particle } P \text{...[1].}$$

Now this expression must be viewed in reference to *attraction*, just as the point $yd\theta dydx$ is viewed in reference to *magnitude:* actual value is in neither case implied—nothing but the proper preparation is made for what is to come into existence. In each case the quantities are analytically exhibited in their zero state; but instead of the vague symbol 0, a character is impressed by the notation, upon this initial, or ultimate, or evanescent state of the quantity, which significantly marks its connection, by the bond of continuity, with what is to be, or with what has been. It is the business of the calculus thus to preserve the distinctive peculiarities of different quantities, even when by continuous diminution, according to prescribed laws, they have passed into their ultimate or zero state. If we integrate the above expression in reference to $d\theta$, from $\theta=0$ to $\theta=2\pi$, our point, or particle P, expands into a ring; and we have

$$2\pi\frac{xydydx}{(y^2+x^2)^{\frac{3}{2}}} = \text{the attraction of a ring of rad.} = y, \text{ at dist.} = x \text{ from the point.}$$

If this be now integrated in reference to dy, the ring becomes converted into a circular section of the solid; so that

$$2\pi\left\{1-\frac{x}{(y^2+x^2)^{\frac{1}{2}}}\right\}dx = \text{the attraction of a circle of rad.} = y, \text{ at dist.} = x \text{ from the point,}$$

the integration extending from $y=0$ to $y=$*anything*.

And finally, the integration of this, within the prescribed limits of x, of which y is a function, will give the attraction of the entire mass, when multiplied by the density of it.

It may be observed here, in reference to the above zero-expressions for a ring—a mere circumference—and for a circle without thickness, that if we eliminate the zero-factors, the attractions of rings and circles will respectively vary as the following quantities, where r is the rad. of the circle, and a the distance of the attracted point, on the central axis, from the plane of the circle:

$$2\pi\frac{ar}{(r+a^2)^{\frac{3}{2}}}, \text{ and } 2\pi\left\{1-\frac{a}{(r^2+a^2)^{\frac{1}{2}}}\right\}.$$

We shall give a practical illustration of the foregoing theory, but before doing so, in order that there may be no obscurity about the process, we shall briefly review what has been done, thus:—

1. Having a body of revolution in prospect, we introduce, in reference

to its generation, the angular quantity θ, and symbolize the point P thus, $d\theta dydx$, which point is not as yet controlled by any conditions, all the symbols being arbitrary and independent.

2. We next integrate [1] in regard to $d\theta$, just as if the other symbols were constants; and taking θ from 0 to 2π, we get the attraction for a ring of particles, the magnitude of this ring being altogether arbitrary, as well as the distance of it from the attracted point: x and y are therefore still as arbitrary and independent as at first.

3. By a second integration, in reference to dy, we fill up, as it were, this ring, and get a circle of any radius y, and at any distance x from the point. Up to this step no controlling conditions, establishing a relation between x and y, have been impressed upon these symbols; but it now becomes necessary, for the purpose of applying the general result thus obtained to the special case before us—to the exclusion of all points not belonging to the particular body of revolution with which we have to deal —to take into account the prescribed relationship between x and y, the origin being at the attracted point, and thus to take the final integral only between the limits of the proposed generating curve. The several steps are, therefore, these:—

$$1.\ \frac{xydydx}{(y^2+x^2)^{\frac{3}{2}}}\int d\theta,\quad 2.\ 2\pi xdx\int\frac{ydy}{(y^2+x^2)^{\frac{3}{2}}},\quad 3.\ 2\pi\int\left\{1-\frac{x}{(y^2+x^2)^{\frac{1}{2}}}\right\}dx;$$

but without actually performing the subordinate integrations, it is usual to symbolize the whole operation thus,

$$\iiint\frac{xyd\theta dydx}{(y^2+x^2)^{\frac{3}{2}}},\ \text{and the variables being at first quite independent,}$$

it matters not in what order the integrations are executed.

Problem 1. A paraboloid of revolution, of uniform density, attracts a particle at the focus: required the attraction of a slice comprehended between two planes perp. to the axis, one passing through the focus, and the other at the distance a from it.

The equation of the generating parabola, when the focus is the origin, is $y^2=4m(x+m)$, which value of y^2, substituted in the general expression (3) above, gives

$$\text{Attraction}=2\pi\int_0^a\left\{1-\frac{x}{(x^2+4mx+4m^2)^{\frac{1}{2}}}\right\}dx=2\pi\int_0^a\left\{1-\frac{x}{x+2m}\right\}dx$$

$$=2\pi\int_0^a\left\{1-1+\frac{2m}{x+2m}\right\}dx=4\pi m\int_0^a\frac{dx}{x+2m}=4\pi m\log\frac{a+2m}{2m}.$$

Problem 2. To find the attraction of a spheroid of uniform density, and very small excentricity, on a particle at its pole. Putting R for the fixed semi-axis,

$$y^2=(1\pm e)^2(2Rx-x^2),$$

the particle being at the origin, the upper sign applying when the spheroid is oblate, and the lower when it is prolate. Now e being regarded—as in the case of the earth—so small that its square and higher powers may be neglected, the preceding equation may be written

$$y^2=(1\pm 2e)(2Rx-x^2).$$

Hence the expression (3) is $2\pi \int \left\{1 - \frac{x}{(2Rx \pm 4Rex \mp 2ex^2)^{\frac{1}{2}}}\right\} dx$

$$= 2\pi \int \left\{1 - \sqrt{\frac{x}{(2R \pm 4Re) \mp 2ex}}\right\} dx$$

$$= 2\pi x - 2\pi \int \{(2R \pm 4Re) \mp 2ex\}^{-\frac{1}{2}} x^{\frac{1}{2}} dx,$$

which, between the limits $x=0$, $x=2R$, is $\frac{4}{3}\pi R\left(1 \pm \frac{4}{5}e\right)$,

the attraction of the spheroid. [The integral is obtained by the formula of reduction, p. 589.]

If $e=0$, that is, if the solid be a sphere, the attraction is $\frac{4}{3}\pi R$.

We leave the student to prove that if the attracted point be without the sphere, and at the distance a from its centre,

the attraction will be $\frac{4}{3}\pi \frac{R^3}{a^2}$.

But he must be apprized that, as these results are purely numerical, they merely express so many times the unit of attraction, whatever this may be assumed to be. Such conclusions, therefore, only show the relative proportions of the attractive forces of different bodies, or how these forces vary for the same body at different distances of the attracted point. If the attracted point be taken anywhere *within* the sphere, the result will show that the attraction towards the centre varies *directly* as the distance of the point therefrom.

It is plain from the above, when the point is *without*, that the attractions of spheres of the same density vary directly as the cubes of their radii, and inversely as the squares of the distances of the point attracted from their centres. Instead of the cubes of the radii, we may obviously substitute the volumes of the spheres, or when their densities are different, their masses. Suppose m to be the mass of the earth of radius r, m_1 that of a planet, M that of the sun, and D the distance between the planet and sun: then we have

$$\frac{m}{r^2} : \frac{M}{D^2} :: g : \frac{Mr^2}{mD^2}g = \text{attraction of sun on the planet.}$$

In like manner, $\frac{m_1 r^2}{mD^2}g$ = attraction of planet on the sun,

$$\therefore \frac{M+m_1}{m} \cdot \frac{r^2}{D^2}g = \text{united attraction of both.}$$

This expresses the attractive force mutually exercised by the sun and planet in terms of the terrestrial attraction g.

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617. **Velocity.**—If the velocity which a body has at t seconds continue uniform for t_1 seconds afterwards, then, calling the increment $t_1 - t$ of the time, Δt, and the corresponding increment of the linear space $s_1 - s$, Δs,

we have for the velocity v at the time t, $v = \frac{\Delta s}{\Delta t}$,

however small Δt, and consequently Δs, which depends on it, may be. But if the velocity during any finite interval of time be variable, then this expression is not the true velocity at the time t, but approaches to it nearer and nearer as the interval Δt diminishes. In fact, if the velocity continuously increase or continuously decrease during the interval Δt, it is plain that the preceding expression for v will always be intermediate between the true value of v at the time t, and the true value at the time t_1; that is, it will be a velocity intermediate between the velocities at the beginning and end of the interval Δt. Now when by continuously diminishing Δt this increment vanishes, the three velocities coalesce, and we therefore have for the variable velocity, at the time t,

$$v=\frac{ds}{dt},$$ whatever be the path s which the body describes.

In a case of uniform velocity this is of course constant, and equal to $\frac{\Delta s}{\Delta t}$, whatever be Δt.

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618. **Acceleration.**—Like as a uniform velocity generates equal increments of space in equal times, so a uniform acceleration of velocity generates equal increments of velocity in equal times. Putting, as usual, f for the amount of this uniform acceleration, we therefore have

$$f=\frac{\Delta v}{\Delta t}.$$ But if the acceleration during the interval Δt

continuously increase or continuously decrease, then this expression will always be an acceleration intermediate between the accelerations at the beginning and end of Δt, and since, when by continuously diminishing Δt this increment vanishes, the three accelerations coalesce, we must have for the acceleration at the time t,

$$f=\frac{dv}{dt}.$$ If the acceleration be uniform, then of course f=a constant.

Measuring s in the direction of the motion, we thus have $v=\frac{ds}{dt}$, $f=\frac{dv}{dt}=\frac{d^2s}{dt^2}$...[1]:

remarkable expressions of the calculus, showing (1) that the first differential coefficient derived from the space in reference to the time of describing it is the velocity generated, and (2) that the second differential coefficient is the acceleration generated, or, as it is usually called, the accelerating force. From these the fundamental equations of motion are readily derived thus:—

Fundamental Equations of Motion.—1. When the acceleration is constant.

From the second of [1], $dv=fdt$, $\therefore v=ft+C$...[2].

If t become 0 when v does, that is, if t be measured from the beginning of the motion, the constant C vanishes, and we infer that the velocities acquired in any times from the commencement are proportional to the times themselves: putting therefore $v=ft$ in the first of [1], we have

$$ds=ftdt,\ \therefore s=\frac{1}{2}ft^2=\frac{1}{2}vt...[3],$$

no constant being added because s vanishes with t. From [3] we infer that the spaces measured from the commencement of motion are as the

squares of the times of describing them. Eliminating t from [2], [3], disregarding the constant C, we have

$$v^2=2fs, \therefore v=\sqrt{(2fs)}\ldots[4],$$

showing that the spaces described are as the squares of the velocities acquired. If the time be reckoned not from the commencement of motion, but from when the body has acquired a velocity v_1, then the constant in [2] will not be 0 but $C=v_1$, since by hypothesis v_1 is what v becomes when $t=0$. In this case, the foregoing equations will be

$$v=ft+v_1,\ s=\frac{1}{2}ft^2+v_1t,\ v^2=2fs+v_1^2,\ \therefore v=\sqrt{(2fs+v_1^2)}\ldots[5].$$

2. When the acceleration is variable.—From equations [1] we have

$$\frac{f}{v}=\frac{dv}{ds},\ \therefore fds=vdv,\ \therefore \int fds=\frac{1}{2}v^2,\ s=\int\frac{v}{f}dv, \text{ also } t=\int\frac{ds}{v}\ldots[6].$$

The general equations [1] from which the foregoing expressions have been deduced, give the velocity and the acceleration of velocity along the path s of the moving body. But referring this path to two rectangular axes in its plane, we have for the component velocities and accelerations in the co-ordinate directions

$$\text{vel. along } X=\frac{dx}{dt},\ \text{vel. along } Y=\frac{dy}{dt}.\quad \text{Accel. along } X=\frac{d^2x}{dt^2},\ \text{along } Y=\frac{d^2y}{dt^2}.$$

Take, for illustration, the case of a projectile: here the equations of motion are

$$\frac{d^2x}{dt^2}=0,\ \frac{d^2y}{dt^2}=-g, \text{ since } y \text{ is measured in direction opposite to that in which } g \text{ acts.}$$

Multiplying by dt, and integrating, the component velocities are

$$\frac{dx}{dt}=C,\ \frac{dy}{dt}=C'-gt.$$

Multiplying again by dt, and integrating, we have

$$x=Ct,\ y=C't-\frac{1}{2}gt^2, \text{ or putting } aC \text{ for } C',$$

$$y=ax-\frac{1}{2}gt^2, \text{ as at page 433, where } \tan\alpha=a.$$

The following problems will serve further to illustrate these formulæ.

PROBLEM 1.—To determine the vertical motion of a heavy body towards the earth, the force of gravity varying inversely as the square of the distance from the centre.

Let $r=$ the rad. of the earth, a the distance of the body from the centre at the commencement of motion, x its distance after the lapse of t seconds: then the acceleration f at the time t will be given by the proportion

$$\frac{1}{r^2}:\frac{1}{x^2}::g:f,\ \therefore f=\frac{r^2}{x^2}g=-\frac{d^2x}{dt^2}, \text{ by [1].}$$

This is taken negatively because x is measured in direction opposite to that of the motion. Referring now to the third of [6], we have

$$v^2=-2\int\frac{r^2g}{x^2}dx=2\frac{r^2g}{x}+C.$$

The constant C depends upon the initial velocity, that is, upon the velocity of projection, or that which the body has at the distance a, where gravity begins to act: if this initial velocity be 0, then $C=-2r^2g\div a$, and thus the velocity the body has at any time t, that is, after having fallen from the distance a to the distance x, is completely determined: it is

$$v^2=2\frac{r^2g}{x}-2\frac{r^2g}{a}=2\frac{r^2g(a-x)}{ax}.$$

When the body arrives at the surface of the earth, that is, when $x=r$, it will have acquired the velocity

$$v=\sqrt{\frac{2rg(a-r)}{a}}, \text{ which, if } a=\infty, \text{ is } v=\sqrt{(2rg)}, \text{ since then } \frac{a-r}{a}=1.$$

Hence, from however great a height the body may fall, it cannot arrive at the earth with so great a velocity as this, so that if it were possible to project a body vertically upwards with such velocity, it would go on to infinity and never stop—supposing, of course, that there is no resisting medium or other disturbing force. Taking the radius of the earth at 3965 miles, the above expression for v would be $v=6{\cdot}9506$ miles; so that if a body were to be projected vertically upwards with a velocity of about 7 miles a second, and were to experience no resistance, it would never return to the earth.

In order to determine the time t, we have

$$dt=-\frac{dx}{v}=-\frac{1}{r}\sqrt{\left(\frac{a}{2g}\frac{x}{a-x}\right)}dx=-\frac{1}{r}\sqrt{\frac{a}{2g}}\cdot\frac{x}{\sqrt{(ax-x^2)}}dx,$$

$$\therefore t=-\frac{1}{r}\sqrt{\frac{a}{2g}}\cdot\int\frac{x}{\sqrt{(ax-x^2)}}dx=\frac{1}{r}\sqrt{\frac{a}{2g}}\left\{\sqrt{(ax-x^2)}+\frac{1}{2}a\cos^{-1}\frac{2x-a}{a}\right\},$$

which integral needs no correction, because $a-x$ and t become 0 at the same time. When $x=r$, that is, when the body reaches the earth's surface,

$$t=\frac{1}{r}\sqrt{\frac{a}{2g}}\left\{\sqrt{(ar-r^2)}+\frac{1}{2}a\cos^{-1}\frac{2r-a}{a}\right\}, \text{ which is } \infty \text{ when } a=\infty.$$

If the force of attraction be assumed to reside in a single point, then x, upon the body falling to this centre of attraction, becomes 0, and (still putting g for the acceleration at a given distance r) the whole time of falling from the distance a will be

$$t=\frac{1}{r}\sqrt{\frac{a}{2g}}\times\frac{1}{2}a\cos^{-1}-1=\frac{1}{r}\frac{\pi}{2\sqrt{2g}}a^{\frac{3}{2}}.$$

In like manner, if the fall be from any other distance a_1, the time t_1 is a similar expression,

$$\therefore t^2:t_1^2::a^3:a_1^3. \text{ That is—}$$

the squares of the times of falling from rest to the centre of attraction are as the cubes of the distances passed through.

Problem 2. A body revolves about a single centre of attraction, to find the relation between the trilinear spaces described by the radius vector and the times of describing them.

When a body, acted upon by a single centre of force, moves in a curve line, we may consider such motion to arise from a primitive impulse given to the body, which impulse alone would have caused it to describe a straight line; but being continuously acted upon by a force out of this

line, it is deflected from its wonted rectilinear course at the very commencement of motion, leaving that course a tangent to its actual path at the point of projection; and since nothing draws the body out of the plane in which the centre of force and the line of projection are situated, the path of the body must be a plane curve. Hence, placing the origin of the rectangular axes at the centre of force, the rectangular components (X, Y) of the acceleration F, at any point of the curve, are

$$X=F\cos\alpha,\quad Y=F\cos\beta$$

$$=F\frac{x}{r},\qquad =F\frac{y}{r},$$

where r is the radius vector, or line from the centre of attraction to the point (x, y) at which the body has arrived in any time t, from the commencement of motion. Hence the equations of motion are

$$\frac{d^2x}{dt^2}=-F\frac{x}{r},\quad \frac{d^2y}{dt^2}=-F\frac{y}{r},$$

the negative signs being prefixed because, F being attractive, the components X, Y, are measured in directions opposite to those of x, y.

To eliminate F, multiply the first of these by y, the second by x, and subtract the products: we thus have

$$\frac{yd^2x-xd^2y}{dt^2}=0,\ \therefore\ \frac{ydx-xdy}{dt}=C.$$

Multiplying this by dt, and integrating, we have (582),

$$2\text{ Sectorial Area}=Ct+C_1,\text{ which being 0 when } t=0,\text{ we have } C_1=0,$$

$$\therefore\text{ Sectorial Area}=\frac{1}{2}Ct.$$

Hence the sectorial areas described are as the times of describing them, so that *equal areas are described about S, the centre of attraction, in equal times*, whatever be the law (F) of the attracting force. This is Newton's remarkable *principle of equal areas;* in the particular case of the solar attraction on the planets, the principle was first announced by Kepler.

Problem 3. To determine the motion of a heavy body P rolling down a smooth curve, situated in a vertical plane.

The motion of the body P is due to that component of the force g of gravity which is in the direction of the curve, and which component is g multiplied by the secant of the angle which a tangent at P makes with the y of it:

$$\text{it is }\therefore g\frac{dy}{ds}\text{ (p. 515).}$$

This $\therefore$ is the acceleration f of P down the curve, $\therefore$ [1] p. 609,

$$-\frac{d^2s}{dt^2}=g\frac{dy}{ds},\ \therefore\ 2\frac{ds}{dt}d\frac{ds}{dt}=-2gdy,\text{ or } 2vdv=-2gdy,$$

$\therefore$ integrating, $v^2=-2gy+C$. [Note. s is measured *opposite* to the direction of motion.]

Let h be the height above the axis of x from which the body begins to descend, then when $y=h$, $v=0$: hence

$$0=-2gh+C,\ \therefore v^2=2g(h-y),\text{ or } v=\sqrt{\{2g(h-y)\}}.$$

Since this expression is independent of x, it follows that if a body

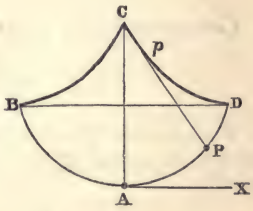
move without resistance from any assigned height h down any curves whatever to points all at the same vertical distance y from a horizontal plane, the acquired velocities will all be the same as that which would be acquired by a vertical fall through the same height, be the curves long or short; and the same would obviously be true were the actuating vertical force other than g. Hence no modification of path can increase the velocity with which a body arrives at any point from any given point, provided the motion be due to a force constant in intensity and direction.

The time of descent will depend on the nature of the curve: it is found thus. By [1],

$$v=-\frac{ds}{dt},\ \therefore dt=-\frac{ds}{v},\ \text{that is, in the case of gravity,}$$

$$dt=\frac{-ds}{\sqrt{\{2g(h-y)\}}},\ \therefore t=\int\frac{-ds}{\sqrt{\{2g(h-y)\}}}\ldots[\text{A}].$$

PROBLEM 4. *The Cycloidal Pendulum.*—The generation and properties of the curve called the cycloid will be discussed at p. 619, where it is shown that the curve is such that the evolute of each half of it is a curve equal and similar to that half. Let BAD be the curve here spoken of, CB, CD, being the two branches of its evolute: these curves, uniting in a cusp at C, are each the same in everything but position as the two equal arcs AB, AD. If, therefore, a simple pendulum be suspended from the point C, and be made to swing between the two equal cycloidal cheeks CB, CD, the bob P will describe an arc AP of the cycloid AD equal and similar to the arc Cp about which the flexible thread wraps itself. If l be the length CA of the thread, and s that of the arc AP, it is shown at page 620 that $s^2=2ly$, the curve being referred to AX, AC, as rectangular axes. The present problem is to find the time in which such a pendulum will make a complete vibration, or pass from P at the vertical height h above A to an equal height above A on the other side of it.



$$\text{Since } s=\sqrt{2ly},\ \therefore ds=\sqrt{\frac{l}{2y}}.dy,\ \therefore\ [\text{A}]\ t=\int\frac{-ds}{\sqrt{\{2g(h-y)\}}}=$$

$$\frac{1}{2}\sqrt{\frac{l}{g}}\int\frac{-dy}{\sqrt{(hy-y^2)}}=-\frac{1}{2}\sqrt{\frac{l}{g}}.\ \text{versin}^{-1}\frac{2}{h}y+C.$$

$$\text{Hence from } y=h \text{ to } y=y, \text{ we have } t=\frac{1}{2}\sqrt{\frac{l}{g}}.\left(\pi-\text{versin}^{-1}\frac{2}{h}y\right),$$

which expresses the number of seconds in descending from the height h to the height y, therefore the time of descent to the lowest point A, where $y=0$, is

$$t=\frac{\pi}{2}\sqrt{\frac{l}{g}},\ \text{so that for a complete vibration the time is } T=\pi\sqrt{\frac{l}{g}}.$$

Since this time depends solely on the length l of the thread, and is independent of the length PA of the semi-arc, it follows that the time will be the same whatever be the extent of the arc of vibration. Moreover,

at the same place, or for the same value of g, the times of vibration of different pendulums are as the square roots of their lengths, and for the same pendulum, at different places, the times are inversely as the square roots of the values of gravity.

Since CA is the radius of curvature of the cycloid at A, an arc, of which A is the middle point, may be taken so small as to differ insensibly from a circular arc described from C, so that a simple pendulum swinging freely from C—the cycloidal cheeks being removed—will perform a vibration in the time T above, if the extent of arc (the *amplitude*) be very small: but we shall consider this matter independently.

PROBLEM 5. *The Simple Pendulum.*—To determine the time of vibration in a circular arc of radius r.

Let the vertical and horizontal axes originate at the lowest point of the curve, and as it will be more convenient, take the former for the axis of x, and the latter for that of y: then for the equation of the circular path we have

$$y^2=2rx-x^2,\ \therefore \left(\frac{dy}{dx}\right)^2=\frac{(r-x)^2}{2rx-x^2},\ \therefore\ ds=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}dx=\frac{rdx}{\sqrt{(2rx-x^2)}}.$$

Consequently, by [A], changing y into x, we have

$$t=\int\frac{-ds}{\sqrt{\{2g(h-x)\}}}=\frac{r}{\sqrt{2g}}\int\frac{-dx}{\sqrt{\{(h-x)(2rx-x^2)\}}}.$$

This cannot be integrated in finite terms, but it may be integrated to any degree of approximation by putting the differential under the form

$$dt=-\frac{1}{2}\sqrt{\frac{r}{g}}\cdot\frac{dx}{\sqrt{(hx-x^2)}}\left(1-\frac{x}{2r}\right)^{-\frac{1}{2}};$$

$$\text{for since } \left(1-\frac{x}{2r}\right)^{-\frac{1}{2}}=1+\left(\frac{x}{2r}\right)+\frac{1.3}{2.4}\left(\frac{x}{2r}\right)^2+\frac{1.3.5}{2.4.6}\left(\frac{x}{2r}\right)^3+\ldots$$

the proposed differential may be replaced by a series of others, all of the same integrable form, namely, all of the form

$$\frac{-x^n dx}{\sqrt{(hx-x^2)}}.\quad \text{Now } \int_h^0\frac{dx}{\sqrt{(hx-x^2)}}=-\pi,$$

$$\int_h^0\frac{xdx}{\sqrt{(hx-x^2)}}=-\frac{1}{2}\pi h,\ \int_h^0\frac{x^2dx}{\sqrt{(hx-x^2)}}=-\frac{1.3}{2.4}\pi h^2,\ \text{and so on;}$$

hence for $2t=T$, the time of a whole vibration, we have

$$T=\pi\sqrt{\frac{r}{g}}\cdot\left\{1+\left(\frac{1}{2}\right)^2\left(\frac{h}{2r}\right)+\left(\frac{1.3}{2.4}\right)^2\left(\frac{h}{2r}\right)^2+\ldots\ldots\right\}.$$

We thus see that if the arc of vibration be so small that $\pi\sqrt{\frac{r}{g}}\cdot\frac{h}{8r}$, and the higher powers may be disregarded, the time of a complete oscillation will be the same as that in the cycloidal pendulum. Hence, for a very small amplitude or arc of vibration, the length of a circular pendulum beating seconds is $r=l=g\div\pi^2$, which

$\therefore$ varies directly as the force of gravity; and from the expression $T=\pi\sqrt{\frac{l}{g}}$,

this force varies at different places inversely as the squares of the times of vibration of the same pendulum.

Of course, a more exact expression for the time is $T=\pi\sqrt{\frac{l}{g}}\cdot\left\{1+\frac{h}{8l}\right\}$.

From careful experiments by Capt. Kater, the length of the seconds' pendulum in the latitude of London was found to be 39·13929 inches, but these and like experiments could not be made by such a simple pendulum as here described, for a heavy particle without sensible magnitude, and a thread without weight, form but an imaginary pendulum: the actual pendulum employed is a heavy mass, the centre of oscillation of which being found, the length of an ideal pendulum that would vibrate in the same time becomes known, and may be conceived to replace it. The valuable property (613) that the centres of oscillation and suspension of a heavy vibrating body are interchangeable, leads to a convenient practical way of finding the exact length of the equivalent simple pendulum. "The process used by Captain Kater was the following. The pendulum consisted of a bar AB of plate brass, about $1\frac{1}{2}$ inch broad, and $\frac{1}{8}$ inch thick. At A was a knife-edge of the hardest steel, its back bearing firmly against solid knees of brass; and at B a similar knife-edge. When the pendulum was in use these edges rested on horizontal plates of agate. At C was a large flat bob: at F was an adjustible weight, and at G a smaller weight, which by means of a screw could be adjusted with very great nicety. The principle of the operation was to observe the number of vibrations per day made by the pendulum when suspended on the knife-edge A, and again when suspended on B. If these were not equal, the sliding weights F and G were moved till they became equal. Then as A, B were at different distances from the centre of gravity, it was certain that B was the centre of oscillation corresponding to the centre of suspension A, and therefore that the distance between A and B was the length of the simple pendulum vibrating in the same time." Airy: "Figure of the Earth," Ency. Met.



Knowing the length of the seconds' pendulum, it will be easy to find the time of vibration of a pendulum of any other length, or the length of a pendulum to vibrate in any other time, the place being the same: thus, l being the length of the seconds' pendulum, and l_1 the length of any other, the number of seconds this last will vibrate in will be

$$\frac{t_1}{1}=\sqrt{\frac{l}{l_1}},\quad t_1=\sqrt{\frac{l_1}{l}}=t_1.$$ Let $l_1=20$ feet $=240$ in., then $t_1=\sqrt{\frac{240}{39.139}}=2\cdot5$ secs.

Again, to find the length of a pendulum oscillating once in 10 secs., we have

$$l_1=39.13929\times10^2=39139.29 \text{ inches.}$$

To find the gain or loss in a day by shortening or lengthening a pendulum.

Let l be the original length, n the number of vibrations it makes in 24 hours, Δl the length by which the pendulum is shortened, and Δn the number of vibrations gained in consequence: then, since the square roots of the lengths are as the times of one vibration, and that these are inversely as the number made in a day, we have

$$\sqrt{l} : \sqrt{(l-\Delta l)} :: \frac{1}{n} : \frac{1}{n+\Delta n}, \therefore \frac{n+\Delta n}{n} = \sqrt{\frac{l}{l-\Delta l}} = \left(1-\frac{\Delta l}{l}\right)^{-\frac{1}{2}} = 1+\frac{\Delta l}{2l}+\ldots;$$

hence, neglecting the subsequent terms because of the smallness of Δl, $\Delta n = n\frac{\Delta l}{2l}$.

Suppose the seconds' pendulum shortened by the 300th part of its length, then

$$\Delta n = 24.60.60\frac{l}{600l} = 24.6 = 144 = \text{no. of vibrations (or seconds) gained in a day.}$$

The foregoing general expression is the same when Δn is the number of vibrations *lost*, in consequence of an *increase* of Δl in the length of the pendulum. Hence, conversely, to find how much the pendulum must be lengthened or shortened in order that it may keep true time, the rule is this:—Multiply twice the length of the pendulum in inches by the number of seconds gained or lost in a day, and divide the product by the number of seconds in a day: the quotient will be the number of inches, or parts of an inch, by which the pendulum is to be lengthened or shortened.

619. Besides the ordinary purposes to which it is applied, the pendulum is also employed for the purpose of finding the heights of mountains, and the depths of mines: thus,

1. Let r be the rad. of the earth, h the height of a mountain, n the no. of vibrations in 24 h., and Δn the number lost at the top of the mountain, on account of the diminution of the force of gravity. Then since this force varies inversely as the distance from the centre

$$g : g' :: \frac{1}{r^2} : \frac{1}{(r+h)^2}. \text{ Also (p. 614) } \frac{1}{n} : \frac{1}{n-\Delta n} :: \frac{1}{\sqrt{g}} : \frac{1}{\sqrt{g'}} :: r : r+h,$$

$$\therefore \frac{r+h}{r} = \frac{n}{n-\Delta n} = 1+\frac{\Delta n}{n} \text{ nearly, } \therefore h = r\frac{\Delta n}{n}.$$

Hence the height varies as the number of vibrations, or seconds, lost.

2. Again: let h be the depth of a mine, n and Δn as before, Δn being still the number of vibrations *lost*, because *below* the surface the attraction varies directly as the distance from the centre: here, therefore,

$$g : g' :: r : r-h. \quad \text{Also } \frac{1}{n} : \frac{1}{n-\Delta n} :: \frac{1}{\sqrt{g}} : \frac{1}{\sqrt{g'}} :: \frac{1}{\sqrt{r}} : \frac{1}{\sqrt{(r-h)}},$$

$$\therefore \frac{r-h}{r} = \left(\frac{n-\Delta n}{n}\right)^2 = 1-2\frac{\Delta n}{n} \text{ nearly, } \therefore h = 2r\frac{\Delta n}{n}.$$

Hence the depth varies as the number of seconds lost.

(1) Suppose that at the top of a mountain a seconds' pendulum loses at the rate of 10 seconds in 24 hours: required the height of the mountain regarding the radius of the earth as 4000 miles.

$$\text{Here } r = 4000,\ n = 24.60.60,\ \Delta n = 10,\ \therefore h = \frac{4000.10}{24.60.60} = \frac{25}{54} = \tfrac{1}{2} \text{ a mile nearly.}$$

(2) Again: suppose the pendulum loses 3 seconds at the bottom of a mine: then

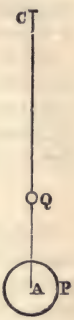
$$h = \frac{8000\times 3}{24.60.60} = \frac{5}{18} \text{ mile} = 1467 \text{ feet.}$$

We shall terminate these inquiries with a problem which will show the application of the formula [2] at p. 603. It will exhibit to the student how such expressions as $\Sigma(r.m)$ and $\Sigma(r^2.m)$, apparently implying the summation of an infinite number of quantities, may be often turned to account without any such usually impracticable summation at all.

PROBLEM.—A pendulum consists of a long slender rod CA, and two balls P, Q, of which the larger P is fixed at A, and the smaller Q is moveable: given the time t of a vibration to find the variation of the place of Q corresponding to a small variation of t.

Disregarding the weight of the rod, suppose the mass of each ball to be compressed into its centre. Let $CQ=x$, $CA=a$, then, by the formula referred to, the length l of the equivalent simple pendulum will be

$$l=\frac{Qx^2+Pa^2}{Qx+Pa}$$; and supposing the latitude of the place to be such that the length of the seconds' pendulum is 39·2 inches, we have



$$39{\cdot}2 : \frac{Qx^2+Pa^2}{Qx+Pa} :: 1 : t^2=\frac{1}{39{\cdot}2}\frac{Qx^2+Pa^2}{Qx+Pa},$$

$$\therefore 2tdt=\frac{Q^2x^2dx+2PQaxdx-PQa^2dx}{39{\cdot}2(Qx+Pa)^2},$$

$$\therefore dx=\frac{78{\cdot}4t(Qx+Pa)^2}{Q^2x^2+2PQax-PQa^2}dt.$$

Now the variations Δt, Δx, of t and x being small, they may be regarded in a matter of this kind as proportional to their differentials; suppose for instance

$a=158$ inches, $x=126{\cdot}9$, $P=20$ lbs., $Q=1$ lb.; then $t=2$ secs.;

let the change in the position of Q be such as to cause a loss of 15 secs. in 24 hours, or 86400 secs., then

86400 : 2 : : 15 : ·000347 the loss in 2 secs. $=\Delta t$: hence

$\Delta x=1{\cdot}84$ inches, the corresponding displacement of Q.

EXAMPLES FOR EXERCISE.

(1) Required the radius of gyration of a cylinder revolving about its axis, the radius of the circular base being r.

(2) Required the rad. of gyration of a sphere of radius $=r$.

(3) Required the rad. of gyration of a cone of altitude a, about its axis.

(4) A sphere oscillates about an axis at the distance a from the centre: required the distance below the centre of oscillation.

(5) A vibrating cone is suspended at its vertex: the alt. of the cone is a, and the rad. of its base na: required the length of the equivalent simple pendulum.

(6) A clock gains three minutes a day: by what part of the length l must its pendulum be lengthened to make it beat seconds?

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* The diameters of P, Q, are so small in comparison with the lengths CA, CQ, that the centres of gravity, and the centres of oscillation of P and Q, in reference to the point of suspension C, may be regarded as coincident. [See inference at bottom of p. 603.]

(7) A clock loses 1 min. a day: how much must the pendulum be shortened to make it keep true time, the place being London?

(8) A seconds' pendulum loses at the rate of 21·6 seconds when at the top of a mountain: required the height of the mountain, taking 4000 miles for the radius of the earth.

(9) If a pendulum be 39·37 inches in length instead of 39·1393, how much will the clock lose in a day?

(10) A seconds' pendulum at the bottom of a mine lost 2 secs. a day: required the depth of the mine.

(11) A sphere, whose radius is two feet, is suspended from a point 10 feet above its centre: in what time would one vibration be made by this pendulum on the top of a mountain 2 miles high, the acceleration of gravity at the bottom being $32\frac{1}{6}$ ft.?

(12) A small weight W is suspended from the extremity of a long slender rod of length a: it is required to find at what distance x from the point of suspension another small weight w must be placed, so that the pendulum may perform its vibrations in the least time possible.

(13) Of similar right cones each attracts a particle at its vertex: prove that the attraction varies as the height of the cone.

(14) An upright cylinder attracts a particle in the prolongation of its axis: prove that the attraction is proportional to the height of the cylinder plus the difference of the distances of the particle from its upper and lower edges.

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620. **The Ballistic Pendulum.**—This is a heavy block of wood suspended by an iron rod to a strong iron axis: it was devised by Robins, and employed by him, and afterwards by Hutton, for the purpose of determining the initial velocities of cannon-balls.

The projectile is fired directly against the vertical face of the suspended block, and enters the wood, which, with the ball in it, is set oscillating by the force of the impact.

Let B be the mass of the ball striking the pendulum with the velocity V, M the mass of the body struck, and M_1 the mass which, placed at the point of impact (regarded as in the vertical through the centre of gravity), might be substituted for M without modifying the motion: then putting a for the distance of M_1, or the point of impact, from the fixed axis, we have

$$M_1 = k^2 M \div a^2 \text{ (p. 601)};$$

and since the momenta are the same before and after impact (482), if v be the initial velocity of M_1 and B united, after impact, we must have

$$\left(B + \frac{k^2 M}{a^2}\right) v = BV, \quad \therefore v = \frac{BVa^2}{Ba^2 + Mk^2} = \text{initial vel. of } M_1.$$

Now l being the length of the equivalent simple pendulum, and θ the angle of a semi-vibration, the velocity of the extremity of l, when it descends to the lowest point, is the same as what would be acquired in falling down the versin θ to radius l; that is,

$$\text{it is } \sqrt{(2gl \text{ versin } \theta)} = 2 \sin \frac{\theta}{2} \sqrt{(gl)},$$

which is $\therefore$ to the initial velocity of M_1, as l is to a, that is,

$$l : a :: 2 \sin \frac{\theta}{2} \sqrt{(gl)} : \frac{BVa^2}{Ba^2 + Mk^2} \therefore \frac{BVal}{Ba^2 + Mk^2} = 2 \sin \frac{\theta}{2} \sqrt{(gl)} \ldots [1].$$

If h be the distance of the centre of gravity of the pendulum, including the ball, from the fixed axis, we shall have (612),

$$l=\frac{Ba^2+Mk^2}{(B+M)h},\ \therefore\ Ba^2+Mk^2=(B+M)hl,$$

$$\therefore\ [1],\ BVal=2\sin\frac{\theta}{2}(B+M)hl\sqrt{(gl)},\ \therefore\ V=2\sin\frac{\theta}{2}\frac{B+M}{B}\frac{h}{a}\sqrt{(gl)}.$$

Suppose, after impact, that the pendulum makes n oscillations in a minute, then (p. 614),

$$\frac{60}{n}=\pi\sqrt{\frac{l}{g}},\ \therefore\ \sqrt{(gl)}=\frac{60}{\pi n}g,\ \therefore\ V=2\sin\frac{\theta}{2}\frac{B+M}{B}\frac{60gh}{\pi na},$$

in which formula weights may be substituted for masses.

NOTE.—The impact is directed as nearly as possible to the centre of percussion, in order that little or no force may be expended in producing shock to the fixed axis. (For the practical details of numerous experiments with the ballistic pendulum, see Hutton's Tracts.)

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621. **The Cycloid.**—If a circle AmY, roll without sliding along a straight line BD, and continue always in the same plane, the curve $BAPD$ described by any point P in its circumference is called a *cycloid*. The following properties of it are obvious (see next figure).

The semi-circle $AmY=YD$, arc $PQ=QD$, $\therefore$ arc $Am=YQ$...[1],

Pmn being drawn parallel to DB: also CQ is parallel to AY.

Now the arc Am being = arc CP, $\therefore$ arc $mY=$ arc PQ,

$\therefore$ the chords PQ, mY are equal and parallel, $\therefore$ $mP=YQ$,

$\therefore$ [1], arc $Am=mP$...[2]. Put now $nP=x$, $An=y$,

also the angle $nOm=\theta$, then $AO-nO=y=r\ (1-\cos\theta)$,

and [2], arc $Am+nm=x=r(\theta+\sin\theta)$.

Or since $nm=\sqrt{(2ry-y^2)}$, and arc $Am=r\cos^{-1}\dfrac{r-y}{r}$,

$$\therefore\ x=r\cos^{-1}\frac{r-y}{r}+\sqrt{(2ry-y^2)},\ \therefore\ \frac{dx}{dy}=\sqrt{\frac{2r-y}{y}}=\frac{(2ry-y^2)^{\frac{1}{2}}}{y}.$$

This latter is the differential equation of the curve, $\dfrac{dx}{dy}$ being the trig. tan. of the angle which a straight line touching the curve at P makes with the axis of y.

Rectification of the curve.—Put s for the arc AP, then we have

$$ds=\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}dy=\sqrt{\left(\frac{2r}{y}\right)}dy,\ \therefore\ s=2(2r)^{\frac{1}{2}}y^{\frac{1}{2}}=\sqrt{8ry}=2\sqrt{2ry};$$

so that when $y=2r$, the arc AD is $4r=$twice the diam. of the generating circle.

Quadrature of the curve.—This is best effected as follows:—

$$\int xdy=xy-\int ydx=xy-\int(2ry-y^2)^{\frac{1}{2}}dy.$$

The second term of this integral, taken from $y=0$, to $y=2r$, evidently expresses the area of the semicircle of radius r, that is,

it is $\frac{1}{2}\pi r^2$; and the first term xy, between these limits, is $YD\times 2r=2\pi r^2$.

Hence the area of the semi-cycloid AYD is $\frac{3}{2}\pi r^2$,

so that the length of the whole cycloid is 4 times the diameter, and its area 3 times the area, of the generating circle.

Properties of the curve.—These are very numerous, we can here notice but one or two of the more important.

1. The tangent to the curve at P is parallel to the chord Am, and $\therefore$ PQ, parallel to mY, is perp. to that tangent.

For the expression for $\frac{dx}{dy}$ above is

$$\frac{dx}{dy}=\frac{\sqrt{(An.nY)}}{An}=\frac{nm}{An}=\tan nAm.$$

Hence the chord CP parallel to Am is tangent to the curve at P.

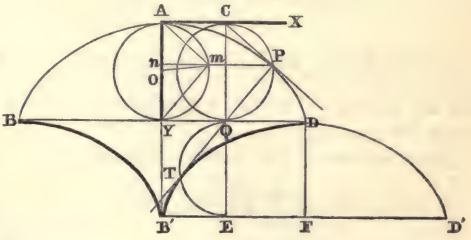
2. The cycloidal arc AP is twice the chord Am of the corresponding circular arc.

For $AP=s=2\sqrt{(AY.An)}=2Am$.

3. The evolute of a semi-cycloid is an equal semi-cycloid.

This property may be proved analytically by the method explained at (592), but the following geometrical proof is likely to be preferred by the student.

Conceive a cycloid $B'DD'$, equal to the cycloid BAD, to be placed in reference to the latter, as in the figure, its vertex being at D. Draw PQ, QT, chords of the generating circles: then by (1) above, QT touches the lower cycloid at T. Also $EB'=$ arc ET, and $FB'=$ arc ETQ, $\therefore$ $FE=DQ=$ arc $TQ=$ arc PQ: hence the angles TQE, PQC, are equal, and $\therefore$ PQT is a straight line, and, as just shown, it is a tangent to the curve at T, and, moreover, $PT=2TQ=$ cycloidal arc TD by (2) above. Consequently the normal PT, at any point P of the semi-cycloid AD, is a tangent to the equal semi-cycloid DB', so that DB' is the evolute of DA, and a string, applied round the arc $B'D$, if unwrapped from D, and kept stretched, will by its extremity P trace out the semi-cycloid DA. The entire length of the string, or of the semi-cycloid, as shown above, is twice the diameter of the generating circle$=4r$: if this length be called l, then $AP^2=s^2=2ly$. In the figure at p. 613, the semicycloids above are inverted, B' being the point of suspension of the cycloidal pendulum.



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622. **The Catenary.**—This is the curve formed by a heavy flexible chain of uniform thickness and density when suspended from its extremi-

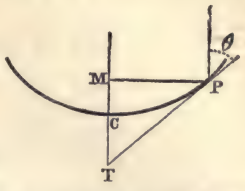
ties, and allowed to hang freely. Let C be the lowest point of the catenary,

a= that length of chain of which the weight equals the tension at C,

t= that length of chain of which the weight equals the tension at P,

x= the vertical line CM, y= the horizontal line, MP.

s=CP.



This portion CP of the chain is kept at rest by the tension at P, the tension at C, and its own weight, which is proportional to its length: these three forces are thus proportional to t, a, and s, respectively, and they act in directions parallel to TP, PM, and MT, $\therefore$ by the triangle of forces,

$$\frac{s}{a}=\frac{MT}{PM}=\cot T=\frac{dx}{dy}\ldots[1],\ \frac{t}{a}=\frac{TP}{PM}=\frac{ds}{dy}\ldots[2].\quad \text{Also}$$

$$\left(\frac{ds}{dy}\right)^2=1+\left(\frac{dx}{dy}\right)^2=\frac{s^2+a^2}{a^2},\ \text{and}\ \therefore \frac{dy}{ds}=\frac{a}{\sqrt{(s^2+a^2)}},\ \therefore\ dy=\frac{ads}{\sqrt{(s^2+a^2)}},$$

and since $s=0$ when $y=0$, the integral is

$$y=a\log\frac{s+\sqrt{(s^2+a^2)}}{a},\ \therefore e^{\frac{y}{a}}=\frac{\sqrt{(s^2+a^2)}+s}{a},\ \text{and}\ \therefore e^{-\frac{y}{a}}=\frac{\sqrt{(s^2+a^2)}-s}{a}$$

$$\therefore s=\frac{a}{2}(e^{\frac{y}{a}}-e^{-\frac{y}{a}})\ldots[3].\ \therefore[1],\frac{dx}{dy}=\frac{1}{2}(e^{\frac{y}{a}}-e^{-\frac{y}{a}})=\cot T,\ \text{or}\ \cot\theta\ldots[4],$$

$$\therefore x=\frac{1}{2}\int(e^{\frac{y}{a}}-e^{-\frac{y}{a}})dy=\frac{a}{2}(e^{\frac{y}{a}}+e^{-\frac{y}{a}})-a\ ;\ldots[5].$$

Differentiating [3], $\frac{ds}{dy}=\frac{1}{2}(e^{\frac{y}{a}}+e^{-\frac{y}{a}})$, $\therefore$ [2], $t=\frac{a}{2}(e^{\frac{y}{a}}+e^{-\frac{y}{a}})\ldots[6]$.

Now if N be the number of which the Nap. log is $\frac{y}{a}$,

then $N=e^{\frac{y}{a}}$, and the preceding equations give

$$s=\frac{a}{2}\left(N-\frac{1}{N}\right)=a\cot\theta,\ \cot\theta=\frac{1}{2}\left(N-\frac{1}{N}\right),x=\frac{a}{2}\left(N+\frac{1}{N}\right)-a=t-a,$$

$$t=\frac{a}{2}\left(N-\frac{1}{N}\right);$$

so that whatever value be assumed for $\frac{y}{a}$ the corresponding value of N becomes known,

and thence the proper values of s, θ, x, and t.

These have been computed for y=100 units of any kind, and a= a succession of equi-different values, as also for a=100, and y= a succession of values 1, 2, 3, &c., and the results tabulated for use in reference to the construction of suspension bridges. For a specimen of these tables and of their practical use, the student is referred to Vol. iii. of the Woolwich Course.

We must here bring these inquiries to a close; and for further applications of the Calculus, must refer the student to works specially devoted

to analytical Mechanics, and other physico-mathematical subjects. The writings of Professor Potter, of University College, London, are among the most recent and best that he can consult.

END OF THE CALCULUS AND ITS APPLICATIONS.

NOTE: ON INTERPOLATION (see p. 162).

In order to be able to insert between the terms of a proposed series other terms, such that all may follow the same general law, it will be necessary to solve the following problem, namely,

PROBLEM. To determine the general relation which exists between two variables x and y, from knowing the particular independent cases,

$$x=\alpha,\ y=\beta;\ x=\alpha_1,\ y=\beta_1;\ x=\alpha_2,\ y=\beta_2,\ \&c.$$

Let the general relation be

$$y=A+Bx+Cx^2+Dx^3+\ldots,$$

then from the preceding known conditions, which we are to suppose to be as many in number as there are coefficients A, B, C, &c., to be determined, we have

$$\left.\begin{aligned} \beta &= A+B\alpha+C\alpha^2+D\alpha^3+\ldots\ldots \\ \beta_1 &= A+B\alpha_1+C\alpha_1{}^2+D\alpha_1{}^3+\ldots \\ \beta_2 &= A+B\alpha_2+C\alpha_2{}^2+D\alpha_2{}^3+\ldots \\ &\&c. \qquad\qquad \&c. \end{aligned}\right\}\ldots[1],$$

$$\therefore \beta_1-\beta = B(\alpha_1-\alpha)+C(\alpha_1{}^2-\alpha^2)+D(\alpha_1{}^3-\alpha^3)+\ldots,$$

$$\beta_2-\beta_1 = B(\alpha_2-\alpha_1)+C(\alpha_2{}^2-\alpha_1{}^2)+D(\alpha_2{}^3-\alpha_1{}^3)+\ldots,$$

$$\&c. \qquad\qquad \&c.$$

$$\left.\begin{aligned} \therefore \frac{\beta_1-\beta}{\alpha_1-\alpha} &= B+C(\alpha_1+\alpha)+D(\alpha_1{}^2+\alpha_1\alpha+\alpha^2)+\ldots\ldots=a \\ \frac{\beta_2-\beta_1}{\alpha_2-\alpha_1} &= B+C(\alpha_2+\alpha_1)+D(\alpha_2{}^2+\alpha_2\alpha_1+\alpha_1{}^2)+\ldots=a_1 \\ &\&c. \qquad\qquad \&c. \end{aligned}\right\}\ldots[2],$$

the values a, a_1, &c., being known because the first member of each equation is known: and A has been eliminated.

Proceeding in a similar way with the group of equations [2], B may be eliminated, and we shall have

$$\frac{a_1-a}{\alpha_2-\alpha}=C+D(\alpha_2+\alpha_1+\alpha)+\ldots=b\ldots[3],$$

and so on. And by continuing this process we may thus eliminate the coefficients one after another, till we come to the last, the value of which may then be determined from the final simple equation. Suppose the general relation to be $y=A+Bx+Cx^2$, and that three conditions are given: the equations [2], [3], become

$$\frac{\beta_1-\beta}{\alpha_1-\alpha}=B+C(\alpha_1+\alpha)=a,\ \frac{\beta_2-\beta_1}{\alpha_2-\alpha_1}=B+C(\alpha_2+\alpha_1)=a_1,\ \frac{a_1-a}{\alpha_2-\alpha}=C=b.$$

Substituting this value of C in the first equation, we have

$B=a-b(\alpha_1+\alpha)$, and since the assumed relation gives $\beta=A+B\alpha+C\alpha^2$,

we get $A=\beta-a\alpha+b\alpha\alpha_1$, so that the relation sought is

$y=\beta-a\alpha+b\alpha\alpha_1+(a-b\alpha_1-b\alpha)x+bx^2$, or $y=\beta+a(x-\alpha)+b(x-\alpha)(x-\alpha_1)$.

If four independent conditions had been given, we should have got

$$y=\beta+a(x-\alpha)+b(x-\alpha)(x-\alpha_1)+c(x-\alpha)(x-\alpha_1)(x-\alpha_2)\ldots[4],$$

and so on. Now the general relation between x and y may evidently be such as not to agree with the form we have here assumed for it: y may not be a rational integral function of x: yet if we regard the equation between x and y as that of some curve, the given independent conditions will denote so many points in that curve, and by proceeding as above, we shall determine a parabolic curve necessarily passing through all these points, and the closer they lie together the more nearly will this parabolic curve approach to coincidence with the assigned curve between the limits of the given co-ordinates. Between these limits, therefore, the general relation sought may be approximated to closer and closer—by replacing it, as above, by a relation of the assumed form—the more numerous the given conditions are, and the smaller the intervals between the successive values of x. Let these successive values α, α_1, α_2, &c., be in arithmetical progression, h being the common difference,

so that $\alpha_1=\alpha+h$, $\alpha_2=\alpha+2h$, $\alpha_3=\alpha+3h$, &c., and let $x=\alpha+h_1$,

$\therefore x-\alpha=h_1$, $x-\alpha_1=h_1-h$, $x-\alpha_2=h_1-2h$, &c.

Put also $\beta_1-\beta=\Delta_1\beta$, $\beta_2-\beta_1=\Delta_1\beta_1$, $\beta_3-\beta_2=\Delta_1\beta_2$, &c.: then [2],

$$a=\frac{\Delta_1\beta}{h},\ a_1=\frac{\Delta_1\beta_1}{h},\ a_2=\frac{\Delta_1\beta_2}{h},\ \&c.$$

In like manner, putting $\Delta_2\beta$, $\Delta_2\beta_1$, &c., for the second differences,

$\Delta\beta_1-\Delta\beta$, $\Delta\beta_2-\Delta\beta_1$, &c., we have by [3],

$$b=\frac{\Delta_2\beta}{1.2h^2},\ b_1=\frac{\Delta_2\beta_1}{1.2h^2},\ \&c.$$

Putting also $\Delta_3\beta$, &c., for the third differences $\Delta_2\beta_1-\Delta_2\beta$, &c., we get,

$$c=\frac{\Delta_2\beta}{1.2.3h^3},\ \&c.$$

Hence the formula [4] becomes

$$y=\beta+\frac{h'}{h}\Delta_1\beta+\frac{h'(h'-h)}{h.2h}\Delta_2\beta+\frac{h'(h'-h)(h'-2h)}{h.2h.3h}\Delta_3\beta+\ldots.$$

Or putting n for the number $h'\div h$, whether this be whole or fractional,

$$y=\beta+n\Delta_1\beta+\frac{n(n-1)}{1.2}\Delta_2\beta+\frac{n(n-1)(n-2)}{1.2.3}\Delta_3\beta+\ldots,$$

which is an extension of the formula at p. 160, to the case of n fractional. An example of its application is given at p. 162.

END OF THE ELEMENTARY COURSE.

ANSWERS TO THE EXAMPLES FOR EXERCISE.

ALGEBRA.

Definitions, p. 4.

(1) 4. (2) 8. (3) 138.
(4) 540. (5) 35. (6) 4.
(7) 0. (8) 3.

Addition, p. 5.

(1) $24a$. (2) $7by+9a$.
(3) $22\frac{1}{2}ax-7\frac{1}{2}cz$.
(4) $11\frac{1}{2}mx-16ny-16\frac{a}{b}$.
(5) $11axy+b$.
(6) $10ax+4$.
(7) $cz-16$.
(8) $-17\frac{1}{2}axy+6\frac{1}{2}bz$.

Case II., p. 6.

(4) $2ax+2y+41$.
(5) $2x+3y+8a+b+20$.
(6) $3abx+10ay$.
(7) $6mx+13ny+8az+20$.
(8) $11\frac{x}{y}+10\frac{y}{x}+3ax+6\frac{a}{b}-3$.
(9) $15abx-19aby-15ay+4abz-12ax+x$.

Subtraction, p. 7.

(1) $2ax+4by+7$.
(2) $-3abx-2z$.
(3) $-5xyz+8xy+6$.
(4) $-5mxy-7nyz+2a-8$.
(5) $15az-6by+9x-1$.
(6) $3ax-1\frac{1}{2}by-2z-6$.
(7) $7py-26az-7n-m$.
(8) $-6cxz-9ey+4kz+a+b-4$.

Simple Equations, p. 10.

(1) $x=7$. (2) $x=9$.
(3) $x=4\frac{2}{3}$. (4) $x=4$.
(5) $x=60$. (6) $x=6$.
(7) $x=-1\frac{13}{17}$. (8) $x=\frac{b+m}{a-c}$.
(9) $x=\frac{4}{a}$. (10) $y=12$.

Questions, p. 11.

(6) $x=210$. (7) $x=40$.
(8) $8\frac{1}{2}$ and $21\frac{1}{2}$.
(9) £50, £30, £20.
(10) 4. (11) $6\frac{2}{3}$.
(12) $3^h \cdot 20^m$ past 12.
(13) $17\frac{1}{2}$ miles.
(14) $21\frac{1}{3}$ days.
(15) 120 gals.
(16) 100 lbs.
(17) 1st. 6 min.; 2nd. 4 min.
(18) 90 of each sort.
(19) 11 halfcr. 29 shilgs.
(20) *A*, £24; *B*, £36; *C*, £56.

Brackets, p. 13.

(1) $[3(a+b)+1](x+y)$.
(2) $5\sqrt{(x-a)}$.
(3) $(3a+8)\sqrt[3]{(x-y)}+(a-3)\sqrt{(x+y)}$.
(4) $3(2z-y^2)-2x^2+a-b$.
(5) $(3a-7)(x-y)+(a-b-4)xy-4z$.
(6) $9(x+y)-(a+b)(x^2-y^2)$.
(7) $(4a-3b)\sqrt{x}+(6a-5b)\sqrt{y}-b-d$.

Multiplication, p. 14.

(1) $18a^5b^2x^2y$.
(2) $-28a^2b^2xy^7$.
(3) $15a^3b^3x^2y^5$.
(4) $-14a^3bx^3y^4$.
(5) $30al^3xy^7$.
(6) $-\frac{3}{4}abx^3y^4z$.

Case II., p. 14.

(1) $12a^3b^2x^3-6a^2bx^2y^2$.

(2) $-12a^2bx^3y^2z-26ab^2xz^4+10abxz$.
(3) $-2a^2bcx+3ab^2cxy-4abc^2xz$.
(4) $4a^2x^2y+8abxy^2+12axy$.
(5) $3abx^5-\frac{9}{4}a^2x^3-\frac{3}{4}ax^2$.
(6) $-a^2b^3x^2y+\frac{5}{6}abcxy^2$.

Case III., p. 16.

(1) $x^4-2a^2x^2+a^4$.
(2) x^6-1.
(3) x^6-a^6.
(4) $x^3+\frac{3}{4}x^2+\frac{3}{4}x+1$.
(5) $6x^3-23x^2+26x-8$.
(6) x^4-y^4.
(7) $2x^4-4\frac{1}{2}x^3+5x^2-4\frac{1}{2}x+2$.
(8) $18ax^3-3(5a+2)x^2-(18a-5)x+6$.
(9) $abx^{2m}+(a^2+b^2)x^{m+n}+abx^{m+n}$.
(10) $\frac{1}{3}x^3+1\frac{1}{6}x^2-\frac{7}{9}x+\frac{4}{3}$.
(11) $x^4-\frac{5}{4}x^3+1\frac{3}{8}x^2-\frac{1}{2}x$.
(12) $a^2x^2-b^2y^2$.
(13) $9a^2x^2-42abxy+49b^2y^2$.

Division, p. 17.

(1) $-3ax^2y$. (2) $-6a^2xy^2$.
(3) $9x^2y^3\sqrt{z}$. (4) $\frac{2b^2x^2}{a}$.
(5) $-\frac{7a^3x^3y}{3bc}$. (6) $-\frac{7a^2x}{4by}$
(7) a^nb^mx. (8) $\frac{6a^4x^4}{b^3}$.

Case II., p. 17.

(1) $a^3-3bx+5x^2$.
(2) $12x-3ax^2+\frac{4}{x}$.
(3) $4ax^2-3by^3+2z^4$. (4) $\frac{3x}{5}-9$.
(5) $8a^2x+a-2xy$. (6) $\frac{1}{2ab}$.

Case III., p. 18.

(1) $2x+3$. (2) $2x^3+4x^2+8x+16$.
(3) $x^3+x^2y+xy^2+y^3$.
(4) $a^4+a^3x+a^2x^2+ax^3+x^4$.
(5) $5x^3+4x^2+3x+2$.
(6) $2x^2+2x+\frac{5(3x-4)}{3(x-1)}$.
(7) $1+x+x^2+x^3+$&c.
(8) $x^2+(a+b)x+a^2+ab+c+\frac{a^3+a^2b+ac+d}{x-a}$.

Fractions, p. 19.

(1) $\frac{axy+b}{y}$. (2) $\frac{x-2a}{x}$.
(3) $\frac{2b}{a+b}$. (4) $\frac{(a+b)^2}{ab}$.
(5) $\frac{x^3-3x^2+8x-9}{x-2}$.
(6) $\frac{2x^2}{x+a}$. (7) $-\frac{2a^4}{x^2+a^2}$.
(8) $\frac{a^2+b^2-c^2}{2ab}$.

Page 20.

(1) $a+2b+\frac{b^2}{a}$. (2) $x+y+\frac{3}{x-y}$.
(3) $a-b+\frac{b^2}{a+b}$. (4) $3x+2+\frac{2}{3ax}$.
(5) $x^2+3x+7+\frac{7(x-1)}{x^2-3x+2}$. (6) 1.
(7) $3x^3+2x^2+2+\frac{14x+1}{4x^2-7x}$.

Addition—Subtraction, p. 21.

(1) $\frac{5a-x}{6}$. (2) $\frac{46x-11}{14}$.
(3) $\frac{4a^2}{a^2-b^2}$. (4) $\frac{1}{1-x^2}$. (5) $\frac{a-x}{a+x}$.

Multiplication—Division, p. 21.

(1) $\frac{3x(2x-3)}{20y}$. (2) $\frac{4cx^3}{5y^2}$. (3) 2.
(4) $\frac{a-x}{b-y}$. (5) $4(x+a)$. (6) 1.
(7) $\frac{12x}{a^2+x^2}$. (8) $\frac{(1+x^2)(1-y^2)}{xy}$.
(9) $\frac{x(x+y)}{x-y}$. (10) $\frac{3(20-x)}{4(3x+4)}$.
(11) $\frac{15}{28\sqrt{(a+x)}}$. (12) $\frac{ab(a^2+b^2)}{a^2+ab+b^2}$.

Simple Equations, p. 27.

(1) $x=4$. (2) $x=1$. (3) $x=47\frac{1}{2}$.
(4) $x=144$. (5) $x=12$.

(6) $x=\frac{1-a^3-a^2}{a^2-a}$. (7) $x=6$.

(8) $x=100$. (9) $x=1$. (10) $x=24$.

(11) $x=1$. (12) $x=\frac{4}{5}$

(13) $x=\left(\frac{2a}{1+a}\right)^2$. (14) $x=\frac{2a\sqrt{b}}{b+1}$.

(15) $x=a(b-1)$.

(16) $x=a\left(1-\sqrt{\frac{1}{(b-1)^2+1}}\right)$.

(17) $x=\frac{b^2}{4a}-a$. (18) $x=\frac{1}{5}$.

(19) $x=1-\frac{1}{\sqrt{(a+1)}}$. (20) $x=\frac{4}{5}a$.

(21) $x=4$. (22) $x=2$. (23) $x=1$.

Simple Equations.

Two Unknowns, p. 30.

(1) $x=5$, $y=7$. (2) $x=7$, $y=16$.
(3) $x=3$, $y=5$. (4) $x=19$, $y=3$.
(5) $x=11$, $y=4$. (6) $x=8$, $y=\frac{1}{2}$.
(7) $x=12$, $y=6$. (8) $x=13$, $y=3$.
(9) $x=\frac{1}{2}$, $y=\frac{1}{3}$. (10) $x=21$, $y=63$.
(11) $x=13$, $y=7$. (12) $x=\cdot02$, $y=2\cdot9$.
(13) $x=-\frac{41}{19}$, $y=-4\frac{84}{95}$.
(14) $x=\left(-\frac{7}{2}\right)^2$, $y=3^2$.

Simple Equations.

Three Unknowns, p. 31.

(1) $x=24$, $y=6$, $z=23$.
(2) $x=1$, $y=12$, $z=60$.
(3) $x=2$, $y=-3$, $z=1$.
(4) $x=12$, $y=20$, $z=30$.
(5) $x=3$, $y=2$, $z=1$.
(6) $x=2$, $y=4$, $z=7$.
(7) $x=10$, $y=2$, $z=3$.
(8) $x=2a$, $y=2b$, $z=2c$.

Simple Equations. *Questions*, p. 34.

(1) *A*, £50; *B*, £30; *C*, £20.

(2) 14 rye, 36 barley. (3) $\frac{3}{8}$.

(4) Man, $21\frac{3}{7}$ days; Woman, 50.
(5) 32*s.*; 14 poor. (6) 18, 22, 10, 40.
(7) $a=3$, $b=-1$, $c=-2$. (8) 240.
(9) $5\frac{115}{271}$ days.
(10) $22\frac{3}{4}$ lb. lighter; $201\frac{1}{4}$ heavier.
(11) *A*, 105 min.; *B*, 210; *C*, 420. All, 60.
(12) 420 oz. copper; 85 oz. tin.
(13) Tin, 74 lb.; lead, 46 lb. (14) 234.
(15) *A*, 8 oz.; *B*, 5 oz.; *C*, 3 oz.
(16) 160 oz.; 623 sovs.
(17) Add 1 h. $5\frac{5}{11}$ min. successively.
(18) 8 gal., 5 gal., and 5 gal.

(19) *A*, $\frac{1}{17}(22a-9b-8c)$; *B*, $\frac{1}{17}(21b-4c-6a)$; *C*, $\frac{1}{17}(20c-3b-4a)$.

(20) $\frac{(a-b)^n}{a^{n-1}}$. (21) 62*s.* and 34*s.*

(22) £81, 41, 21, 11, 6.

Square Root of a Polynomial, p. 39.

(1) $2x^2-x-1$. (2) $3x^2-a$.
(3) $3x^3-2x^2+x-4$.
(4) $2x^3+3x^2-x+1$.

(5) $2xy^2-3x^2y+2x^3$. (6) $\frac{x}{y}+\frac{y}{x}+1$.

Cube Root of a Polynomial, p. 40.

(1) x^2-2x+1. (2) x^2+2x-4.
(3) $2x+3$. (4) x^2-ax+a^2.

Surds. *Reduction*, p. 41.

(1) $\frac{1}{7}\sqrt{14}$. (2) $2\sqrt{15}$. (3) $3\sqrt{-6}$.

(4) $2ax^2y^3\sqrt{3}$. (5) $\frac{x}{y^2}\sqrt{y}$.

(6) $\frac{a}{3b^3}\sqrt{6ab}$. (7) $\frac{1}{4ab^2}\sqrt{2b}$.

(8) $x\sqrt[3]{(a+bx)}$. (9) $(a+b)\sqrt{3x}$.
(10) $\sqrt{(a^2-x^2)}$.

Surds. *Addition—Subtraction*, p. 42.

(1) $10\sqrt{2}$. (2) $\frac{2}{5}\sqrt{10}$. (3) $4a^2\sqrt{3x}$.

(4) $25\sqrt{2}$. (5) $(a-x)\sqrt{x}$.

(6) $a\sqrt{ab}$. (7) $\frac{67}{60}\sqrt{15}$.

(8) $a\sqrt{(a+b)}$. (9) $2\sqrt{-1}$.
(10) $2(6\sqrt{2}+\sqrt{6})+ax\sqrt{b}$.

Surds. *Multiplication—Division*, p. 42.

(1) $6\sqrt[6]{432}$. (2) $7\sqrt[3]{15}$. (3) $\frac{3}{8}\sqrt{3}$.

(4) $32\sqrt[3]{2}$. (5) $\sqrt[6]{3}$. (6) $\frac{2}{3}\sqrt[6]{1800}$.

(7) $\sqrt{6}-\frac{5}{3}\sqrt[6]{108}$. (8) $\frac{1}{a}\sqrt[4]{a^3b}$.

(9) $a^{\frac{m-n}{mn}}$. (10) $\frac{1}{2}(\sqrt[6]{12}+\sqrt[4]{2})$.

Binomial Surds, p. 43.

(1) $-\frac{3}{17}(2\sqrt{7}+3\sqrt{5})$.

(2) $\frac{1}{6}(3+\sqrt{3})$. (3) $\frac{4\sqrt{3}-3\sqrt{2}}{5}$.

(4) $\frac{x(a+\sqrt{x})}{a^2-x}$. (5) $\frac{13\sqrt{10}-42}{37}$.

(6) $\frac{3(\sqrt[4]{5}-\sqrt[4]{3})(\sqrt{5}+\sqrt{3})}{2}$.

(7) $2(\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4})$.

(8) $\frac{a\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}}{x+y}$. (9) $2\sqrt{3}$.

Binomial Surds. *Square Root*, p. 44.

(1) $\frac{1}{2}(\sqrt{2}+\sqrt{6})$. (2) $\frac{1}{2}(\sqrt{26}+\sqrt{6})$.

(3) $\sqrt{6}-2$. (4) $\sqrt{5}-\sqrt{2}$.

(5) $\sqrt{7}+1$. (6) $2\sqrt{7}+\sqrt{14}$.

(7) $\sqrt{(1+\sqrt{5})}$.

Imaginary Quantities, p. 45.

(1) $\sqrt{21}+2\sqrt{-7}$.

(2) $6+\sqrt{10}+(2\sqrt{2}-3\sqrt{5})\sqrt{-1}$.

(3) $\frac{8}{5}-\frac{27}{10}\sqrt{-1}$.

(4) $6(1+\sqrt{2})+(9\sqrt{2}-4)\sqrt{-1}$.

(5) $3(\sqrt{5}-\sqrt{6})\sqrt{-1}$.

(6) $\frac{1}{6}(3\sqrt{2}+2\sqrt{3})(1-2\sqrt{-1})$.

(7) $-(117+44\sqrt{-1})$.

(8) $(a^3-3ab^2)\pm(3a^2b-b^3)\sqrt{-1}$.

Quadratic Equations, p. 49.

(1) 4, −25. (2) 5, 21.

(3) 4, $-21\frac{1}{3}$. (4) 3, $-\frac{3}{5}$.

(5) $4\frac{1}{4}$, $\frac{1}{4}$. (6) 25, 3. (7) 18, 3.

(8) 1, 9. (9) $2\pm\sqrt{5}$.

(10) ± 3, $\sqrt{-1}$.

(11) 9, −2, $\frac{7\pm\sqrt{173}}{2}$.

(12) $\frac{3\pm\sqrt{5}}{3}$, $\frac{9\pm\sqrt{-83}}{5}$.

(13) 3, 1, $\frac{-11\pm\sqrt{-71}}{8}$.

(14) $-1\pm\frac{3}{4}\sqrt{2}$. (15) 2^2, $(-1)^2$.

(16) 1, $\left(-\frac{1}{2}\right)^3$. (17) 60, 235.

(18) 1, $\frac{1}{8}(-3\pm\sqrt{-55})$. (19) 2, −3.

(20) $\frac{1}{2}(1\pm\sqrt{3})$, $\frac{1}{2}(1\pm\sqrt{-7})$.

(21) 1, $\frac{-3\pm\sqrt{5}}{2}$.

(22) $\frac{1}{2}\{-2\pm\sqrt{3}\pm\sqrt{(3\mp 4\sqrt{3})}\}$.

Quadratics. *Questions*, p. 52.

(1) *A*, 9 m.; *B*, 10 m. (2) £2, £3.
(3) $30\mp 10\sqrt{5}$, and $-10\pm 10\sqrt{5}$.
(4) No.
(5) Hyp. 20; Base 12; Perp. 16.
(6) 4 and 7. (7) 8.
(8) *A*, 30 h.; *B*, 20 h. (9) 15 and 24.

Simultaneous Quadratics, p. 57.

(1) $\begin{cases} x=\pm\frac{4}{3}\sqrt{6},\ \pm 3 \\ y=\pm\frac{1}{6}\sqrt{6},\ \pm 1. \end{cases}$

(2) $\begin{cases} x=\pm 3,\ \pm\frac{1}{6}\sqrt{2} \\ y=\pm 2,\ \pm\frac{7}{6}\sqrt{2}. \end{cases}$ (3) $\begin{cases} x=\pm 3. \\ y=\pm 2. \end{cases}$

(4) $\begin{cases} x=\pm 6,\ \mp 25\sqrt{-330} \\ y=\pm 5,\ \pm 30\sqrt{-330}. \end{cases}$

(5) $\begin{cases} x=8,\ 4 \\ y=4,\ 8. \end{cases}$ (6) $\begin{cases} x=\frac{1}{2},\ -1 \\ y=-1,\ \frac{1}{2}. \end{cases}$

(7) $\begin{cases} x=1,\ 3,\ 2\pm 5\sqrt{-1} \\ y=3,\ 1,\ 2\mp 5\sqrt{-1}. \end{cases}$

(8) $\begin{cases} x=4,\ -2,\ 1\pm\sqrt{-15} \\ y=2,\ -4,\ -1\pm\sqrt{-15}. \end{cases}$

(9) $\begin{cases} x=8,\ 4 \\ y=4,\ 8. \end{cases}$ (10) $\begin{cases} x=9,\ 6\frac{1}{4} \\ y=4,\ 6\frac{1}{4}. \end{cases}$

(11) $\begin{cases} x=\frac{a}{2}\left\{1\pm\sqrt{\left(1-\frac{4}{ab}\right)}\right\} \\ y=\frac{b}{2}\left\{1\pm\sqrt{\left(1-\frac{4}{ab}\right)}\right\}. \end{cases}$

(12) $\begin{cases} x=4,\ -1,\ \frac{17\pm\sqrt{-283}}{2} \\ y=1,\ -4,\ \frac{-17\pm\sqrt{-283}}{2}. \end{cases}$

(13) $\begin{cases} x=2 \\ y=1. \end{cases}$ (14) $\begin{cases} x=2\frac{1}{2} \\ y=1\frac{1}{2}. \end{cases}$

Miscellaneous Questions, p. 57.

(1) 19, 9, 5.

(2) $\frac{1}{2}(3\pm\sqrt{5})$, $\frac{1}{2}(1\pm\sqrt{5})$.

(3) 36. (4) 841 and 121. (5) 61 or 12.

(6) Fore, 4 yds.; hind, 5 yds.

(7) $\frac{1}{2}\sqrt{5}$ and $\frac{1}{4}(5+\sqrt{5})$.

(8) Time, 4 h. and 6 h. Vel. $\frac{5}{6}$ miles.

(9) The roots of equa. $z^2-\frac{(1\pm\sqrt{5})m}{2}z+\frac{(1\pm\sqrt{5})m^2}{2}=0$.

(10) 5, 12. (11) $S=p^2-2q$.

(12) $S=p^4-4p^2q+2p^2$.

(13) $\frac{3+\sqrt{(9+8n)}}{2}$. (14) 2304, 1296.

(15) No. (16) 6, 7, 8, 9.

(17) 93. (18) $\frac{a\pm\sqrt{(a^2-4b}}{2}$.

(19) $\frac{3\pm\sqrt{5}}{2}$, $\frac{1\pm\sqrt{5}}{2}$.

(20) ±1, ±3, ±5.

Arith. Prog., p. 62.

(1) 31. (2) $\frac{1}{4}$. (3) 0. (4) -3.

(5) 140 or 280. (6) $\frac{5}{12}$, $\frac{1}{2}$, $\frac{7}{12}$, $\frac{2}{3}$, $\frac{3}{4}$.

(7) $-3\frac{1}{3}$. (8) 4.

Geom. Prog., p. 64.

(1) This is an Arith. Prog.; 34.

(2) 5465. (3) 49. (4) $2\frac{1}{12}$.

(5) $\frac{x}{x-1}$. (6) ±3, $4\frac{1}{2}$, $\pm6\frac{3}{4}$.

(7) 6, 18, 54, 162, 486. (8) $1\frac{1}{3}$.

Harmonic Prog., p. 65.

(1) 4, 6. (2) 3, $2\frac{2}{3}$, 12.

(3) 3, 1. (4) 3, 4, 6.

Prop. and Prog. *Questions*, p. 66.

(1) 4 days. (2) 6 m. 6 fur. 20 yds.

(5) 7 nights; 63 yds.

(6) £5825. 8s. $5\frac{1}{4}$d.

(7) $\frac{9172}{33300}$, or $\frac{2293}{8325}$.

(8) 2 gal. from A, 12 from B.

(10) 1, 3, 5. (11) 1, 2, 4.

(12) 3, 9, 27, 81. (13) 2, 10.

(14) 234.

Piling of Shot, p. 69.

(1) 364 and 650. (2) 392. (3) 3640.

(4) 23405. (5) 6146.

Binomial Theorem, p. 76.

(5) $(2a^2)^4\left\{1+4\left(\frac{3x^3}{2a^2}\right)+6\left(\frac{3x^3}{2a^2}\right)^2+4\left(\frac{3x^3}{2a^2}\right)^3+\left(\frac{3x^3}{2a^2}\right)^4\right\}$.

(6) $a^6-15a^4b^2+15a^2b^4-b^6+(6a^5b-20a^3b^3+6ab^5)\sqrt{-1}$.

(7) $32a^5x^5-240a^4x^4y+720a^3x^3y^2-1080a^2x^2y^3+810axy^4-243y^5$.

(8) $\frac{1}{a^2}\left(1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^2}+\ldots\right)$.

(9) $1-3x^2+6x^4-10x^6+\ldots$.

(10) $\frac{1}{a^3}\left(1+\frac{6x}{a}+\frac{24x^2}{a^2}-\frac{80x^3}{a^3}+\ldots\right)$.

Binomial Theorem, *Case II.*, p. 76.

(1) $b+\frac{x}{2b}-\frac{x^2}{2.4b^3}+\frac{3x^3}{2.4.6b^5}-\ldots$

(2) $b+\frac{x}{3b^2}-\frac{2x^2}{3.6b^5}+\frac{2.5x^3}{3.6.9b^8}-\ldots$

(3) $\frac{1}{\sqrt{a}}\left(1+\frac{x}{2a}+\frac{3x^2}{2.4a^2}+\frac{3.5x^3}{2.4.6a^3}+\ldots\right)$.

(4) $a^{\frac{2}{3}}\left(1+\frac{2x}{3a}-\frac{x^2}{3^2a^2}+\frac{4x^3}{3^4a^3}-\dots\right)$.

(5) $\frac{1}{\sqrt{a}}\left(a^2-\frac{3x}{2^2}-\frac{3x^2}{2^5a^2}-\frac{5x^3}{2^7a^4}-\dots\right)$.

(6) $a^{-\frac{2}{3}}\left\{1-\frac{2}{3}\left(\frac{x}{a}\right)+\frac{5}{3^2}\left(\frac{x}{a}\right)^2-\frac{5.8}{3.3^3}\left(\frac{x}{a}\right)^3+\dots\right\}$.

(7) $2+\frac{1}{3.2^2}-\frac{2}{3.6.2^5}+\frac{5}{3.6.9.2^7}-\dots$

(8) $2-\frac{1}{3.2}+\frac{2}{3.6.2^3}-\frac{2.5}{3.6.9.2^5}+\dots$

(9) $a+\frac{x}{2a}+\frac{3x^2}{2.4a^3}+\frac{3.5x^3}{2.4.6a^5}+\dots$

(10) $\frac{1}{\sqrt[3]{3}}\left\{1+\left(\frac{7}{9}\right)x+\frac{4}{3.6}\left(\frac{7}{3}\right)^2x^2+\frac{4.7}{3.6.9}\left(\frac{7}{3}\right)^3x^3+\dots\right\}$.

Interest, Annuities, &c., p. 88.

(1) 4 per c. (2) 12·04. (3) 22½ yrs.
(4) £742. 5s. (5) £681. 11s. 7d.
(6) £551. 15s. 3d. (7) 25.
(8) £822. 14s. 6d. (9) 7 yrs. nearly.
(10) 24418270.

Permutations, p. 90.

(1) 362880. (2) 3628800. (3) 15120.
(4) 1260. (5) 64. (6) 340.

Combinations, p. 93.

(1) 330. (2) 84. (3) 252.
(4) 4, 70. (5) 22. (6) 7.
(7) 49639590. (8) 93600.

Probabilities, p. 99.

(1) $\frac{1}{8}$. (2) $\frac{3}{8}, \frac{7}{8}, \frac{5}{8}$. (3) 199 to 125.

(4) $\frac{1}{4}$. (5) $\frac{10}{25}$. (6) $\frac{31}{32}$. (7) $\frac{27}{1000}$.

(8) $\frac{1}{120}$. (9) $3W, 7B$. (10) $\frac{7}{24}$.

(11) $\frac{29}{70}$. (12) $\frac{3}{5}$ and $\frac{2}{5}$.

Theory of Equations, p. 104.

(1) $Q=3x^3+4x^2+13x+19$; $R=19$.
(2) $Q=7x^3+8x^2+28x+90$; $R=258$.
(3) $Q=5x^3-37x^2+218x-1302$; $R=7799$.
(4) $Q=7x^3-38x$; $R=-814$.
(5) $Q=x^4+11x^3+47x^2+205x+830$; $R=3306$.

Trans. of Equations, p. 112.

(1) $x^3-6x^2-9x+54=0$.
(2) $x^3-7x+6=0$.
(3) $6x^3+51x^2+148x+146=0$.
(4) $12x^3+42x^2-25x+3\cdot5=0$.
(5) $19x^4+206x^3+793x^2+1232x+580=0$.
(6) $x^3-7x+7=0$.
(7) $x^2-\frac{1}{4}A_1{}^2+N=0$.
(8) $2x^4-5x^2+3x-1=0$.
(9) $x^3-x^2-34x-56=0$.
(10) $x^3-6x^2+11x-6=0$. (11) 2, −3.
(12) 2, 3, 6. (14) $3x^4-6x^2+2=0$.
(15) $1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}$.
(16) $\pm1, \pm\sqrt{-1}$.
(17) $x^6-3x^4+2x^2-7x-5=0$.
(18) $a_5=-(A_4+a_1+a_2+a_3+a_4)$.

Solution of Equations, p. 134.

(1) 3·2213261. (2) 2·4384783.
(3) 1·0560897. (4) 2·209753.
(5) 2·35833. (6) 2·3027756.
(7) 1·284724. (8) 8765.
(9) 2·080083823. (10) 9·886016.
(11) ·316604. (12) 1·283137.
(13) −2·35833. (14) −5·018424.
(15) −3·9073786. (16) −1·876837.
(17) −1·584418. (18) −5·2467839.
(19) 2·557351, 23·213112, 23·229537.
(20) 2·8580833, ·44332769, ·6060183, −3·9073785.
(21) All imaginary.
(22) Two positive; two imaginary.
(23) All imaginary.
(24) One positive; four imaginary.

Decomposition of Equations, p. 141.

(1) $x=-1\pm\frac{\sqrt{3}\pm\sqrt{(15-4\sqrt{3})^2}}{2}$.

(2) $x=2\pm\frac{\sqrt{\{2(9\pm\sqrt{33}\}}}{2}$.

(4) $x=3, \frac{3\pm\sqrt{21}}{2}$.

(5) $\begin{cases} x=\frac{2-\sqrt{-1}\pm\sqrt{\{-1-4\sqrt{-1}\}}}{2}. \\ x=\frac{2+\sqrt{-1}\pm\sqrt{\{-1+4\sqrt{-1}\}}}{2}. \end{cases}$

(6) $x^2+\left\{\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}-b+\frac{2c}{a}\right)}\right\}x+\frac{c}{a}$.

Recurring Equations, p. 144.

(1) $x=1, \frac{-3\pm\sqrt{5}}{2}$.

(2) $x=2, \frac{1}{2}, \frac{5\pm12\sqrt{-1}}{13}$.

(3) $\frac{1\pm\sqrt{-3}}{2}; \frac{1}{2}, 2; \frac{1}{2}, 2$.

(4) $-1, \frac{1+\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}$.

(5) $\frac{-1\pm\sqrt{-1}}{\sqrt{2}}, \frac{1\pm\sqrt{-1}}{\sqrt{2}}$.

(6) $1, -1, -1, +\sqrt{-1}, -\sqrt{-1}$.

Vanishing Fractions, p. 145.

(1) $\frac{1}{3}$. (2) $\frac{3}{5}$. (3) $\frac{1}{2}$. (4) $\frac{1}{2}b$.

(5) $\frac{4}{3}$. (6) $\frac{7}{6}$.

Maxima and Minima, p. 150.

(1) $\frac{1}{2}a$ and $\frac{1}{2}a$. (2) No maximum : the *least* possible are $\frac{1}{2}a$, $\frac{1}{2}a$.

(3) $\sqrt{a}$ and $\sqrt{a}$. (4) $\frac{1}{2}$.

(5) When $x=\frac{5}{6}$. (6) $x=\frac{1}{4}$.

(7) No. (8) 1 and 2. (9) 8 and 4.

(10) $\frac{2}{3}$. (11) $x=\frac{a}{13}$. (12) $x=\frac{3a}{4}, \frac{3}{a}$.

Indeterminate Equations, p. 153.

(1) $\begin{cases} x=9, 20, 31, 42. \\ y=19, 14, 9, 4. \end{cases}$

(2) $\begin{cases} x=56. \\ y=9. \end{cases}$

(3) 4. (4) No. (5) 190.

(6) $\begin{cases} x=2, 4, 6, 8, \&c. \\ y=8, 11, 14, 17, \&c. \end{cases}$

Indeterminate Equations, p. 154.

(1) 10. (2) 24. (3) 1, 3, 6, or 2, 1, 7.

(4) $x=5, y=1, z=2$.

(5) 8, 10, 2 oz., or 10, 5, 5 oz.

(6) $\begin{cases} x=15, 50. \\ y=82, 40. \\ z=15, 30. \end{cases}$

Least Integers, p. 155.

(1) 43. (2) 503.

(3) 104. (4) 7691.

Indeterminate Coefficients, p. 158.

(1) $4x(x+1)-8x+1$.

(2) $\frac{2}{x+3}-\frac{1}{x+2}$.

(3) $\frac{2}{x-1}+\frac{3}{x+2}$, and $\frac{2}{3(x-1)}+\frac{x-1}{3(x^2+x+1)}$.

(4) $\frac{3}{x}+\frac{2}{x-1}-\frac{1}{x+2}-\frac{2}{x^2}$.

Finite Series, p. 160.

(1) $\frac{n(n+1)(2n+1)}{2.3}$.

(2) $\left(\frac{n(n+1)}{2}\right)^2$. (3) $\frac{n(n+1)(n+2}{2.3}$.

(4) 430. (5) n^2.

(6) $\frac{n(n+1)(n+2)(n+3)}{4}$.

(7) $\frac{n(n+1)(n+2)(n+3)}{2.3.4.}$.

(8) $n^2(2n^2-1)$.

Infinite Series, p. 163.

(1) $\frac{3}{4}$. (2) $\frac{1}{4}$. (3) $\frac{1}{4}$. (4) $\frac{11}{18}$.

(5) $\frac{1}{12}$. (6) $\frac{5}{4}$ (7) $\frac{1}{8}$.

(8) $\frac{5}{24}$. (9) $\frac{89}{36}$. (10) $\frac{1}{12}$.

Recurring Series, p. 166.

(1) $\frac{1+x}{(1-x)^2}$. (2) $\frac{1+x}{1-x-x^2}$.

(3) $\frac{1+x}{(1-x)^3}$.

(4) $\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$.

Convergency of Series, p. 169.

(3) Divergent. (4) $x=0$, and $x=1$.

(5) Limit$=\frac{1}{192}$.

PLANE TRIGONOMETRY.

Right-angled Triangles, p. 179.

(1) 164·4, 296. (2) 60°9′.
(3) 95·365, 120·097.
(4) 384·05, 480·036. (5) 40°16′, 72.
(6) $A=54°36'\ 14''$, $C=35°\ 23'\ 46''$, $AC=92·23$.
(7) $C=37°\ 3'$, $AB=138·24$, $BC=104·34$.
(8) 63°26′6″, 26°33′54″. (9) 66·78 ft.
(10) 8000 miles. (11) 219·8 ft.
(12) 56·649 ft. (13) Dist. 4100·4 ft. Height, 289·125 ft.
(14) Diam. 7916 miles. Dist. 136 miles.
(15) 935·75 yards.

Oblique Triangles, *Case I.*, p. 184.

(1) 22°37′, 351,648.
(2) $a=309·595$, $b=436·844$.
(3) $C=61°13'47''$, $a=101·617$.
(4) $B=45°13'55''$, or $134°46'5''$.
(6) 41°28′, 106°32′, 72·36 ; or 138°32′, 9°28′, 12·415.

Oblique Triangles, *Case II.*, p. 158.

(1) $A=103°19'21''$; $B=44°22'13''$; $c=288·894$.
(2) $A=53°54'11''$; $B=99°53'33''$; $c=9·729$.
(3) $B=26°52'42''$, $C=21°0'6''$, $a=3387·974$.
(4) $A=52°15'25''$, $B=39°14'35''$, $c=632·08$.
(5) $A=18°21'20''$, $C=33°34'40''$, $b=2400·036$.
(6) $A=98°9'16''$, $B=36°49'40''$, $c=1194·7$.

Oblique Triangles, *Case III.*, p. 189.

(1) $B=54°33'24''$.
(2) 41°24′35″, 55°46′16″, 82°49′9″.
(3) $A=40°33'12''$, $B=78°13'1''$, $C=61°13'47''$.
(4) $A=37°59'53''$, $B=52°37'47''$, $C=89°22'20''$.

Miscellaneous Problems, p. 189.

(3) $DA=301·01$, $DC=629·101$, $DB=719·522$.
(4) 1·49 mile.
(5) Height$=210·44$ ft. Dist.$=250·79$.
(6) $DA=5$m., $DC=4·892$m., $DB=7$m.
(7) 521 ft. (8) 84 ft.
(9) $PA=709·33$, $PC=1042·66$, $PB=934$ yds.
(11) $h=53·964$ ft. (12) 44·46 yds.

Applications of Formulæ, p. 206.

(13) $A=15°$.

(14) $\sin A=\frac{1}{2}\sqrt{(-2\pm\sqrt{5})}$.

(15) $A=15°$. (16) $x=18°$ or $54°$.
(17) $x=45°$. (18) $m^2-n^2=4\sqrt{mn}$.

Trigonometrical Series, &c., p. 224.

(1) $\cos\theta+\sqrt{-1}\sin\theta$.

(2) $\cos\frac{2}{5}\theta-\sqrt{-1}\sin\frac{2}{5}\theta$.

(4) $\sin\{\theta+\frac{1}{2}(n-1)(\alpha+\varpi)\}\times \sin\frac{1}{2}n(\alpha+\varpi)\div\sin\frac{1}{2}(\alpha+\varpi)$.

(5) $S=\tan n\theta$.

(6) $S=\cot\frac{1}{2}x-\cot 2^{n-1}x$.

(7) $S=\cot\theta-2^n\cot 2^n\theta$.
(8) $S=(x^{n+2}\cos n\theta-x^{n+1}\cos(n+1)\theta+x\cos\theta-x^2)\div(x^2-2x\cos\theta+1)$.

Imaginary Roots of ± 1, p. 226.

(1) $\sqrt[3]{N}$, $\frac{-1\pm\sqrt{-3}}{2}\sqrt[3]{N}$.

(2) $-\sqrt[3]{N}, \frac{1\pm\sqrt{-3}}{2}\sqrt[3]{N}.$

(3) $1, \frac{\sqrt{5}-1}{4} \pm \frac{\sqrt{(-10-2\sqrt{5})}}{4}$,

$-\frac{\sqrt{5}+1}{4} \pm \frac{\sqrt{(-10+2\sqrt{5})}}{4}.$

(4) $x^{2n}-2x^n\cos\theta+1=$

$(x^2-2x\cos\frac{\theta}{n}+1)\times$

$x^2-2x\cos\frac{2\pi+\theta}{n}+1)\times$

$(x^2-2x\cos\frac{4\pi+\theta}{n}+1)$, &c.

to n factors, in each of which the coef. of x does not exceed 2.

SPHERICAL TRIGONOMETRY.

Right-angled Triangles, p. 238.

(1) $c=70°23'42''$. (2) $B=66°59'25''$.
(3) $A=103°52'48''$.
(4) $b=69°9'48''$, $B=79°56'4''$.
(5) $b=50°30'31''$ or $129°29'29''$.
(6) $b=59°38'27''$, $B=66°20'40''$.
(7) $a=4°35'26''$, $c=11°35'49''$, $B=66°58'1''$.
(8) Hyp$=41°24'35''$.
(9) $B=61°50'30''$.

Quadrantal Triangles, p. 239.

(1) $48°0'9''$ and $25°25'20''$.
(2) $D=65°27'9''$, $BD=110°12'44''$, $B=81°1'58''$.
(3) $42°56'12''$, and $132°2'44''$.
(4) $AB=70°8'39''$, $BD=73°17'29''$, $B=96°13'23''$.

Oblique Triangles, *Case I.*, p. 241.

(1) $A=45°12'52''$.
(2) $A=63°10'58''$.
(3) $A=100°11'22''$.
(4) $A=51°30'$, $B=59°16'$, $C=131°30'$.
(5) See the work at p. 241.

Oblique Triangles, *Case II.*, p. 242.

(1) $b=50°10'30''$. (2) $a=70°0'3''$.
(3) $a=50°54'32''$, $b=37°47'18''$, $c=74°51'50''$.

Oblique Triangles, *Case III.*, p. 245.

(1) $42°15'12''$ and $34°14'58''$.
(2) $A=30°28'11''$, $B=130°3'11''$, $c=40°$.
(3) $A=130°5'22''$, $B=32°26'6''$, $c=51°6'12''$.

Oblique Triangles, *Case IV.*, p. 246.

(1) $a=40°0'10''$, $b=50°10'30''$.
(2) $a=84°14'28''$, $b=44°13'44''$.
(3) $a=76°35'36''$, $c=50°10'30''$, $B=34°15'3''$.

Oblique Triangles, *Case V.*, p. 248.

(1) $B=59°15'47''$, $C=131°29'46''$, $c=120°48'$.
(2) $C=53°30'56''$.

Oblique Triangles, *Case VI.*, p. 250.

(1) $b=52°39'4''\frac{1}{2}$, $c=112°22'58''\frac{1}{2}$, $C=119°15'$.
(2) $C=36°45'28''$. (3) $a=63°49'32''$.

MENSURATION.

Surfaces—Parallellograms: Triangles, p. 257.

(1) 760. (2) $21\frac{7}{12}$. (3) $108F.5\frac{2}{3}I.$
(4) 20·869. (5) $112\frac{16}{27}$.

Triangles, p. 258.

(1) $66\frac{3}{8}$. (2) 11·654*F*.
(3) 61*A*. 1*R*. 39*P*. (4) 29*A*. 8*P*.

Quadrilaterals, p. 260.

(1) $13\frac{13}{24}$. (2) 439·3*Y*.
(3) 17*A*. 2*R*. 21*P*. (4) 324·76*Y*.

Equidistant Ordinates, p. 261.

(1) 19·416*F*. (2) 874*I*. (3) 85*F*.

Regular Polygons, p. 262.

(1) 1091*F*. 7Pts. 6*I*. (2) $\frac{1}{4}a^2\sqrt{3}$.
(3) 1267·04*F*. (4) 1973·9*F*.

Circles—Sectors, p. 263.

(1) 1·069. (2) 2·1231*f*.
(3) 11·4595*F*. (4) 804·3986*F*.
(5) 69·156*m*. (6) 1*Y*. 8*F*. $73\frac{1}{4}I.$

Segments of Circles, p. 265.

(1) 104·356*F*. (2) 164·39*F*.
(3) 1834·9*F*. (4) 636·375*F*.
(5) 44·728. (6) 24.6526*f*.

Circular Rings, p. 266.

(1) 235·62*F*. (2) 4205·28*F*.
(3) 412·335*F*.

MENSURATION.

Solids—Prisms, Cylinders, p. 267.

(1) 21$\frac{1}{9}$. (2) £6. 13s. 5$\frac{1}{4}$d. (3) 60.
(4) 1·5217*F*. (5) 48·1459.
(6) 106*F*. 479$\frac{1}{2}$*I*.

Surface of Prism or Cylinder, p. 268.

(1) 1800*F*. (2) 91·948*F*.
(3) £3. 14s. 8d.

Volume of Pyramid and Cone, p. 268.

(1) 71·052*F*. (2) 22·5609*F*.
(3) 27·5276. (4) 40·437 in.

Surface of Pyramid and Cone, p. 269.

(1) 90*F*. (2) 76·737*F*. (3) 61·194*F*.

Volume of Frustum, p. 270.

(1) 19$\frac{1}{2}$. (2) 527·7888. (3) 7004·41.
(4) 6213096 tons.

Surface of Frustum, p. 270.

(1) 3262$\frac{1}{2}$*F*. (2) 90·7245*F*.
(3) 119·58*F*.

Volume of Sphere, p. 271.

(1) 904·7808*F*. (2) 108·665*I*.
(3) ·14786.

Surface of Sphere and Segment, p. 273.

(1) 198, 943, 687. (2) 27·489*F*.
(3) 42*F*. 69*I*.

Vol. of Segment and Zone, p. 274.

(1) 253070·54*I*. (2) 61·7848*F*.

Weight of Shot and Shells, p. 276.

(1) 24lb. nearly. (2) 5·54 in.

Page 277. (1) 112·215 lb. (2) 84$\frac{1}{8}$ lb.
Page 277. (1) 5$\frac{5}{191}$ lb. (2) 11$\frac{15}{16}$.

Maxima and Minima, p. 284.

(1) Isosceles. (2) Isosceles. (3) 45°.

(4) Slant height$=\frac{3}{2}$diam. of base.

(5) Bisect the sides of the angle A.
(6) Height=3 radius.

(7) Alt.$=\frac{2}{3}$diameter.

(8) Alt.$=\frac{2}{3}r\sqrt{3}$.

(9) Length of side$=\frac{2}{9}$diam.$\times$depth.

(10) Radius$=2\frac{11}{12}$ inches.

ANALYTICAL GEOMETRY.

Equations of Straight Lines, p. 292.

(1) $y=\frac{1}{2}x-\frac{5}{2}$. (2) $y=\frac{2}{5}x-4$.

(3) $y=\frac{7}{3}x-7$. (4) $y+3=a(x-8)$.

(5) $y=ax-6$. (6) $y=\frac{4}{3}x$.

Intersections, &c., p. 294.

(1) $x=-\frac{1}{2}$, $y=-\frac{1}{2}$; $\tan=\frac{4}{3}$.

(2) $y=-\frac{3}{2}x-\frac{21}{2}$. (3) $y=\frac{2}{3}x+\frac{1}{3}$.

Problems on the Straight Line, p. 300.

(1) $y=x+y'-x'$, or $y=-x+y'+x'$.

(2) $\frac{1}{10}\sqrt{5}$. (3) $y-\frac{ab'-a'b}{a-a'}=m\left(x-\frac{b'-b}{a-a'}\right)$.

(4) $y=\frac{ab'-a'b}{b'-b}x$.

Problems on the Circle, p. 308.

(1) A circle of half the diameter.
(2) $y-1=(3\pm2\sqrt{2})(1-x)$.

(6) A circle of centre $\left(\frac{am}{m-1}, 0\right)$,

and radius$=\frac{a\sqrt{m}}{m-1}$.

(8) $4(1+a_1^2)(b_1^2+bb_1+c)=(2a_1b_1+a_1b+a)^2$.

CONIC SECTIONS.

Problems and Theorems, p. 325.

(1) Equa. of asymptotes, $x=2$, $y=-1$.

Point, $y=-\frac{1}{2}$.

(2) A parabola. (3) An hyperbola.

(4) An hyperbola. (5) An ellipse.

(6) An ellipse.

STATICS.

Examples, p. 371.

(1) R=$\sqrt{19}$ lb. (2) R=135·65 lb.

(3) P=10, 14·14, and 5·17 lb.

(5) 2·354, and 5·18 ft. (6) 6·57 lb.

(7) 30·62 lb. (8) 60·9 and 49·02 lb.

(9) $\tan\theta=\frac{\cos\alpha}{2-\sin\alpha}$.

(10) $P=W\frac{\sqrt{\{a^2b^2+(a-b)^2c^2\}}}{bc}$.

Statics. *Examples*, p. 382.

(1) Centre of gr. $3\frac{1}{2}$ ft. from the 5 lb., $\frac{7}{26}$ ft. nearer.

(3) $\frac{a}{6}\sqrt{7}$, $\frac{a}{6}\sqrt{19}$, $\frac{a}{6}\sqrt{13}$.

(4) $\frac{a}{2}(3-\sqrt{3})$. (5) $\frac{1}{6}r$.

Statics. *Examples*, p. 395.

(1) $T=\frac{1}{2}W\cot\theta$. (2) $R=-6$, $\theta=60°$.

(3) $\theta=45°$.

(4) $P=90$ lb., $T=330$ lb. nearly.

(5) $2\frac{a\sqrt{W}+b\sqrt{W'}}{\sqrt{(W+W')}}$. (6) $\frac{Wa}{b}$.

(7) $\tan\theta=\frac{W'\tan\alpha'-W\tan\alpha}{(W+W')\tan\alpha\tan\alpha'}$.

(8) $\tan\theta=\frac{Wr}{wc}$.

(9) $\cot\theta=\frac{W+W'\cos\alpha}{W'\sin\alpha}$.

(10) $\cos[W_1, W_2]=\frac{W_3^2-W_2^2-W_1^2}{2W_1W_2}$.

$\cos[W_2, W_3]=\frac{W_1^2-W_2^2-W_3^2}{2W_2W_2}$

$\cos[W_3, W_1]=\frac{W_2^2-W_1^2-W_3^2}{2W_3W_1}$

Statics. *Examples*, p. 402.

(1) $7\frac{2}{7}$ lb. (2) 11 lb. 6 oz.

(3) *A*, $88\frac{1}{17}$ lb.; *B*, $61\frac{13}{17}$ lb. (4) 135°.

(5) $AC=\frac{aw}{2(W+w)}$.

(6) $\frac{3}{5}$ in., 7 lb., and 82 lb.

(7) 157·6 lb. (8) $445\frac{1}{4}$ lb.

Statics. *Examples*, p. 404.

(1) 1000 lb. (2) 4 ft. 2 in.

(3) 88·54 lb. (4) $10\frac{2}{3}$ lb.

Statics. *Examples*, p. 407.

(1) 7. (2) 4 lb. (3) $9\frac{1}{2}$ lb. (4) 100 lb.

Statics. *Examples*, p. 409.

(2) 522·19 lb. (3) 4·828 ft.

Statics. *Examples*, p. 410.

(1) 4523·76 lb. (2) 19301 lb.

(3) $\frac{3}{5}$ in.

Statics. *Examples*, p. 419.

(1) $38\frac{1}{2}$. (2) 192. (3) $\frac{1}{10}$ ft.

(4) ·072 ft. (5) 567 ft. (6) 75·4 ft.

(7) 2412. (8) 24000. (9) 600.

(10) 3·183 in., and 12·732 in.

DYNAMICS.

Vertical Motion, p. 429.

(1) 5 secs. (2) 34875 ft. (3) 42·8 ft.

(4) 64·4 ft. (5) 1·2 secs., and 4·7 secs.

(6) $83\frac{1}{3}$ ft. (8) $\frac{b^2}{4a}$ ft.

Inclined Planes, p. 431.

(1) 32·2 ft. (2) $2\frac{1}{3}$ secs. (3) 203·4 ft

Projectiles, p. 436.

(1) $r=24\cdot84$ft., $H=1\cdot61$ft., $t=\cdot64$ sec.

(3) $v_1=60\cdot9$ft., $H=43\cdot3$ft., $t=3\cdot28$sec.

(4) 60°.

(5) $\dfrac{w_1v_1\sin\alpha_1+\ldots+w_nv_n\sin\alpha_n}{w_1+\ldots+w_n}t-\frac{1}{2}gt^2$.

Gunnery, p. 438.

(1) 871 yds. (2) 24°16′, or 65°44′.

(3) $h_1=1802$ ft., $v_1=340$ ft., charge $=2\cdot96$ lb.

(4) 4332 ft. (5) $6\frac{3}{4}$ lb.

(6) 4244. and 6745 ft.

(7) 4·925 lb. (8) 32 lb. (9) 35°.

Centrifugal Force, p. 442.

(1) ·04308 ft. (2) ·0542 ft.

(3) ·03964 ft. and ·05328 ft.

(4) 8·025 ft. per sec.

Moving Forces, p. 450.

(1) 86·94 ft. (2) $t=4\cdot935$ secs., $v=4\cdot815$ ft.

(3) $\dfrac{80}{121}W$. (4) $\dfrac{Wv_1+(W-W')}{W+W'}gt$.

(5) 2·99 secs. (6) $l=2\dfrac{W'}{W}h$.

(7) $t=4\cdot73$ secs. (8) $s=499\frac{1}{2}$ ft.

(9) $Q=\dfrac{PR}{r^2}\{\sqrt{(R^2+Rr)}-R\}$.

(10) $R+\dfrac{Q}{P}\{\sqrt{(1+\dfrac{P}{Q})}+1\}r$.

Impact, p. 455.

(1) $V=2\frac{10}{17}$ ft.; A's loss, $\dfrac{7}{17}$ft.;

B's gain $\dfrac{10}{17}$ft.

(2) Vel. of $A=4\cdot56$ ft.; vel. of $B=2\cdot04$ ft.

(3) $e=\cdot836$. (4) $155\frac{19}{31}$ ft.

(5) From the second given point P draw PD perp. to the plane, and produce it to E, making $DE=PD\div e$: draw a line from the first given point to E, and it will pierce the plane in the required point.

HYDROSTATICS.

Examples, p. 465.

(1) 128 in. (2) 20250 lb.

(7) Depth $=r\sqrt{\dfrac{2}{3}}$.

(8) Depth of chord $\dfrac{4}{5}h$: vertex downward.

Specific Gravity, p. 475.

(1) $S=4\cdot5$. (2) $S=13\cdot5$. (3) $\frac{3}{4}$ vol.

(4) $S=\cdot75$. (5) 2 to 3.

(6) $\dfrac{1}{12}$ of foot. (8) $S=1$.

(9) Copper 100 lb.; Tin 12 lb.

(10) $1\frac{8}{15}$ grs.

(11) Tin 74 lb. Lead 46 lb.

(12) ·37895 cub. ft.

Air-pump, Barometer, &c., p. 488.

(1) $\left(\dfrac{10}{11}\right)^4$. (2) $11\frac{109}{121}$ lb.

(3) $17\frac{7}{25}$ lb. (4) $33\frac{3}{4}$ ft. (5) 34·56 ft.

(6) 2·039 ft.

(7) $26°\frac{3}{5}$, and $120°\frac{1}{5}$. (8) $3\frac{3}{4}$ in.

(9) ·5765 ft.

DIFFERENTIAL CALCULUS.

Algebraic Functions, p. 496.

(1) $\dfrac{dy}{dx}=6x^2-8x+3$.

(2) $\dfrac{dy}{dx}=-6x$. (3) $\dfrac{dy}{dx}=6x^2+12x^3$.

(4) $\dfrac{dy}{dx}=-\dfrac{2b}{x^3}$. (5) $\dfrac{dy}{dx}=\dfrac{2a^2x}{(a^2+x^2)^2}$.

(6) $\dfrac{dy}{dx}=\dfrac{2x(a-2x^3)}{(a+x^3)^3}$.

(7) $\dfrac{dy}{dx}=\dfrac{5}{2}bx\sqrt{(a+bx^2)}$.

(8) $\dfrac{dy}{dx}=\dfrac{1}{(1-x^2)^{\frac{3}{2}}}$.

(9) $\dfrac{dy}{dx}=\dfrac{a}{2\sqrt{x}}-\dfrac{b}{x}$.

(10) $\dfrac{dy}{dx}=\dfrac{\cdot6(1-x^2)}{(3+x^2)^2\sqrt{x}}$.

(11) $\frac{dy}{dx}=\frac{2\{x+\sqrt{(x^2-1)}\}^2}{\sqrt{(x^2-1)}}$.

(12) $\frac{dy}{dx}=\frac{1}{a^2}\left\{2x+\frac{a^2+2x^2}{\sqrt{(a^2+x^2)}}\right\}$.

Log. and Exp. Functions, p. 499.

(1) $\frac{dy}{dx}=1+\log x$.

(2) $\frac{dy}{dx}=\frac{1}{\log x}\left(1-\frac{1}{\log x}\right)$.

(3) $\frac{dy}{dx}=\log a.\, a^{cx}c$.

(4) $\frac{dy}{dx}=\frac{a}{x\sqrt{(a^2+x^2)}}$.

(5) $\frac{dy}{dx}=\frac{1}{\sqrt{(x^2\pm a^2)}}$.

(6) $\frac{dy}{dx}=\frac{\sqrt{ax}}{(a-x)x}$.

(7) $\frac{dy}{dx}=\frac{a^{\log x}\log a}{x}$.

(8) $\frac{dy}{dx}=nx^{n-1}e^x$.

(10) $\frac{dy}{dx}=-\frac{e^{\frac{\sqrt{(1+x^2)}}{x}}}{x^2\sqrt{(1+x^2)}}$.

Trig. Functions, p. 502.

(1) $\frac{dy}{dx}=-a\sin ax$. (2) $\frac{dy}{dx}=\frac{1}{x}-\sin x$.

(3) $\frac{dy}{dx}=(1+x\sec^2 x)e^{\tan x}$.

(4) $\frac{du}{dx}=-\operatorname{cosec}^2 x^y\left(\frac{y}{x}+\frac{dy}{dx}\log x\right)x^y$.

(5) $\frac{dy}{dx}=\frac{a}{\sqrt{(1-a^2x^2)}}$.

(6) $\frac{dy}{dx}=\frac{n}{x\sqrt{(x^{2n}-1)}}$.

(7) $\frac{dy}{dx}=\frac{e^{\frac{x}{2}}}{\sqrt{(2-e^x)}}$.

(8) $\frac{dy}{dx}=\frac{2}{x\sqrt{(m^2x^4-1)}}$.

(9) $\frac{dy}{dx}=-\frac{2}{1+x^2}$. (10) $\frac{dy}{dx}=\frac{1}{1+x^2}$.

(11) $\frac{dy}{dx}=\frac{1}{1+x^2}$. (12) $\frac{dy}{dx}=\sec x$.

Integration, p. 505.

(1) $-\frac{a}{2x^2}+C$.

(2) $ax-\frac{b}{2x^2}+\frac{2}{5}x^{\frac{5}{2}}+C$.

(3) $\frac{1}{2}(4x^2+3)^{\frac{3}{2}}+C$.

(4) $\frac{3}{5}\log\{C(x^5\pm a)\}$.*

(5) $-2\frac{\sqrt{(2ax-x^2)}}{x}+C$.

(6) $\frac{1}{a}\sqrt{\left(\frac{a}{2a-x}\right)}$. (Omitting C).

(7) $\frac{\sqrt{(a^9+6x^9)}}{27}$. (8) $-\frac{(a^2+x^2)^{\frac{3}{2}}}{3a^2x^3}$.

(9) $x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^n}{n}$.

(10) $-\left(\frac{3x^4}{2}+\frac{6x^5}{5}+x^6+\frac{6x^7}{7}+\frac{3x^8}{4}+\frac{2x^9}{3}\right)$.

Maclaurin's Theorem, p. 525.

(1) $a^x=(1+Ax+\frac{A^2x^2}{1\cdot 2}+\frac{A^3x^3}{1.2.3}+\ldots)1^x$.

(2) $\sin^{-1}x=x+\frac{x^3}{2.3}+\frac{1\cdot 3x^5}{2.4.5}+\ldots$

(3) $\cos^{-1}x=\frac{\pi}{2}-x-\frac{x^3}{2.3}-\frac{1\cdot 3x^5}{2.4.5}-\ldots$

(4) $(\tan x)^4=x^4+\frac{4x^6}{3}+\frac{6x^8}{5}+\ldots$

(5) $\cos^n x+ni\cos^{n-1}x\sin x+\frac{n(n-1)}{2}i^2\cos^{n-2}x\sin^2 x+\ldots$

(6) $(a+bx+cx^2+\ldots)^n=a^n+na^{n-1}bx+\left.\frac{n(n-1)}{2}a^{n-2}b^2+na^{n-1}c\right|x^2+\left.\frac{n(n-1)(n-2)}{2.3}a^{n-3}b^3+n(n-1)a^{n-2}bc+na^{n-1}d\right|x^3+\ldots$

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* This example belongs to the case of art. 536.

Compound Functions, p. 533.

(1) $\left\{\frac{du}{dx}\right\}=3(x^2+ay)+3(ax+y^2)\frac{dy}{dx}$.

(2) $\left\{\frac{du}{dx}\right\}=\log y+\frac{x}{y}\frac{dy}{dx}$.

(3) $\left\{\frac{du}{dx}\right\}=\frac{1}{\sqrt{(x^2-2ax)}}$.

(4) $\left\{\frac{du}{dx}\right\}=(\cos x)^{\sin x}\times$

$$\left\{\cos x\log\cos x-\frac{\sin^2 x}{\cos x}\right\}.$$

(5) $\left\{\frac{du}{dx}\right\}=\sqrt{\frac{1-x}{1+x}}$

Implicit Functions, p. 534.

(1) $\frac{dy}{dx}=-1$.

(2) $\frac{dy}{dx}=-\frac{x}{y}$, $\frac{d^2y}{dx^2}=-\frac{y^2+x^2}{y^3}=-\frac{a^2}{y^3}$.

(3) $\frac{dy}{dx}=\frac{y}{y-x}$, $\frac{d^2y}{dx^2}=\frac{y(y-2x)}{(y-x)^3}=$

$$-\frac{a^2}{(y-x)^3}.$$

(4) $\frac{dy}{dx}=\frac{3x-2b-a}{\sqrt{(x-b)}}$.

(5) $\frac{dy}{dx}=-\frac{y}{\sqrt{(a^2-y^2)}}$.

(6) $y=1+\frac{x}{3}-\frac{x^3}{3^4}+\ldots$

Vanishing Fractions, p. 539.

(1) 0. (2) $-\frac{3}{2}$. (3) 0. (4) -1.

(5) $\frac{1}{2}$. (6) 0. (7) 0. (8) 1.

(9) $\frac{1}{2}$. (10) -1. (11) 1. (12) 1.

Vanishing Fractions, p. 541.

(1) $\frac{dy}{dx}=\sqrt[3]{\frac{b}{a}}$. (2) $\frac{dy}{dx}=\sqrt{2}$, or 0.

(3) $\frac{dy}{dx}=\pm\sqrt{\frac{a}{b}}$, or 0.

(4) $\frac{dy}{dx}=3$, -1, -2.

Maxima and Minima, p. 547.

(4) $a=b$. (5) $x=e$.

(6) The equal conjugates.

(7) Rad. of base $=\frac{1}{4}\sqrt{2}\times$alt.

(8) $r=\frac{a}{\pi}$. (9) $4\frac{11}{16}$ in. (10) 109°28′.

(11) 116°14′21″.

(12) $x=2$, a min.; $y=-2$: that is, for adjacent values of x, y will be a greater negative number than for $x=2$.

The Pendulum, &c., p. 616.

(1) $\frac{r}{2}\sqrt{2}$. (2) $r\sqrt{\frac{2}{5}}$.

(3) $a\sqrt{\frac{3}{10}}$. (4) $\frac{2r^2}{5a}$.

(5) $\frac{4+n^2}{5}a$. (6) $\frac{1}{240}l$.

(7) ·055 in. (8) 1 mile.

(9) 33·885 secs. (10) 978 ft.

(11) 1·73 secs.

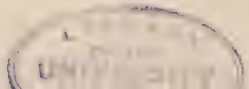
(12) $x=\frac{a}{w}\{\sqrt{(W^2+Ww)}-W\}$.

Page 572. The co-ordinates of the four cusps are

$$\alpha=0,\ \beta=\pm\frac{a^2-b^2}{b},\text{ and}$$

$$\beta=0,\ x=\pm\frac{a^2-b^2}{a}.$$

THE END.



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